Z-transform

The Z-transform is an important tool for analysis of linear-time invariant systems in frequency domain.

The Z-transform plays the same role in the analysis of discrete-time signals and LTI systems as the Laplace transform does in the analysis of continuous-time signals & LTI systems.

For ex in Z-domain the convolution of two time domain signals is equivalent to multiplication of their corresponding Z-transforms.

This property greatly simplifies the analysis of the response of an LTI system to various signals. In addition, the Z-transform provides us with a means of characterizing an LTI system & its response to various signals by its pole-zero locations.

The Z-transform

The Direct Z-transform

The Z-transform of a discrete-time signal $x(n)$ is defined as the power series

$$\boxed{X(Z) \equiv \sum_{n=-\infty}^{\infty} x(n) Z^{-n}}$$

where Z is a complex variable. The relation is sometimes called the direct Z-transform because it transforms the time-domain signal $x(n)$ into its complex plane representation $X(Z)$

For convenience, the Z-transform of a signal $x(n)$ is denoted by

$$X(Z) \equiv Z\{x(n)\}$$

whereas the relationship between $x(n)$ & $X(Z)$ is indicated by

$$x(n) \xleftrightarrow{Z} X(Z)$$

Since, the Z-transform is an infinite power series, it exists only for those values of Z for which this series converges.

The region of convergence (ROC) of $X(Z)$ is the set of all values of Z for which $X(Z)$ attains a finite value.

Determine the Z-transform of the following finite-duration signals

(i) $x_1(n) = \{1, 2, \underset{\uparrow}{5}, 7, 0, 1\}$

$$X_1(Z) = \sum_{n=-\infty}^{\infty} x_1(n) Z^{-n}$$

$$= 1 \times Z^2 + 2 \times Z + 5 + 7Z^{-1} + 1 \times Z^{-3}$$

$$X(Z) = Z^2 + 2Z + 5 + 7Z^{-1} + Z^{-3}$$

ROC: entire Z-plane except $Z=0$ & $Z=\infty$

(ii) $x_2(n) = \{0, 0, \underset{\uparrow}{1}, 2, 5, 7, 0, 1\}$

$$X_2(Z) = 1 + 2Z^{-1} + 5Z^{-2} + 7Z^{-3} + Z^{-5}$$

ROC: entire Z-plane except $Z=0$

(iii) $x_3(n) = \{3, 2, -1, 0, 1, \underset{\uparrow}{3}\}$

$$X_3(Z) = 3Z^5 + 2Z^{+4} - 1Z^3 + Z^{+1} + 3$$

ROC: entire Z-plane except $Z=\infty$

Z-transform & ROC of infinite duration sequence

Determine the Z-transform and ROC of the signal $x(n) = a^n u(n)$

$x(n)$ consists of infinite no. of nonzero values

$X(Z) = \quad x(n) = \{a, a^2, a^3, \ldots (a)^n \ldots\}$

The Z-transform of infinite power series

$$X(Z) = \sum_{n=-\infty}^{\infty} x(n) Z^{-n}$$

$$= \sum_{n=0}^{\infty} a^n Z^{-n} = \sum_{n=0}^{\infty} (aZ^{-1})^n$$

This is an infinite series & we know

$$\sum_{n=0}^{\infty} r^n = 1 + r + r^2 + \cdots \frac{1}{1-r} \text{ if } |r|<1$$

therefore $|aZ^{-1}| < 1$ or $|Z| > a$, $X(Z)$ converges & is given by

$$X(Z) = \frac{1}{1-aZ^{-1}} = \frac{Z}{Z-a}$$

Find the z-transform and ROC of the signal
$x(n) = -a^n u(-n-1)$

the given signal is of infinite duration & anticausal. We know

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

$$\boxed{\begin{aligned} u(-n-1) &= 0 \quad \text{for } n \geq 0 \\ &= 1 \quad \text{for } n < 1 \end{aligned}}$$

$$X(z) = \sum_{n=-\infty}^{-1} (-a^n) z^{-n} \qquad \left(\frac{1}{az}\right)^{-n}$$

$$= \sum_{n=1}^{\infty} (-a^n) z^{-n}$$

$$= -\sum_{l=1}^{\infty} (a^{-1}z)^l$$

where $l = -n$
$n = \infty, l = \infty$
$n = -1, l = 1$

Using the formula

$$A + A^2 + A^3 + \ldots = A(1 + A + A^2 + \ldots) = \frac{A}{1-A}$$

$$= -\frac{a^{-1}z}{1 - a^{-1}z} = \frac{-1}{\frac{1}{a^{-1}z} - 1} = +\frac{1}{1 - az^{-1}}$$

the above series converges

$$|a^{-1}z| < 1, \text{ or } |z| < |a|$$

therefore $x(n) = -a^n u(-n-1) \xleftrightarrow{z} X(z) = \dfrac{1}{1 - az^{-1}}$

ROC: $|z| < |a|$

$$\left(\frac{1}{az}\right)^{-n}$$

$$\frac{1}{1-x} - 1 = \frac{1 - x + x}{1 - x}$$

$$= \frac{-1}{az^{-1} - 1}$$

$$= \frac{-1}{-(1 - az^{-1})}$$

Determine the Z-transform of the signal
$x(n) = a^n u(n) + b^n u(-n-1)$

The Z-transform is given by

$$X(z) = \sum_{n=0}^{\infty} a^n z^{-n} + \sum_{n=-\infty}^{-1} b^n z^{-n}$$

$$= \sum_{n=0}^{\infty} (az^{-1})^n + \sum_{l=1}^{\infty} (b^{-1}z)^l$$

The first power series converges if $|az^{-1}| < 1$ or $|z| > |a|$.

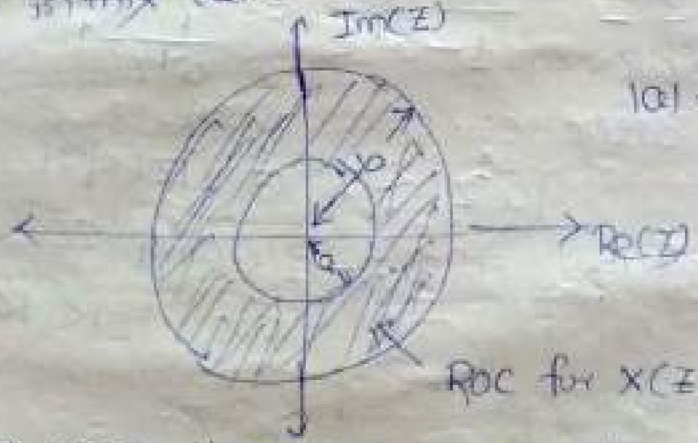
The second power series converges if $|b^{-1}z| < 1$ or $|z| < |b|$.

In determining the convergence of $X(z)$, we have to consider two different cases.

case-I $|b| < |a|$

In this case the two ROC above donot overlap as shown. Consequently, we can't find values of z for which both power series converge simultaneously. Clearly in this case $X(z)$ doesnot exist.

case-II $|a| < |b|$



In this case there is a ring in the Z-plane where both power series converges simultaneously as shown in fig. Then we obtain

$$X(z) = \frac{1}{1-az^{-1}} - \frac{1}{1-bz^{-1}}$$

$$= \frac{z}{z-a} - \frac{z}{z-b} \qquad \text{ROC:- } |a| < |z| < |b|$$

Properties of Z transform.

The Z-transform is a very powerful tool for the study of discrete time signals & systems. The power of this transform is a consequence of some very important properties that the transform possesses.

① **Linearity** If $x_1(n) \xleftrightarrow{Z} X_1(Z)$

& $x_2(n) \xleftrightarrow{Z} X_2(Z)$

then $x(n) = a_1 x_1(n) + a_2 x_2(n) \xleftrightarrow{Z} X(Z) = a_1 X_1(Z) + a_2 X_2(Z)$

for any constants a_1 & a_2.

Proof $Z\{a_1 x_1(n) + a_2 x_2(n)\}$

$$= \sum_{n=-\infty}^{\infty} \{a_1 x_1(n) + a_2 x_2(n)\} Z^{-n}$$

$$= a_1 \sum_{n=-\infty}^{\infty} x_1(n) Z^{-1} + a_2 \sum_{n=-\infty}^{\infty} x_2(n) Z^{-n}$$

$$= a_1 X_1(Z) + a_2 X_2(Z)$$

> Determine the Z-transform & ROC of the signal
> $$x(n) = [3(2)^n - 4(3)^n] u(n)$$

If we define the signals

$$x_1(n) = (2)^n u(n)$$

$$x_2(n) = (3)^n u(n)$$

$x(n)$ can be written as

$$x(n) = 3x_1(n) - 4x_2(n)$$

Accordingly, its Z transform is

$$X(Z) = 3X_1(Z) - 4X_2(Z)$$

We known that

$$a^n u(n) \xleftrightarrow{Z} \frac{1}{1-aZ^{-1}} \quad \text{ROC } |Z| > |a|$$

$$x_1(n) = 2^n u(n) \xleftrightarrow{Z} X_1(Z) = \frac{1}{1-2Z^{-1}} \quad |Z| > 2$$

$$x_2(n) = 3^n u(n) \xleftrightarrow{Z} X_2(Z) = \frac{1}{1-3Z^{-1}} \quad \text{ROC: } |Z| > 3$$

The intersection of ROC of $X_1(Z)$ & $X_2(Z)$ is $|Z| > 3$.

$$X(Z) = \frac{1}{1-2Z^{-1}} - \frac{1}{1-3Z^{-1}} \quad \text{ROC: } |Z| > 3$$

Determine the Z-transform of the signals
(a) $x(n) = (\cos \omega_0 n)u(n)$ (b) $x(n) = (\sin \omega_0 n)u(n)$

