Boolean Algebra

Boolean Algebra introduced by George Boole in 1854, is an algebraic structure defined by a set of elements, B, together with two binary operators '+' & '·', provided the following postulates (Huntington postulates) are satisfied.

1. The closure property holds with respect to the operators '+' & '·'.

2. (a) There is an identity element with respect to +, designated by 0:

$$x+0 = x = 0+x$$

(b) There is an identity element with respect to ·, designated by 1:

$$x \cdot 1 = x = 1 \cdot x$$

(3) It is commutative with respect to '+' & '·'

$$x+y = y+x \quad \& \quad x \cdot y = y \cdot x$$

(4) (a) '·' is distributive over '+'.

$$x \cdot (y+z) = (x \cdot y) + (x \cdot z)$$

(b) + is distributive over '·'

$$x+(y \cdot z) = (x+y) \cdot (x+z).$$

(5) For every element $x \in B$, $\exists$ an element $x' \in B$ such that $x + x' = 1$ & $x \cdot x' = 0$.
x' is called the complement of x.

(6) There exists at least two elements $x, y \in B$ such that $x \neq y$.

Boolean algebra can be formulated with the choice of the elements of B.
Two-valued Boolean algebra is defined on a set of two elements, $B = \{0, 1\}$.
The rules for the two binary operators '+' & '·' are the same as the AND, OR & NOT operators as follows.

AND

x	y	$x \cdot y$
0	0	0
0	1	0
1	0	0
1	1	1

OR

x	y	$x+y$
0	0	0
0	1	1
1	0	1
1	1	1

NOT

x	x'
0	1
1	0

Now let us check the postulates for the set $B = \{0, 1\}$.

(1) closure :

It is obvious from the table, as the result of each operation is either 1 or 0 & $0, 1 \in B$.

(2) Identity elements :

(a) $0+0 = 0$ & $0+1 = 1+0 = 1$

[0 is the identity with respect to +]

(b) $1.1 = 1$ & $1.0 = 0.1 = 0$

[1 is the identity with respect to 0]

(3) commutative law :

Commutative laws are satisfied from the symmetry of the binary operator tables.

(4) Distributive law :

The values of $x.(y+z)$ & $(x.y) + (x.z)$ are shown in the following table, which shows that the distributive law is satisfied.

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x	y	z	$y+z$	$x\cdot(y+z)$	$x\cdot y$	$x\cdot z$	$(x\cdot y)+(x\cdot z)$
0	0	0	0	0	0	0	0
0	0	1	1	0	0	0	0
0	1	0	1	0	0	0	0
0	1	1	1	0	0	0	0
1	0	0	0	0	0	0	0
1	0	1	1	1	0	1	1
1	1	0	1	1	1	0	1
1	1	1	1	1	1	1	1

(5) Complement →

The complements are calculated as follows.

(a) $x+x'=1$,

Since $0+0'=0+1=1$

& $1+1'=1+0=1$

(b) $x.x'=0$

Since $0.0'=0.1=0$

& $1.1'=1.0=0$

(6) postulate 6 is satisfied as the two valued Boolean algebra has two distinct elements 1 & 0 with $1\neq 0$.

DUALITY

The duality principle states that every algebraic expression remains valid if we interchange the operators & identity.

To obtain the dual of an algebraic expression, the OR & AND operators are interchanged, 1 is replaced by 0 & 0 is replaced by 1.

e.g. the following two ~~opeso~~ expressions are dual of each other.

$x+0=x$ & $1.x=x$.

Theorems of Boolean Algebra

1 → (a) $x+x=x$ (b) $xx=x$

2 → (a) $x+1=1$ (b) $x0=0$

3 → $(x')'=x$,
Involution

4 → (a) $x+(y+z)=(x+y)+z$ (b) $(xy)z=x(yz)$

5 → (a) $(x+y)'=x'y'$; (b) $(xy)'=x'+y'$
De Morgan's law

6 → (a) $x+xy=x$, (b) ~~x+yz~~ $x(x+y)=x$
Absorption law

Theorem-1 :→

(a) $x+x=x$

Pf :→ $(x+x) = (x+x)1$ [Since $x1=x$]

$= (x+x)(x+x')$ [$\because x+x'=1$]

$= x+xx'$ [Distributive law]

$= x+0$ $\because xx'=0$

$= x$

(b) $xx=x$

Pf :→ $xx = xx+0$ [$\because x+0=x$]

$= xx+xx'$ [$\because xx'=0$]

$= x(x+x')$ [Distributive law]

$= x1$ [$\because x+x'=1$]

$= x$ [$\because x1=x$]

Theorem-2 →

(a) $x+1=1$

Pf :→ $x+1 = 1(x+1)$ [$\because x1=x$]

$= (x+x')(x+1)$ [$\because x+x'=1$]

$= x+x'1$

$= x+x'$ [$\because x1=x$]

$= 1$ [$\because x+x'=1$]

(b) $x0=0$

Pf → We know that $x+1=1$.
Using the duality principle, the dual of the expression $x+1=1$ is $x0=0$, which is a valid expression.

Theorem - 3 →

$$(x')'=x$$

Pf → We know that

$$x+x'=1 \quad \& \quad xx'=0,$$

which defines the complement of x.
The complement of x' is x.
Since the complement of x' is unique & is denoted by $(x')'$,
we have $(x')'=x$.