(a) By using Euler's identity, the signal $x(n)$ can be expressed as

$$x(n) = (\cos \omega_0 n)\, u(n)$$

$$= \frac{1}{2} e^{j\omega_0 n} u(n) + \frac{1}{2} e^{-j\omega_0 n} u(n)$$

$$X(Z) = \frac{1}{2} Z\{e^{j\omega_0 n} u(n)\} + \frac{1}{2} Z\{e^{-j\omega_0 n} u(n)\}$$

let $\alpha = e^{\pm j\omega_0}$ $|\alpha| = |e^{\pm j\omega_0}| = 1$ $|e^{j\omega_0} z^{-1}| < 1$

$$= \frac{1}{2}\left[\sum_{n=0}^{\infty} (e^{j\omega_0} z^{-1})^n + \sum_{n=0}^{\infty} (e^{-j\omega_0} z^{-1})^n\right]$$ ROC: $|Z| > 1$

$$a^n u(n) \xleftrightarrow{Z} \frac{1}{1 - a z^{-1}}$$ ROC: $|Z| > |a|$

$$X(Z) = \frac{1}{2} \frac{1}{1 - e^{j\omega_0} Z^{-1}} + \frac{1}{2} \frac{1}{1 - e^{-j\omega_0} Z^{-1}}$$

$$= \frac{1 - Z^{-1}\cos\omega_0}{1 - 2Z^{-1}\cos\omega_0 + Z^{-2}}$$ ROC: $|Z| > 1$

(b) By Euler identity

$$x(n) = (\sin \omega_0 n)\, u(n)$$

$$= \frac{1}{2j}\left[e^{j\omega_0 n} u(n) - e^{-j\omega_0 n} u(n)\right]$$

$$X(Z) = \frac{1}{2j}\left(\frac{1}{1 - e^{j\omega_0} Z^{-1}} - \frac{1}{1 - e^{-j\omega_0} Z^{-1}}\right)$$

$$= \frac{Z^{-1}\sin\omega_0}{1 - 2Z^{-1}\cos\omega_0 + Z^{-2}}$$ ROC: $|Z| > 1$

(ii) Time shifting

If $x(n) \xleftrightarrow{Z} X(Z)$

then $x(n-k) \xleftrightarrow{Z} Z^{-k} X(Z)$

Proof

$$Z\{x(n-k)\} = \sum_{n=-\infty}^{\infty} x(n-k) Z^{-n}$$

$$= Z^{-k} \sum_{n=-\infty}^{\infty} x(n-k) Z^{-(n-k)}$$

$X(Z) = \sum_{n=-\infty}^{\infty} x(n) Z^{-n}$

let $n-k = l$

$$Z\{x(n-k)\} = Z^{-k} \sum_{l=-\infty}^{\infty} x(l) Z^{-l}$$

$$= Z^{-k} X(Z)$$

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Find the Z transform of the sequence
$$x(n) = \left(\frac{1}{3}\right)^{n-1} u(n-1)$$

$$a^n u(n) \xleftrightarrow{Z} \frac{1}{1-aZ^{-1}} \quad \text{for } |Z| > |a|$$

$$\left(\frac{1}{3}\right)^n u(n) \xleftrightarrow{Z} \frac{1}{1-\left(\frac{1}{3}\right)Z^{-1}} = \frac{Z}{Z-1/3}$$

using time shifting property
$$Z\{x(n-1)\} = Z^{-1}X(Z)$$

$$Z\{\underbrace{\left(\frac{1}{3}\right)^{n-1} u(n-1)}_{x(n-k)}\} = Z^{-1}\frac{Z}{Z-\frac{1}{3}} = \frac{1}{Z-\frac{1}{3}}$$

$$x_1(n-k) \xleftrightarrow{Z} Z^{-k}X_1(Z)$$

(iii) Scaling in Z-domain

If $x(n) \xleftrightarrow{Z} X(Z)$

then $a^n x(n) \longleftrightarrow X(a^{-1}Z)$

Proof:-
$$Z\{a^n x(n)\} = \sum_{n=-\infty}^{\infty} a^n x(n) Z^{-n}$$
$$= \sum_{n=-\infty}^{\infty} x(n)(a^{-1}Z)^{-n}$$
$$= X(a^{-1}Z)$$

Determine the Z-transform of the signals
(a) $x_1(n) = a^n (\cos\omega_0 n) u(n)$
(b) $x(n) = a^n (\sin\omega_0 n) u(n)$

(a) $$Z\{(\cos\omega_0 n)u(n)\} = \frac{1-Z^{-1}\cos\omega_0}{1-2Z^{-1}\cos\omega_0+Z^{-2}}$$

using scaling property
$$Z\{a^n x(n)\} = X(a^{-1}Z)$$

$$Z\{a^n(\cos\omega_0 n)u(n)\}$$
$$= \frac{1-(a^{-1}Z)^{-1}\cos\omega_0}{1-2(a^{-1}Z)^{-1}\cos\omega_0+(a^{-1}Z)^{-2}}$$
$$= \frac{1-aZ^{-1}\cos\omega_0}{1-2aZ^{-1}\cos\omega_0+a^2Z^{-2}} \quad |Z|>|a|$$

(b) $Z\{\sin\omega_0 n\, u(n)\} = \dfrac{Z^{-1}\sin\omega_0}{1-2Z^{-1}\cos\omega_0 + Z^{-2}}$

$Z\{a^n \sin\omega_0 n\, u(n)\}$

$$= \frac{(a^{-1}Z)^{-1}\sin\omega_0}{1-2(a^{-1}Z)^{-1}\cos\omega_0 + (a^{-1}Z)^{-2}}$$

$$= \frac{aZ^{-1}\sin\omega_0}{1-2aZ^{-1}\cos\omega_0 + a^2 Z^{-2}} \qquad |Z| > |a|$$

(iv) Time reversal

If $x(n) \xleftrightarrow{Z} X(Z)$

then $x(-n) \xleftrightarrow{Z} X(Z^{-1})$

Proof: $Z\{x(-n)\} = \sum_{n=-\infty}^{\infty} x(-n) Z^{-n}$

$l = -n$

$$= \sum_{l=-\infty}^{\infty} x(l)(Z^{-1})^{-l}$$

$$= X(Z^{-1}) \quad \text{ROC:-}$$

Determine the Z-transform of the signal $x(n) = u(-n)$

$u(n) \xleftrightarrow{Z} \dfrac{1}{1-Z^{-1}}$ ROC $|Z|>1$

By using time reversal property

$u(-n) \xleftrightarrow{Z} \dfrac{1}{1-(Z^{-1})^{-1}} = \dfrac{1}{1-Z}$

ROC $|Z| < 1$

(v) Differentiation in Z-domain

If $x(n) \xleftrightarrow{Z} X(Z)$

then $n x(n) \xleftrightarrow{Z} -Z\dfrac{d}{dZ}X(Z)$

Proof $X(Z) = \sum_{n=-\infty}^{\infty} x(n) Z^{-n}$

Differentiating both sides, we have

$$\frac{d}{dZ}X(Z) = \sum_{n=-\infty}^{\infty} x(n)\frac{d}{dZ}Z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} x(n)\,(-n)\,Z^{-n-1}$$

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$$= -\frac{1}{z}\sum_{n=-\infty}^{\infty} x(n)\, n\, z^{-n}$$

multiplying bothsides with $-z$ we get

$$-z\frac{d}{dz}X(z) = \sum_{n=-\infty}^{\infty} n x(n) z^{-n}$$

$$= Z\{n x(n)\}$$

Determine the z-transform of the signal $x(n) = na^n u(n)$

$$a^n u(n) \xleftrightarrow{Z} \frac{1}{1-az^{-1}} \quad ROC\ |z|>|a|$$

$$na^n u(n) \xleftrightarrow{Z} \sum_{n=-\infty}^{\infty}\{n a^n u(n)\} z^{-n}$$

$$= -z\frac{d}{dz}\frac{1}{1-az^{-1}}$$

$$= -z\frac{(1-az^{-1})\cdot 0 - 1\cdot(-a)(-1)z^{-2}}{(1-az^{-1})^2}$$

$$= \frac{-\cancel{z}\,a}{z^{\cancel{2}}(1-az^{-1})^2} = \frac{az^{-1}}{(1-az^{-1})^2}$$

$|z|>|a|$

(vi) Convolution of two sequences 257

If $x_1(n) \xleftrightarrow{Z} X_1(Z)$

$x_2(n) \xleftrightarrow{Z} X_2(Z)$

then $x(n) = x_1(n) * x_2(n) \xleftrightarrow{Z} X(Z) = X_1(Z)\, X_2(Z)$

Proof: the convolution of $x_1(n)$ & $x_2(n)$

$$x(n) = \sum_{k=-\infty}^{\infty} x_1(k)\, x_2(n-k)$$

the z-transfer $X(Z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$

$$= \sum_{n=-\infty}^{\infty}\left[\sum_{k=-\infty}^{\infty} x_1(k)\, x_2(n-k)\right] z^{-n}$$

Interchanging the order of summation & applying time-shifting property

$$X(Z) = \sum_{k=-\infty}^{\infty} x_1(k)\left[\sum_{n=-\infty}^{\infty} x_2(n-k) z^{-n}\right]$$

$$= X_2(Z)\sum_{k=-\infty}^{\infty} x_1(k) z^{-k} = X_2(Z)\, X_1(Z)$$

Compute the convolution $x(n)$ of the signals
$x_1(n) = \{1, -2, 1\}$
$x_2(n) = \begin{cases} 1, & 0 \le n \le 5 \\ 0, & \text{elsewhere} \end{cases}$

$$X_1(Z) = 1 - 2Z^{-1} + Z^{-2}$$
$$X_2(Z) = 1 + Z^{-1} + Z^{-2} + Z^{-3} + Z^{-4} + Z^{-5}$$

$$X(Z) = X_1(Z)\, X_2(Z)$$
$$= 1 - Z^{-1} - Z^{-6} + Z^{-7}$$

Hence $x(n) = \{\underset{\uparrow}{1}, -1, 0, 0, 0, 0, -1, 1\}$

(vii) Correlation of two sequences

If $x_1(n) \xleftrightarrow{Z} X_1(Z)$
$x_2(n) \xleftrightarrow{Z} X_2(Z)$

then $r_{x_1 x_2}(l) = \sum_{n=-\infty}^{\infty} x_1(n) x_2(n-l) \xleftrightarrow{Z} R_{x_1 x_2}(Z)$
$$= X_1(Z)\, X_2(Z^{-1})$$

Proof: we know that
$$r_{x_1 x_2}(l) = x_1(l) * x_2(-l)$$

Using the convolution & time reversal properties, we easily obtain
$$R_{x_1 x_2}(l) = Z\{x_1(l)\}\, Z\{x_2(-l)\}$$
$$= X_1(Z)\, X_1(Z^{-1})$$

Determine the autocorrelation of sequence of the signal $x(n) = a^n u(n)$, $-1 < a < 1$

$$R_{xx}(Z) = Z\{r_{xx}(l)\} = X(Z)\, X(Z^{-1})$$

$$X(Z) = \frac{1}{1 - aZ^{-1}} \qquad \text{ROC: } |Z| > |a|$$

$$X(Z^{-1}) = \frac{1}{1 - aZ} \qquad \text{ROC: } |Z| < \frac{1}{|a|}$$

$$R_{xx}(Z) = \frac{1}{1 - aZ^{-1}} \cdot \frac{1}{1 - aZ}$$
$$= \frac{1}{1 - a(Z + Z^{-1}) + a^2} \qquad |a| < |Z| < \frac{1}{|a|}$$

Determine the signal $x(n)$ whose Z-transform is given by $X(z) = \log(1+az^{-1})$ $|z| > |a|$

By taking the first derivative of $X(z)$

$$\frac{d}{dz}X(z) = \frac{1}{1+az^{-1}} \cdot -az^{-2}$$

thus $$-z\frac{d}{dz}X(z) = az^{-1}\left[\frac{1}{1-(-a)z^{-1}}\right]$$

$$a^n u(n) \xleftrightarrow{z} \frac{1}{1-az^{-1}}$$

The inverse Z-transform of the term in the bracket $(-a)^n u(n)$

The multiplication by z^{-1} implies a time delay by one same (time shifting property) which results in $(-a)^{n-1} u(n-1)$

Finally from the differentiation property we have $$n x(n) = a(-a)^{n-1} u(n-1)$$

$$x(n) = (-1)^{n+1}\frac{a^n}{n} u(n-1)$$

Inversion of the Z-transform

The inverse Z-transform is formally given by

$$\boxed{x(n) = \frac{1}{2\pi j}\oint_c X(z)\, z^{n-1}\, dz}$$

where the integral is a contour integral over a closed path C that encloses the origin & lies within the region of convergence of $X(z)$. For simplicity, C can be taken as a circle in the ROC of $X(z)$ in the Z-plane.

The inverse Z-transform by Power Series Expansion

Find the inverse Z-transform of $X(z) = \frac{z+0.2}{(z+0.5)(z-1)}$ $|z| > 1$

From the ROC, we find that $x(n)$ is a causal sequence since ROC is the exterior of a circle

$$X(Z) = \frac{Z+0.2}{Z^2-0.5Z-0.5}$$

$$\begin{array}{r|l} & Z^{-1}+0.7Z^{-2}+0.85Z^{-3}+0.775Z^{-4} \\ \hline Z^2-0.5Z-0.5 & Z+0.2 \\ & Z-0.5-0.5Z^{-1} \\ \hline & 0.7+0.5Z^{-1} \\ & 0.7-0.35Z^{-1}-0.35Z^{-2} \\ \hline & 0.85Z^{-1}+0.35Z^{-2} \\ & 0.85Z^{-1}-0.425Z^{-2}-0.425Z^{-3} \\ \hline & 0.775Z^{-2}+0.425Z^{-3} \\ & 0.775Z^{-2}-0.387Z^{-3}-0.387Z^{-4} \\ \hline \end{array}$$

ie. $X(Z) = Z^{-1}+0.7Z^{-2}+0.85Z^{-3}+0.775Z^{-4}+\cdots$

$= \sum_{n=0}^{\infty} x(n) Z^{-n}$

$x(0)=0$, $x(1)=1$, $x(2)=0.7$, $x(3)=0.85$, $x(4)=0.775$

$x(n) = \{\underset{\uparrow}{0}, 1, 0.7, 0.85, 0.775, \cdots\}$

Find the inverse Z-transform of $X(Z) = \frac{Z}{(Z-3)(Z-4)}$ $|Z|<3$

anticausal ROC is interior of a circle

$$\begin{array}{r|l} & \frac{Z}{12}+\frac{7}{144}Z^2+\frac{37}{1728}Z^3 \\ \hline 12-7Z+Z^2 & Z \\ & Z-\frac{7}{12}Z^2+\frac{Z^3}{12} \\ \hline & \frac{7}{12}Z^2-\frac{Z^3}{12} \\ & \frac{7}{12}Z^2-\frac{49}{144}Z^3+\frac{7}{144}Z^4 \\ \hline & \frac{37}{144}Z^3-\frac{7Z^4}{144} \\ & \frac{37Z^3}{144}-\frac{259}{1728}Z^4+\frac{37}{1728}Z^5 \\ \hline \end{array}$$

$X(Z) = \frac{1}{12}Z+\frac{7}{144}Z^2+\frac{37}{1728}Z^3+\cdots$

$x(n) = \{\cdots \frac{37}{1728}, \frac{7}{144}, \frac{1}{12}, \underset{\uparrow}{0}\}$

The inverse Z-transform by partial-fraction Expansion:

Find the inverse Z-transform of $X(Z) = \frac{1+3Z^{-1}}{1+3Z^{-1}+2Z^{-2}}$, $|Z| > 2$

$$X(Z) = \frac{1+3Z^{-1}}{1+3Z^{-1}+2Z^{-2}}$$

$$= \frac{Z(Z+3)}{Z^2+3Z+2} = \frac{Z(Z+3)}{(Z+1)(Z+2)}$$

Dividing $\frac{X(Z)}{Z} = \frac{Z+3}{(Z+1)(Z+2)}$

The above equation can be written in partial fraction form as

$$\frac{X(Z)}{Z} = \frac{C_1}{Z+1} + \frac{C_2}{Z+2}$$

ie $C_1 = (Z+1) \frac{Z+3}{(Z+1)(Z+2)} \Big|_{Z=-1}$

$= 2$

$C_2 = (Z+2) \frac{X(Z)}{Z} \Big|_{Z=-2}$

$= \cancel{(Z+2)} \frac{Z+3}{(Z+1)\cancel{(Z+2)}} \Big|_{Z=-2}$

$= -1$

therefore $\frac{X(Z)}{Z} = \frac{2}{Z+1} - \frac{1}{Z+2}$

$$X(Z) = 2\frac{Z}{Z+1} - \frac{Z}{Z+2}$$

ROC $|Z| > 2$ the sequence is causal

$$\boxed{x(n) = 2(-1)^n u(n) - (-2)^n u(n)}$$

Note $\frac{X(Z)}{Z} = \frac{C_1}{Z-P_1} + \frac{C_2}{Z-P_2} + \cdots + \frac{C_N}{Z-P_N}$

We can determine the coefficients $C_1, C_2, \cdots, C_N$ by using the formula

$$C_K = \frac{(Z-P_K)X(Z)}{Z} \Big|_{Z=P_K}$$

$K = 1, 2, \cdots, N$

Find the inverse z transform of $X(z) = \frac{z(z^2-4z+5)}{(z-3)(z-1)(z-2)}$ for ROC (i) $2<|z|<3$ (ii) $|z|>3$ (iii) $|z|<1$

$$X(z) = \frac{z(z^2-4z+5)}{(z-1)(z-2)(z-3)}$$

$$\frac{X(z)}{z} = \frac{z^2-4z+5}{(z-1)(z-2)(z-3)}$$

$$= \frac{c_1}{z-1} + \frac{c_2}{z-2} + \frac{c_3}{z-3}$$

$$c_1 = (z-1)\frac{X(z)}{z}\Big|_{z=1}$$

$$= \cancel{(z-1)}\frac{z^2-4z+5}{\cancel{(z-1)}(z-2)(z-3)}\Big|_{z=1}$$

$$= \frac{2}{2} = 1$$

$$c_2 = (z-2)\frac{X(z)}{z}\Big|_{z=2}$$

$$= \cancel{(z-2)}\frac{z^2-4z+5}{(z-1)\cancel{(z-2)}(z-3)}\Big|_{z=2}$$

$$= -1$$

$$c_3 = (z-3)\frac{X(z)}{z}\Big|_{z=3}$$

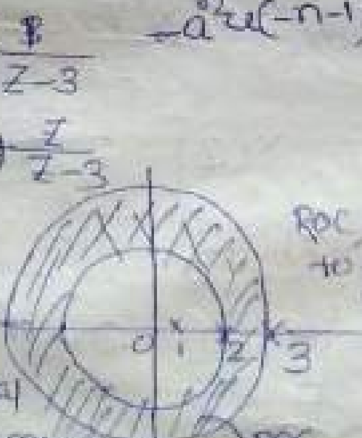
$$= \cancel{(z-3)}\frac{z^2-4z+5}{(z-1)(z-2)\cancel{(z-3)}}\Big|_{z=3}$$

$$= 1$$

$$\frac{X(z)}{z} = \frac{1}{z-1} - \frac{1}{z-2} + \frac{1}{z-3} \qquad -a^n u(-n-1) \longleftrightarrow \frac{1}{1-az^{-1}}$$

$$X(z) = \frac{z}{z-1} - \frac{z}{z-2} + \frac{z}{z-3}$$

(i) $2 < |z| < 3$



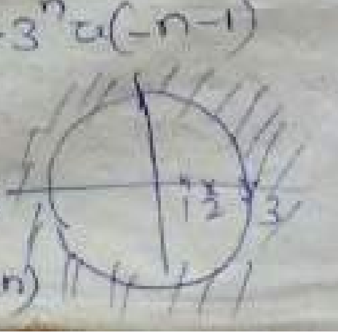
ROC is exterior to the pole $z=1$ & $z=2$

the signal $x(n)$ is two sided. The poles $z=1$ & $z=2$ provide causal part & pole $z=3$ the anticausal part.