These theorems can also be proved using a truth table.

Truth table for De Morgan's Theorem.

x	y	$x+y$	$(x+y)'$	x'	y'	$x'y'$
0	0	0	1	1	1	1
0	1	1	0	1	0	0
1	0	1	0	0	1	0
1	1	1	0	0	0	0

We observe that both the expressions $(x+y)'$ & $x'y'$ have the same truth values.

Thus $(x+y)' = x'y'$

Other theorems can be proved similarly.

Boolean Functions

A Boolean funct^n is a Boolean expression of one or more variables binary operators + & '.

As each variable can take the values of 0 or 1, the truth value of a Boolean funct^n is 0 or 1.

e.g.

$F_1 = xyz'$

$$= \begin{cases} 1 & \text{if } x=1, y=1 \text{ \& } z'=1 \\ 0, & \text{otherwise} \end{cases}$$

$F_2 = x+y+z'$

$$= \begin{cases} 0, & \text{if } x=0, y=0, z'=0 \\ 1, & \text{otherwise} \end{cases}$$

Simplification of Boolean funct^n's

The objective of simplification of a Boolean funct^n is to minimize the number of literals.

Ex Simplify the following Boolean funct^n's

(a) $F_1 = x + x'y + xy$

$= x + y(x' + x)$

$= x + y1 \quad [\because x' + x = 1]$

$= x + y \quad [\because x1 = x]$

(b) $F_2 = (xy' + w'z)(wx' + yz')$

$= \cancel{xy'wx' + xy'yz' + w'zwx' + w'zyz'}$

$= xy'wx' + xy'yz' + w'zwx' + w'zyz'$

$= y'w(xx') + xz'(y'y) + (w'w)zx' + w'y(zz')$

$= y'w0 + xz'0 + 0zx'0 + w'y0$

$= 0$

(c) $F_3 = x + x'y$

$= (x + x')(x + y) \quad$ [Distributive law]

$= 1(x + y) \quad [\because x + x' = 1]$

$= x + y$

Canonical Form

For given n input variables, infinite number of logical expressions are possible. Many of these logic expressions are equivalent.

The canonical form of each Boolean funct^n is unique, hence it is useful in the identification of ~~logical~~ equivalent logical expressions as they have the same canonical form.
Here we shall discuss two canonical forms: sum of minterms & product of maxterms.

Let us consider two binary variables x & y combined with AND operation. Since a binary variable may apper in the normal form (x) or in the complement form (x'), there are four possible combinations:

$x'y'$, $x'y$, xy' & xy

Each of these terms is called a minterm ~~or~~ or standard product.

Similarly, for two binary variables x & y combined with an 'OR' operation, are

$x'+y'$, $x'+y$, $x+y'$ & $x+y$.

Each of these four terms is called a maxterm or standard sum.

Maxterm & minterm are complement to each other.

The n variables can be combined to form 2^n minterms or maxterms.

Variables			Minterms		Maxterms	
x	y	z	Term	Symbol	Term	Symbol
0	0	0	$x'y'z'$	m_0	$x+y+z$	M_0
0	0	1	$x'y'z$	m_1	$x+y+z'$	M_1
0	1	0	$x'yz'$	m_2	$x+y'+z$	M_2
0	1	1	$x'yz$	m_3	$x+y'+z'$	M_3
1	0	0	$xy'z'$	m_4	$x'+y+z$	M_4
1	0	1	$xy'z$	m_5	$x'+y+z'$	M_5
1	1	0	xyz'	m_6	$x'+y'+z$	M_6
1	1	1	xyz	m_7	$x'+y'+z'$	M_7

Table-A

A Boolean funct^n expressed in the form of a sum of minterms or product of maxterms is said to be in the canonical form.

A Boolean function can be expressed as a sum (OR) of minterms or product (AND) of maxterms using a truth table

Ex

Express the function

$F = xz + x'y$ as a sum of minterms.

Sol^n ⇒

x	y	z	x'	xz	$x'y$	$F = xz + x'y$
0	0	0	1	0	0	0
0	0	1	1	0	0	0
0	1	0	1	0	1	1
0	1	1	1	0	1	1
1	0	0	0	0	0	0
1	0	1	0	1	0	1
1	1	0	0	0	0	0
1	1	1	0	1	0	1

There are four 1's in the above truth table.

The minterms corresponding to the 1's in the truth table-A are

$x'yz'$, $x'yz$, $xy'z$ & xyz.

$\therefore$ The function as a sum of minterms can be written as

$$F = x'yz' + x'yz + xy'z + xyz.$$

Note $\Rightarrow$ To express a Boolean funct^n as a product of maxterms directly from the truth table, we first construct the truth table of the Boolean funct^n, and then write the maxterms for each combination of variables that produces a '0' in the funct^n. Finally, taking AND of all these maxterms provides the required form.

Ex $\Rightarrow$ Express the funct^n $F = xz + x'y$ as a product of maxterms.

Sol^n $\Rightarrow$ From the truth table-A, the maxterms corresponding to 0's in the truth table are

$x+y+z$, $x+y+z'$, $x'+y+z$ & $x'+y'+z$

The funct^n as a product of maxterms can be written as

$$F = (x+y+z)(x+y+z')(x'+y+z)(x'+y'+z).$$