$$x(n) = u(n) - 2^n u(n) - 3^n u(-n-1)$$

(ii) $|z| > 3$

$x(n)$ is causal ROC is exterior to all the three poles



$$x(n) = u(n) - 2^n u(n) + 3^n u(n)$$

(iii) $|z|<1$

$x(n)$ is anticausal & all the three terms are anticausal terms

$$x(n) = -u(-n-1) + (2)^n u(-n-1) - 3^n u(-n-1)$$

Note - If $X(z) = \frac{1}{(z-p_2)^2(z-p_1)}$ the equation can be written as

$$\frac{X(z)}{z} = \frac{c_1}{z-p_1} + \frac{c_2}{z-p_2} + \frac{c_3}{(z-p_2)^2}$$

where $c_1 = (z-p_1)\frac{X(z)}{z}\Big|_{z=p_1}$

$$c_2 = \frac{d}{dz}(z-p_2)^2\frac{X(z)}{z}\Big|_{z=p_2}$$

$$c_3 = (z-p_2)^2\frac{X(z)}{z}\Big|_{z=p_2}$$

Determine the causal signal $x(n)$ having z-transform $X(z) = \frac{1}{(1-2z^{-1})(1-z^{-1})^2}$

$$X(z) = \frac{1}{(1-2z^{-1})(1-z^{-1})^2} = \frac{1}{(1-\frac{2}{z})(1-\frac{1}{z})^2}$$

$$= \frac{z^3}{(z-2)(z-1)^2}$$

$$\frac{X(z)}{z} = \frac{z^2}{(z-2)(z-1)^2} = \frac{c_1}{z-2} + \frac{c_2}{z-1} + \frac{c_3}{(z-1)^2}$$

where $c_1 = (z-2)\frac{X(z)}{z}\Big|_{z=2}$

$$= \cancel{(z-2)}\frac{z^2}{\cancel{(z-2)}(z-1)^2}\Big|_{z=2} = 4$$

$$c_2 = \frac{d}{dz}(z-1)^2\frac{X(z)}{z}\Big|_{z=1}$$

$$c_3 = (z-1)^2\frac{z^2}{(z-2)(z-1)^2}\Big|_{z=1} = -1$$

$Z(u(n)) = \frac{1}{1-z^{-1}}$

$$= \frac{d}{dz}\cancel{(z-1)^2}\frac{z^2}{(z-2)\cancel{(z-1)^2}}\Big|_{z=1}$$

$n\,u(n) \xleftrightarrow{Z} -z\frac{d}{dz}\left\{\frac{z}{z-1}\right\}$

$$= \frac{(z-2)2z - z^2}{(z-2)^2}\Big|_{z=1} = -3$$

$$\frac{X(z)}{z} = \frac{4}{z-2} - \frac{3}{z-1} - \frac{1}{(z-1)^2}$$

$$X(z) = \frac{4z}{z-2} - \frac{3z}{z-1} - \frac{z}{(z-1)^2}$$

$$x(n) = 4(2)^n u(n) - 3u(n) - n\,u(n)$$

The System Function of a Linear Time-invariant System

The output of a linear time-invariant system to an input sequence $x(n)$ can be obtained by computing the convolution of $x(n)$ with the unit sample response of the system

in Z-domain $\boxed{Y(Z) = H(Z)\,X(Z)}$

where $Y(Z)$ is the Z-transform of the output sequence $y(n)$, $X(Z)$ is the Z-transform of the input sequence $x(n)$ and $H(Z)$ is the Z-transform of the unit sample response $h(n)$.

If we know $h(n)$ and $x(n)$ we can determine their corresponding Z-transforms $H(Z)$ & $X(Z)$ multiply them to obtain $Y(Z)$ & therefore determine $y(n)$ by evaluating the inverse Z-transform of $Y(Z)$

Alternatively if we know $x(n)$ and we observe the output $y(n)$ of the system, we can determine the unit sample response by first solving for $H(Z)$ from the relation

$$\boxed{H(Z) = \frac{Y(Z)}{X(Z)}}$$

and then evaluating the inverse Z-transform of $H(Z)$

since $$\boxed{H(Z) = \sum_{n=-\infty}^{\infty} h(n)\, Z^{-n}}$$

It is clear that $H(Z)$ represents the Z-domain characterization of a system, whereas $h(n)$ is the corresponding time-domain characterization of the system

The transform $H(Z)$ is called the system function

When the system is described by a linear constant-coefficient difference equation of the form

$$y(n) = -\sum_{k=1}^{N} a_k\, y(n-k) + \sum_{k=0}^{M} b_k\, x(n-k)$$

In this case the system function can be determined directly by computing the Z-transform of both sides. Thus by applying the time-shifting property, we obtain

$$Y(z) = -\sum_{k=1}^{N} a_k Y(z) z^{-k} + \sum_{k=0}^{M} b_k X(z) z^{-k}$$

$$Y(z)\left(1+\sum_{k=1}^{N} a_k z^{-k}\right) = \left(\sum_{k=0}^{M} b_k z^{-k}\right) X(z)$$

$$\frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{M} b_k z^{-k}}{1+\sum_{k=1}^{N} a_k z^{-k}}$$

equivalently

$$\boxed{H(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{1+\sum_{k=1}^{N} a_k z^{-k}}}$$

This is the general form of the system function of a system described by a linear constant coefficient difference equation.

Therefore, a linear time-invariant system described by a constant-coefficient difference equation has a rational system function.

Determine the system function and the unit sample response of the system described by the difference equation

$$y(n) = \frac{1}{2} y(n-1) + 2x(n)$$

By computing the z-transform,

$$Y(z) = \frac{1}{2} z^{-1} Y(z) + 2X(z)$$

hence the system function

$$\frac{Y(z)}{X(z)} = H(z) = \frac{2}{1-\frac{1}{2}z^{-1}} = \frac{4z}{2z-1}$$

this system has a pole at $z = \frac{1}{2}$ and a zero at the origin.

The inverse Z-transform

$$h(n) = 2\left(\frac{1}{2}\right)^n u(n)$$

This is the unit sample response of the system.

Analysis of Linear time-invariant Systems in the Z-domain:

In this section describes the use of the system function in the determination of the response of the system to some excitation signal.

Response of Systems with Rational System functions

let us consider a pole-zero system described by the general linear constant coefficient difference equation

$$y(n) = -\sum_{k=1}^{N} a_k y(n-k) + \sum_{k=0}^{M} b_k x(n-k)$$

and corresponding system function

$$H(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{1 + \sum_{k=1}^{N} a_k z^{-k}}$$

let us represent $H(z)$ as a ratio of two polynomials $B(z)/A(z)$ where $B(z)$ is the numerator polynomial that contains zeros of $H(z)$ and $A(z)$ is the denominator polynomial that contains the poles of $H(z)$

$$\boxed{H(z) = \frac{B(z)}{A(z)}}$$

let us assume that the input signal $x(n)$ has a rational Z-transform $X(z)$ of the form

$$\boxed{X(z) = \frac{N(z)}{Q(z)}}$$

If the system is relaxed, ie initial conditions for the difference equation are zero, $y(-1) = y(-2) = \cdots = 0$ the Z-transform of the output of the system has the form:

$$Y(z) = H(z)\,X(z)$$

$$\boxed{Y(z) = \frac{B(z)\,N(z)}{A(z)\,Q(z)}}$$

Now suppose that the system contains simple poles $p_1, p_2, \cdots, p_N$ and Z-transform of the input signal contains poles $q_1, q_2, \cdots, q_L$ where $p_k \neq q_m$ for all $k = 1, 2, \cdots N$ & $m = 1, 2, \cdots L$

In addition, we assume that the zeros of the numerator polynomial $B(z)$ and $N(z)$ do not coincide with the poles $\{p_k\}$ and $\{q_k\}$

So that there is no pole-zero cancellation Then a partial fraction expansion of $Y(z)$ yields

$$Y(z) = \sum_{k=1}^{N} \frac{A_k}{1-p_k z^{-1}} + \sum_{k=1}^{L} \frac{Q_k}{1-q_k z^{-1}}$$

The inverse transform of $Y(z)$ yields the output signal from the system in the form.

$$y(n) = \sum_{k=1}^{N} A_k (p_k)^n u(n) + \sum_{k=1}^{L} Q_k (q_k)^n u(n)$$

We observe that the output sequence $y(n)$ can be subdivided into two parts

The first part is a function of poles $\{p_k\}$ of the system and is called the natural response of the system The influence of the input signal on this part of the response is through the scale factors $\{A_k\}$

The second part of the response is a function of poles $\{q_k\}$ of the system and is called forced response of the system. The influence of the input signal on this part of the response is through scale factors $\{Q_k\}$

The one-sided z-transform

The two-sided z-transform requires that the corresponding signals be specified for the entire time range $-\infty < n < \infty$. This requirement prevents its use for a very useful family of practical problems, namely the evaluation of the output of non-relaxed systems.

The one sided z-transform can be used to solve difference equation with initial conditions

Defination & properties

The one-sided or unilateral z-transform of a signal $x(n)$ is defined by

$$X^+(z) = \sum_{n=0}^{\infty} x(n) z^{-n}$$

or $\quad Z^+\{x(n)\}$

$$x(n) \xleftrightarrow{Z^+} X^+(z)$$

The one-sided Z-transform differs from the two-sided transform in the lower limit of summation which is always zero, whether or not the signal $x(n)$ is zero for $n<0$ (i.e. causal)

Shifting property

Case 1 Time delay

if $x(n) \xleftrightarrow{Z^+} X^+(Z)$

then $x(n-k) \xleftrightarrow{Z^+} Z^{-k}\left[X^+(Z) + \sum_{n=1}^{k} x(-n) Z^{n}\right]$

$k>0$

case-2 Time advance

if $x(n) \xleftrightarrow{Z^+} X^+(Z)$

then $x(n+k) \xleftrightarrow{Z^+} Z^{k}\left[X^+(Z) - \sum_{n=0}^{k-1} x(n) Z^{-n}\right]$ $k>0$

Response of Pole-Zero System with Non Zero Initial conditions

Suppose the signal $x(n)$ is applied to the Pole-Zero system at $n=0$. Thus the signal $x(n)$ is assumed to be causal. The effects of all previous input signals to the system are reflected in the initial conditions $y(-1), y(-2), \cdots, y(-N)$. Since the input $x(n)$ is causal & we are interested in determining the output $y(n)$ for $n\geq 0$, we can use the one-sided Z-transform

$$y(n) = -\sum_{k=1}^{N} a_k y(n-k) + \sum_{k=0}^{M} b_k x(n-k)$$

Thus one sided Z-transform becomes

$$Y^+(Z) = -\sum_{k=1}^{N} a_k Z^{-k}\left[Y^+(Z) + \sum_{n=1}^{k} y(-n) Z^{n}\right] + \sum_{k=0}^{M} b_k Z^{-k} X^+(Z)$$

Since $x(n)$ is causal, we can set $X^+(Z) = X(Z)$

$$Y^+(Z) = \frac{\sum_{k=0}^{M} b_k Z^{-k}}{1+\sum_{k=1}^{N} a_k Z^{-k}} X(Z) - \frac{\sum_{k=1}^{N} a_k Z^{-k} \sum_{n=1}^{k} y(-n) Z^{n}}{1+\sum_{k=1}^{N} a_k Z^{-k}}$$

$$= H(Z)X(Z) + \frac{N_0(Z)}{A(Z)}$$

where $N_0(z) = -\sum_{k=1}^{N} a_k z^{-k} \sum_{n=1}^{k} y(-n) z^{n}$

Determine the unit step response of the system whose difference equation is $y(n) - 0.7y(n-1) + 0.12y(n-2) = x(n-1) + x(n-2)$ if $y(-1) = y(-2) = 1$

$y(n) - 0.7y(n-1) + 0.12y(n-2) = x(n-1) + x(n-2)$

Taking z-transform both sides.

$$Y(z) - 0.7\left[z^{-1}Y(z) + y(-1)\right] + 0.12\left[z^{-2}Y(z) + z^{-1}y(-1) + y(-2)\right] = z^{-1}X(z) + x(-1) + z^{-2}X(z) + z^{-1}x(-1) + x(-2)$$

Substituting $y(-1) = y(-2) = 1$

& $x(-1) = x(-2) = 0$ for step input

$$Y(z) - 0.7\left[z^{-1}Y(z) + 1\right] + 0.12\left[z^{-2}Y(z) + z^{-1} + 1\right] = z^{-1}X(z) + z^{-2}X(z)$$

for step input $X(z) = \frac{1}{1-z^{-1}}$

$$Y(z)\left[1 - 0.7z^{-1} + 0.12z^{-2}\right] - 0.58 + 0.12z^{-1} = \frac{z^{-1}}{1-z^{-1}} + \frac{z^{-2}}{1-z^{-1}}$$

$$Y(z) = \frac{z^{-1} + z^{-2}}{(1-z^{-1})(1 - 0.7z^{-1} + 0.12z^{-2})} + \frac{0.58 - 0.12z^{-1}}{1 - 0.7z^{-1} + 0.12z^{-2}}$$

$\frac{1}{z} + \frac{1}{z^2} = \frac{z+1}{z^2}$

$1 - \frac{1}{z} = \frac{z-1}{z}$

$$= \frac{\frac{1+z}{z^2}\, z}{(z-1)\left(1 - \frac{0.7}{z} + \frac{0.12}{z^2}\right)} + \frac{0.58z - 0.12}{z\left(1 - \frac{0.7}{z} + \frac{0.12}{z^2}\right)}$$

$$= \frac{(1+z)/z \quad z^{2}}{(z-1)(z^2 - 0.7z + 0.12)} + \frac{(0.58z - 0.12)\, z^2}{z(z^2 - 0.7z + 0.12)}$$

$$\frac{Y(z)}{z} = \frac{z+1}{(z-1)(z-0.4)(z-0.3)} + \frac{0.58z - 0.12}{(z-0.4)(z-0.3)}$$

$$\frac{z+1}{(z-1)(z-0.4)(z-0.3)} = \frac{C_1}{z-1} + \frac{C_2}{z-0.4} + \frac{C_3}{z-0.3}$$

$$C_1 = (z-1)\frac{z+1}{(z-1)(z-0.4)(z-0.3)}\bigg|_{z=1}$$

$$= 4.762$$

$$C_2 = (z-0.4)\frac{z+1}{(z-1)(z-0.4)(z-0.3)}\bigg|_{z=0.4} = -23.33$$

$$C_3 = \cancel{(z-0.3)}\frac{(z+1)}{(z-1)(z-0.4)\cancel{(z-0.3)}}\Bigg|_{z=0.3}$$

$$= \frac{+1.3}{0.07} = 18.57$$

$$\frac{0.58z-0.12}{(z-0.4)(z-0.3)} = \frac{A}{(z-0.4)} + \frac{B}{(z-0.3)}$$

$$A = \cancel{(z-0.4)}\frac{0.58z-0.12}{\cancel{(z-0.4)}(z-0.3)}\Bigg|_{z=0.4}$$

$$= \frac{0.112}{0.1} = 1.12$$

$$B = \cancel{(z-0.3)}\frac{0.58z-0.12}{(z-0.4)\cancel{(z-0.3)}}\Bigg|_{z=0.3}$$

$$= \frac{0.054}{-0.1} = -0.54$$

$$\frac{Y(z)}{z} = \frac{4.762}{z-1} - \frac{23.33}{z-0.4} + \frac{18.57}{z-0.3} + \frac{1.12}{z-0.4} - \frac{0.54}{z-0.3}$$

multiplying both sides with z and taking inverse laplace transform we got

$$y(n) = 4.762u(n) - 22.21(0.4)^n u(n) + 18.03(0.3)^n u(n)$$

Transcient and Steady-state Responses

The response of a system to a given input can be separated into two components the natural response & the forced response

The natural response of a causal system has the form

$$y_{nr}(n) = \sum_{k=1}^{N} A_k (P_k)^n u(n)$$

where $\{P_k\} = k = 1, 2, \dots N$ are the poles of the system and $\{A_k\}$ are scale factors that depends on the initial conditions & on the characteristics of input sequence.

If $|P_k| < 1$ for all k, then $y_{nr}(n)$ decays to zero as n approaches infinity in such a case we refer to the natural response of the system as transient response

The rate at which $y_{nr}(n)$ decays toward zero depends on the magnitude of the pole positions. If all the poles have small magnitudes, the decay is very rapid. On the other hand, if one or more poles are located near the unit circle, the corresponding terms in $y_{nr}(n)$ will decay slowly toward zero & the transient will persist for a relatively long time.

The forced response of the system has the form

$$y_{fr}(n) = \sum_{k=1}^{L} Q_k (q_k)^n u(n)$$

where $\{q_k\}$, $k = 1, 2, \ldots L$ are the poles in the forcing function & $\{Q_k\}$ are scale factors that depend on the input sequence and on the characteristics of the system. If all the poles of the input signal fall inside the unit circle, $y_{fr}(n)$ will decay toward zero as n approaches infinity

On the other hand when the causal input signal is a sinusoid, the poles fall on the unit circle & consequently the forced response is also a sinusoid that persists for all $n \geq 0$.

In this case the forced response is called the steady-state response of the system

Causality & Stability

A causal linear time-invariant system is one whose sample response h(n) satisfies the condition $h(n) = 0 \quad n<0$

ROC of the Z-transform of a causal sequence is the exterior of a circle.

A linear time-invariant system is causal if and only if the ROC of the system function is the exterior of a circle of radius $r<\infty$ including the point $z=\infty$

The stability of a linear time-invariant system can also be expressed in terms of the characteristics of the system function.

A necessary & sufficient condition for a linear time-invariant system to be BIBO stable is

$$\sum_{n=-\infty}^{\infty} |h(n)| < \infty$$

In turn, this condition implies that H(z) must contain the unit circle within its ROC since

$$H(z) = \sum_{n=-\infty}^{\infty} h(n) z^{-n}$$

$$|H(z)| \leq \sum_{n=-\infty}^{\infty} |h(n) z^{-n}| = \sum_{n=-\infty}^{\infty} |h(n)||z^{-n}|$$

when evaluated on the unit circle $|z|=1$

$$|H(z)| \leq \sum_{n=-\infty}^{\infty} |h(n)|$$

Hence, if the system is BIBO stable the unit circle is contained in the ROC of H(z). Therefore a linear-time invariant system is BIBO stable if and only if the ROC of the system function includes the unit circle.

A causal system is characterized by a system function H(z) having ~~as~~ a ROC the exterior of some circle of radius r. For a stable system ROC must have ~~a system function~~ include the unit circle. Consequently a causal and stable must have a system function that converges for $|z| > r < 1$.

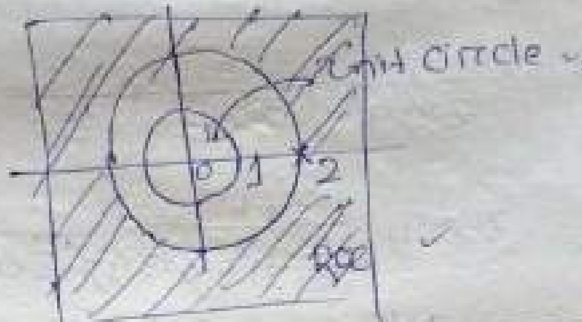
A causal linear time invariant system is BIBO stable if and only if all the poles of $H(z)$ are inside the unit circle.

Find the stability of the system whose impulse response $h(n) = (2)^n u(n)$

$$h(n) = (2)^n u(n)$$

$$H(z) = \frac{1}{1-2z^{-1}} = \frac{z}{z-2} \quad |z|>2$$

the ROC is $|z|>2$ it does not contain unit circle therefore, the system is unstable



Compute the response of the system $y(n)=0.7y(n-1) - 0.12y(n-2) + x(n-1) + x(n-2)$ to input $x(n)=nu(n)$ Is the system is stable

$$y(n) = 0.7\,y(n-1) - 0.12\,y(n-2) + x(n-1) + x(n-2)$$

Taking Z-transform

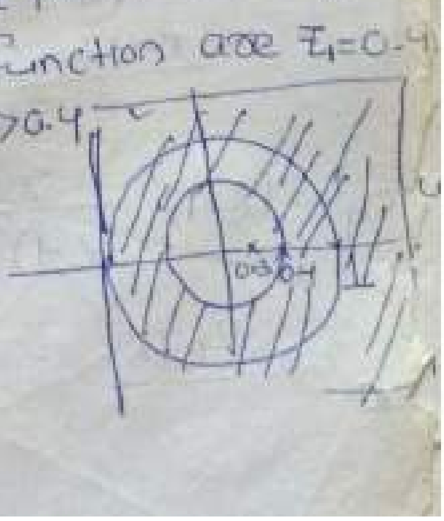
$$Y(z) = 0.7\,z^{-1}Y(z) - 0.12\,z^{-2}Y(z) + z^{-1}X(z) + z^{-2}X(z)$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{z^{-1}+z^{-2}}{1-0.7z^{-1}+0.12z^{-2}}$$

$$= \frac{z+1}{z^2-0.7z+0.12} = \frac{z+1}{(z-0.4)(z-0.3)}$$

the poles of the system function are $z_1=0.4$, $z_2=0.3$ & the ROC $|z|>0.4$

the poles are lying inside the unit circle so the system is stable



For input $x(n) = nu(n)$

$$X(z) = \frac{z}{(z-1)^2}$$

$$\frac{Y(z)}{X(z)} = \frac{z+1}{(z-0.4)(z-0.3)}$$

$$Y(z) = \frac{z+1}{(z-0.4)(z-0.3)} \frac{z}{(z-1)^2}$$

$$\boxed{\frac{Y(z)}{z} = \frac{z+1}{(z-1)^2(z-0.4)(z-0.3)}}$$

$$\frac{Y(z)}{z} = \frac{c_1}{z-0.4} + \frac{c_2}{z-0.3} + \frac{c_3}{z-1} + \frac{c_4}{(z-1)^2}$$

$$c_1 = \cancel{(z-0.4)} \frac{z+1}{(z-1)^2\cancel{(z-0.4)}(z-0.3)} \bigg|_{z=0.4}$$

$$= \frac{1.4}{0.036} = 38.89$$

$$c_2 = \cancel{(z-0.3)} \frac{z+1}{(z-1)^2(z-0.4)\cancel{(z-0.3)}} \bigg|_{z=0.3}$$

$$= -\frac{1.3}{0.049} = -26.53$$

$$c_3 = \frac{d}{dz} \cancel{(z-1)^2} \frac{z+1}{\cancel{(z-1)^2}(z-0.4)(z-0.3)} \bigg|_{z=1}$$

$$= \frac{d}{dz} \frac{z+1}{z^2-0.7z+0.12} \bigg|_{z=1}$$

$$= \frac{z^2-0.7z+0.12-(z+1)(2z-0.7)}{(z^2-0.7z+0.12)^2} \bigg|_{z=1}$$

$$= \frac{0.42-2.6}{0.1764} = \underline{-12.36}$$

$$c_4 = \cancel{(z-1)^2} \frac{z+1}{\cancel{(z-1)^2}(z-0.4)(z-0.3)} \bigg|_{z=1}$$

$$= \underline{4.76}$$

$$\frac{Y(z)}{z} = \frac{38.89}{z-0.4} - \frac{26.53}{z-0.3} - \frac{12.36}{z-1} + \frac{4.76}{(z-1)^2}$$

taking the inverse z-transform

$$y(n) = 38.89\,(0.4)^n u(n) - 26.53\,(0.3)^n u(n) - 12.36\,u(n) + 4.76\,n\,u(n)$$

A linear time-invariant system is characterised by the system function $H(z) = \frac{3-4z^{-1}}{1-3.5z^{-1}+1.5z^{-2}}$

Specify the ROC of $H(z)$ & determine $h(n)$ for the following conditions (a) The system is stable (b) the system is causal (c) The system is anticausal

$$H(z) = \frac{3-4z^{-1}}{1-3.5z^{-1}+1.5z^{-2}}$$

$$= \frac{4z-3z^2}{z^2-3.5z+1.5} = \frac{z(3z-4)}{z^2-3.5z+1.5z}$$

$$\frac{H(z)}{z} = \frac{3z-4}{(z-0.5)(z-3)} = \frac{c_1}{(z-0.5)} + \frac{c_2}{z-3}$$

$$c_1 = (z-0.5)\frac{H(z)}{z}\Big|_{z=0.5}$$

$$= \cancel{(z-0.5)}\frac{3z-4}{\cancel{(z-0.5)}(z-3)}\Big|_{z=0.5}$$

$$= -2.5/-2.5 = 1$$

$$c_2 = (z-3)\frac{H(z)}{z}\Big|_{z=3}$$

$$= \cancel{(z-3)}\frac{3z-4}{(z-0.5)\cancel{(z-3)}}\Big|_{z=3}$$

$$= \frac{5}{2.5} = 2$$

$$\boxed{H(z) = \frac{z}{z-0.5} + \frac{2z}{z-3}}$$

the system has poles at $z=0.5$ & $z=3$

(a) Since the system is stable, its ROC must include the unit circle & hence it is

$\frac{1}{2} < |z| < 3$

$$h(n) = \left(\tfrac{1}{2}\right)^n u(n) + -2(3)^n u(-n-1)$$

(b) Since the system is causal the ROC is $|z| > 3$

$$h(n) = \left(\frac{1}{2}\right)^n u(n) + 2(3)^n u(n)$$

This system is unstable

(c) If the system is anticausal the ROC is $|z| < 0.5$

$$h(n) = -\left[\left(\frac{1}{2}\right)^n + 2(3)^n\right] u(-n-1)$$

the system is unstable

Pole-zero Cancellation

When a Z-transform has a pole that is at the same location as a zero, the pole is cancelled by the zero & consequently, the term containing that pole in the reverse Z-transform vanishes. Such pole-zero cancellations are very important in the analysis of pole-zero systems.

Determine the unit sample response of the system characterized by the difference equation

$$y(n) = 2.5y(n-1) - y(n-2) + x(n) - 5x(n-1) + 6x(n-2)$$

$$Y(z) = 2.5z^{-1}Y(z) - z^{-2}Y(z) + X(z) - 5z^{-1}X(z) + 6z^{-2}X(z)$$

$$Y(z)\left[1 - 2.5z^{-1} + z^{-2}\right] = X(z)\left[1 - 5z^{-1} + 6z^{-2}\right]$$

$$H(z) = \frac{1 - 5z^{-1} + 6z^{-2}}{1 - 2.5z^{-1} + z^{-2}}$$

$$= \frac{z^2 - 5z + 6}{z^2 - 2.5z + 1} = \frac{(z^2 - 5z + 6)}{(z-0.5)(z-2)}$$

$$\frac{z^2 - 5z + 6/z}{(z-0.5)(z-2)} = \frac{A}{z-0.5} + \frac{B}{z-2}$$

$$A = (z-0.5)\frac{z - 5 + 6z^{-1}}{(z-0.5)(z-2)}\Bigg|_{z=0.5}$$

$$= \frac{7.5}{1.5}$$

$$z^2 - 5z + 6 = (z-3)(z-2)$$

$$\frac{(z-3)(z-2)}{(z-0.5)(z-2)}$$

$$H(Z) = \frac{Z-3}{Z-1/2}$$

$$= \frac{2Z-6}{2Z-1} = \frac{2Z-1-5}{2Z-1}$$

$$= 1 - \frac{5}{2Z-1}$$

$$= 1 - \frac{2.5Z^{-1}}{1-0.5Z^{-1}}$$

$$h(n) = \delta(n) - 2.5\,(0.5)^{n-1}u(n-1)$$

$x(n) \overset{Z}{\longleftrightarrow} X(Z)$

$a^n u(n) \longleftrightarrow \frac{1}{1-aZ^{-1}}$

$x(n-k) \overset{Z}{\longleftrightarrow} Z^{-k}X$

$a^{n-1}u(n-1) \overset{Z}{\longleftrightarrow} Z^{-1}\frac{1}{1-aZ}$

Determine the response of the system $y(n) = \frac{5}{6}y(n-1) - \frac{1}{6}y(n-2) + x(n)$ to the input signal $x(n) = \delta(n) - \frac{1}{3}\delta(n-1)$

$$y(n) = \frac{5}{6}y(n-1) - \frac{1}{6}y(n-2) + x(n)$$

$$Y(Z) = \frac{5}{6}Z^{-1}Y(Z) - \frac{1}{6}Z^{-2}Y(Z) + X(Z)$$

$$H(Z) = \frac{1}{1-\frac{5}{6}Z^{-1}+\frac{1}{6}Z^{-2}}$$

$$= \frac{Z^2}{Z^2 - 5/6\,Z + \frac{1}{6}} = \frac{Z^2}{(Z-\frac{1}{2})(Z-\frac{1}{3})}$$

$$x(n) = \delta(n) - \frac{1}{3}\delta(n-1)$$

$$X(Z) = 1 - \frac{1}{3}Z^{-1} \quad 1-\frac{1}{3Z} \text{ zero at } Z = \frac{1}{3}$$

$$= \frac{3Z-1}{3Z} = \cancel{(Z-\tfrac{1}{3})} \quad 1-\frac{1}{3}Z^{-1} \quad 3(Z-\tfrac{1}{3})$$

$$Y(Z) = H(Z)\,X(Z)$$

$$= \frac{Z^2}{(Z-\frac{1}{2})\cancel{(Z-\frac{1}{3})}} \cdot \frac{3\cancel{(Z-\frac{1}{3})}}{3\cancel{Z}}$$

$$= \frac{Z}{Z-\frac{1}{2}}$$

$$y(n) = \left(\frac{1}{2}\right)^n u(n)$$

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The Discrete Fourier Transform

DFT

$$X(K) = \sum_{n=0}^{N-1} x(n)\, e^{-j2\pi kn/N}$$

$$K = 0, 1, 2, \cdots, N-1$$

IDFT

$$x(n) = \frac{1}{N} \sum_{K=0}^{N-1} X(K)\, e^{j2\pi kn/N},$$

$$n = 0, 1, 2, \cdots, N-1$$

Find the DFT of a sequence $x(n) = \{1, 1, 0, 0\}$ and find the IDFT of $Y(K) = \{1, 0, 1, 0\}$

$$x(n) = \{1, 1, 0, 0\}$$

$\underline{N = 4}$

We have $X(K) = \sum_{n=0}^{N-1} x(n)\, e^{-j2\pi nK/4}$

$$K = 0, 1, \cdots N-1$$

$$X(0) = \sum_{n=0}^{3} x(n) = 1 + 1 + 0 + 0 = 2$$

$$X(1) = \sum_{n=0}^{3} x(n)\, e^{-j\pi n/2}$$

$$= x(0) + x(1)\, e^{-j\pi/2} + x(2)\, e^{-j\pi} + x(3)\, e^{-j3\pi/2}$$

$$= 1 + \cos \pi/2 - j \sin \pi/2$$

$$= 1 - j$$

$$X(2) = \sum_{n=0}^{3} x(n)\, e^{-j\pi n}$$

$$= x(0) + x(1)\, e^{-j\pi} + x(2)\, e^{-j2\pi} + x(3)\, e^{-j3\pi}$$

$$= 1 + \cos\pi - j\sin\pi$$

$$= 1 - 1 = 0$$

$$X(3) = \sum_{n=0}^{3} x(n)\, e^{-j3\pi n/2}$$

$$= x(0) + x(1) e^{-j3\pi/2} + x(2) e^{-j3\pi} + x(3) e^{-9\pi/2}$$

$$= 1 + 1 \cdot \cos 3\pi/2 - j \sin 3\pi/2$$

$$= 1+j$$

$$X(K) = \{2, 1-j, 0, 1+j\}$$

$$Y(K) = \{1, 0, 1, 0\} \quad \underline{N=4}$$

$$y(n) = \frac{1}{N} \sum_{K=0}^{N-1} Y(K)\, e^{j2\pi Kn/N} \quad n = 0, 1, 2, \ldots, N-1$$

$$y(0) = \frac{1}{4}\left[Y(0) + Y(1) + Y(2) + Y(3)\right]$$

$$= \frac{1}{4}\left[1+0+1+0\right] = \frac{1}{2}$$

$$y(1) = \frac{1}{N} \sum_{K=0}^{3} Y(K)\, e^{j\pi/2 K}$$

$$= \frac{1}{4}\left[Y(0) + Y(1) e^{j\pi/2} + Y(2) e^{j\pi} + Y(3) e^{j3\pi/2}\right]$$

$$= \frac{1}{4}\left[1 + 0 + \cos\pi + j\sin\pi + 0\right]$$

$$= 0$$

$$y(2) = \frac{1}{4} \sum_{K=0}^{3} Y(K)\, e^{j\pi K}$$

$$= \frac{1}{4}\left[Y(0) + Y(1) e^{j\pi} + Y(2) e^{j2\pi} + Y(3) e^{j3\pi}\right]$$

$$= \frac{1}{4}\left[1 + 0 + 1\cos 2\pi + j\sin 2\pi + 0\right\}$$

$$= \frac{1}{4}\left[\frac{1}{2} + \ldots\right.$$

$$= \frac{1}{2}$$

$$y(3) = \frac{1}{4} \sum_{K=0}^{3} Y(K)\, e^{j3\pi/2 K}$$

$$= \frac{1}{4}\left[Y(0) + Y(1) e^{3\pi/2} + Y(2) e^{j3\pi} + Y(3) e^{j9\pi/2}\right]$$

$$= \frac{1}{4}\left[1 + 0 + \cos 3\pi + j\sin 3\pi + 0\right]$$

$$= \frac{1}{4}\left[1 + 0 + \ldots\right.$$

$$= 0$$

$$y(n) = \{0.5, 0, 0.5, 0\}$$

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DFT as a Linear Transformation

The formulas for DFT & IDFT are given by

$$X(k) = \sum_{n=0}^{N-1} x(n) e^{-j2\pi kn/N} \quad k=0,1,\ldots,N-1$$

$$\boxed{X(k) = \sum_{n=0}^{N-1} x(n) W_N^{kn}} \quad k=0,1,\ldots N-1$$

IDFT $$x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k) e^{j2\pi kn/N} \quad n=0,1,\ldots N-1$$

$$\boxed{x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k) W_N^{-kn}} \quad n=0,1,\ldots N-1$$

where, by defination

$$\boxed{W_N = e^{-j2\pi/N}}$$

which is an Nth root of unity and known as twiddle factor.

Note that the computation of each point of the DFT can be accomplished by N complex multiplication and (N-1) complex additions.

Hence the N-point DFT values can be computed in a total of N^2 complex multiplications and N(N-1) complex additions.

The DFT & IDFT are linear transformation on sequences $\{x(n)\}$ & $\{X(k)\}$ respectively.

let us define an N-point vector x_N of the signal sequences $x(n)$, $n=0,1,\ldots N-1$, an N-point vector X_N of frequency samples & an $N \times N$ matrix W_N as

In matrix form N-point DFT may be expressed as

$$\boxed{X_N = W_N x_N}$$

W_N is the matrix of linear transformation & W_N is a symmetric matrix

$$W_N = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & W_N & W_N^2 & & W_N^{N-1} \\ \cdot & W_N^2 & W_N^4 & & W_N^{2(N-1)} \\ \vdots & & \vdots & & \vdots \\ 1 & W_N^{N-1} & W_N^{2(N-1)} & & W_N^{(N-1)(N-1)} \end{bmatrix}$$

$$x_N = \begin{bmatrix} x(0) \\ x(1) \\ \vdots \\ x(N-1) \end{bmatrix}, \quad X_N = \begin{bmatrix} X(0) \\ X(1) \\ \vdots \\ X(N-1) \end{bmatrix}$$

$\cancel{X(0) = x(0)\, W_N^0 = 1}$

$\cancel{X(k) = x(1)\, W_N^k}$

IDFT can be expressed in matrix form

$$\boxed{x_N = \frac{1}{N} W_N^{*} X_N}$$

Compute the DFT of the four-point sequence $x(n) = (0\ 1\ 2\ 3)$

The first step is to determine the matrix W_4

$$W_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & W_4 & W_4^2 & W_4^3 \\ 1 & W_4^2 & W_4^4 & W_4^6 \\ 1 & W_4^3 & W_4^6 & W_4^9 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & W_4^1 & W_4^2 & W_4^3 \\ 1 & W_4^2 & W_4^0 & W_4^2 \\ 1 & W_4^3 & W_4^2 & W_4^1 \end{bmatrix}$$

By exploiting the periodicity property of W_4

$$\boxed{W^{\alpha} = W^{\alpha \pm N} = W^{\alpha \pm 2N}}$$

$$W_4^4 = W_4^0, \quad W_4^6 = W_4^2, \quad W_4^9 = W_4^1$$

$$W_4^1 = e^{-j2\pi/4} = e^{-j\pi/2} = \cos\pi/2 - j\sin\pi/2 = -j$$

$$W_4^2 = -1 \qquad W_4^3 = j$$

$$W_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix}$$

then

$$X_4 = W_4 x_4$$

$$= \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} (0\,1\,2\,$$

$$= \begin{bmatrix} 6 \\ -2+2j \\ \vdots \end{bmatrix}$$

Relationship of DFT to other transforms

① Relationship to Fourier Transform

The fourier transform $X(e^{j\omega})$ of a finite duration sequence $x(n)$ having length N is given by

$$\boxed{X(e^{j\omega}) = \sum_{n=0}^{N-1} x(n) e^{-j\omega n}}$$

where $X(e^{j\omega})$ is a continuous function of ω.

The discrete Fourier transform of $x(n)$ is given by

$$\boxed{X(k) = \sum_{n=0}^{N-1} x(n) e^{-j2\pi kn/N}} \quad k = 0, 1, 2, \cdots N-1$$

Comparing the above equations, we find that DFT of $x(n)$ is a sampled version of fourier transform of the sequence and is given by

$$X(k) = X(e^{j\omega})\Big|_{\omega = \frac{2\pi k}{N}} \quad k = 0, 1, \cdots N-1$$

② Relationship to Z-transform

let us consider a sequence $x(n)$ of finite duration N with Z-transform

$$\boxed{X(Z) = \sum_{n=0}^{N-1} x(n) Z^{-n}}$$

we have $x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k) e^{j2\pi kn/N}$

Substituting the value of $x(n)$ we get

$$X(Z) = \sum_{n=0}^{N-1} \left[\frac{1}{N} \sum_{k=0}^{N-1} X(k) e^{j2\pi kn/N} \right] Z^{-n}$$

$$= \frac{1}{N} \sum_{k=0}^{N-1} X(k) \sum_{n=0}^{N-1} e^{j2\pi kn/N} Z^{-n}$$

$$= \frac{1}{N} \sum_{k=0}^{N-1} X(k) \sum_{n=0}^{N-1} \left(e^{j2\pi k/N} Z^{-1} \right)^n$$

$$\boxed{X(Z) = \frac{1 - Z^{-N}}{N} \sum_{k=0}^{N-1} \frac{X(k)}{1 - e^{j2\pi k/N} Z^{-1}}}$$

Properties of DFT

Periodicity If $x(n)$ and $X(k)$ are an N-point DFT pair, then

$$x(n+N) = x(n) \quad \text{for all } n$$

$$X(K+N) = X(K) \quad \text{for all } K$$

Linearity If

$$x_1(n) \xleftrightarrow[N]{DFT} X_1(K)$$

and

$$x_2(n) \xleftrightarrow[N]{DFT} X_2(K)$$

then for real-valued or complex valued constants a_1 and a_2

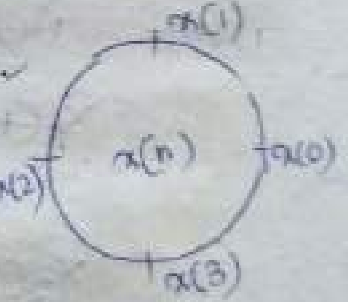
$$a_1 x_1(n) + a_2 x_2(n) \xleftrightarrow[N]{DFT} a_1 X_1(K) + a_2 X_2(K)$$

Circular symmetries of a sequence

In general the circular shift of the sequence can be represented as the index modulo N. Thus we can write

$$x'(n) = x(n-K, \text{modulo } N)$$

$$\boxed{x'(n) = x((n-K))_N}$$



Ex if $K=2$ & $N=4$ we have

$$x'(n) = x((n-2))_4$$

which implies that

$$x'(0) = x((-2))_4 = x(2) \qquad x'(3) = x((1))_4 = x(1)$$

$$x'(1) = x((-1))_4 = x(3)$$

$$x'(2) = x((0))_4 = x(0)$$

Hence $x'(n)$ is simply $x(n)$ shifted circularly by two units in time, where the counter close wise direction has been arbitrarily selected as the positive direction. Thus we conclude that a circular shift of an N-point sequence is equivalent to a linear shift of its periodic extension and vice versa.

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An N-point sequence is called circularly even if it is symmetric about the point zero on the circle. This implies that

$$\boxed{x(N-n) = x(n)} \quad 1 \le n \le N-1$$

An N-point sequence is called circularly odd if it is antisymmetric about the point zero on the circle. This implies that

$$\boxed{x(N-n) = -x(n)} \quad 1 < n \le N-1$$

The time reversal of an N-point sequence is attained by reversing the samples about the point zero on the circle. Thus the sequence $x((-n))_N$ is simply given as

$$\boxed{x((-n))_N = x(N-n)} \quad 0 \le n \le N-1$$

This time reversal is equivalent to plotting $x(n)$ in a clock wise direction on a circle.

Multiplication of two DFTs & Circular Convolution

Suppose we have two finite duration sequences of length N, $x_1(n)$ & $x_2(n)$. Their respective N-point DFTs are

$$X_1(K) = \sum_{n=0}^{N-1} x_1(n)\, e^{-j2\pi nk/N} \quad K = 0, 1, \dots N-1$$

$$X_2(K) = \sum_{n=0}^{N-1} x_2(n)\, e^{-j2\pi nK/N} \quad K = 0, 1, \dots N-1$$

If we multiply the two DFTs together, the result is a DFT say $X_3(K)$ of a sequence $x_3(n)$ of length N.

We have

$$X_3(K) = X_1(K)\, X_2(K) \quad K = 0, 1, \dots N-1$$

The IDFT of $\{X_3(K)\}$ is

$$x_3(m) = \frac{1}{N} \sum_{K=0}^{N-1} X_3(K)\, e^{j2\pi Km/N}$$

$$= \frac{1}{N} \sum_{K=0}^{N-1} X_1(K)\, X_2(K)\, e^{j2\pi Km/N}$$

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$$x_3(m) = \frac{1}{N}\sum_{k=0}^{N-1}\left[\sum_{n=0}^{N-1} x_1(n)\, e^{-j2\pi kn/N}\right]\left[\sum_{l=0}^{N-1} x_2(l)\, e^{-j2\pi kl/N}\right] e^{j2\pi km/N}$$

$$= \frac{1}{N}\sum_{n=0}^{N-1} x_1(n) \sum_{l=0}^{N-1} x_2(l) \left[\sum_{k=0}^{N-1} e^{j2\pi k(m-n-l)/N}\right]$$

We know that
$$\sum_{k=0}^{N-1} a^k = \begin{cases} N, & a=1 \\ \frac{1-a^N}{1-a}, & a \neq 1 \end{cases}$$

where a is defined as $a = e^{j2\pi(m-n-l)/N}$

We observe that $a=1$ when $m-n-l$ is a multiple of N.

$$\sum_{k=0}^{N-1} a^k = \begin{cases} N, & l = m-n+pN = ((m-n))_N \quad p \text{ is an integer} \\ 0, & \text{otherwise} \end{cases}$$

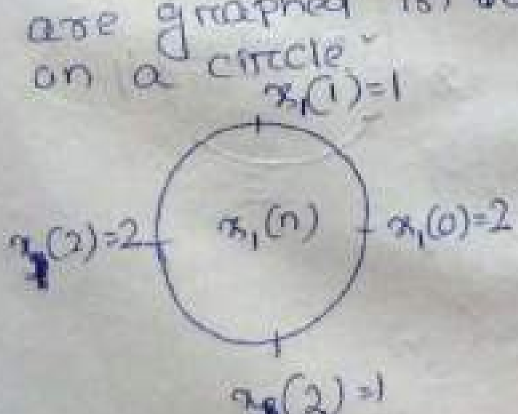
$$\boxed{x_3(m) = \sum_{n=0}^{N-1} x_1(n)\, x_2((m-n))_N} \quad m = 0, 1, \ldots N-1$$

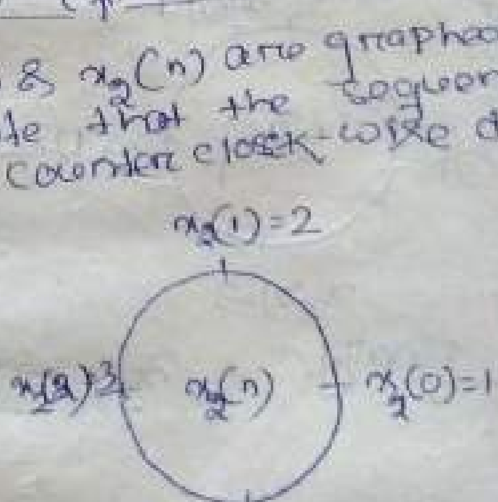
The convolution sum involves the index $((m-n))_N$ and is called circular convolution.

Thus we conclude that multiplication of the DFTs of two sequences is equivalent to the circular convolution of the two sequences in the time domain.

Perform the circular convolution of the following two sequences. $x_1(n) = \{\underset{\uparrow}{2}, 1, 2, 1\}$ $x_2(n) = \{\underset{\uparrow}{1}, 2, 3, 4\}$

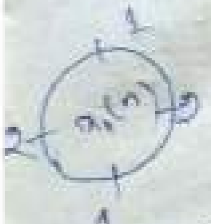
The sequences $x_1(n)$ & $x_2(n)$ are graphed as illustrated. We note that the sequences are graphed in a counterclockwise direction on a circle.





$x_3(m)$ is obtained by circularly convolving $x_1(n)$ with $x_2(n)$ as

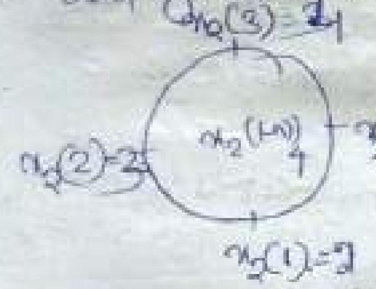
$$x_3(m) = \sum_{n=0}^{N-1} x_1(n)\, x_2((m-n))_N \qquad m = 0, 1, \cdots N-1$$



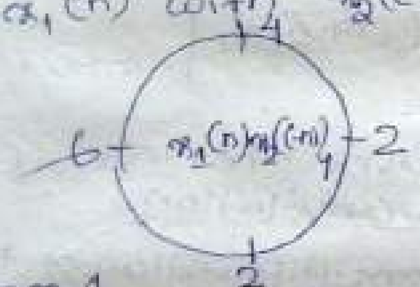
$$x_3(0) = \sum_{n=0}^{3} x_1(n)\, x_2((-n))_N$$

$x_2((-n))_4$ is simply the sequence $x_2(n)$ folded and graphed on a circle as

in ~~simply~~ other words the folded sequence is simply $x_2(n)$ graphed in a clockwise direction



The product sequence is obtained by multiplying $x_1(n)$ with $x_2((-n))_4$ point by point



Finally we sum the values in the product sequence to obtain

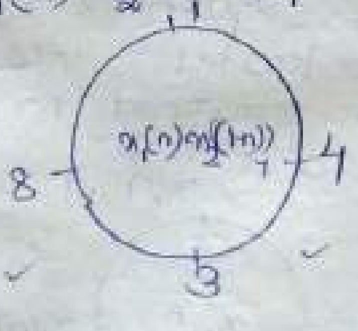
$x_3(0) = \underline{14}$

For m=1

$$x_3(1) = \sum_{n=0}^{3} x_1(n)\, x_2((1-n))_4$$

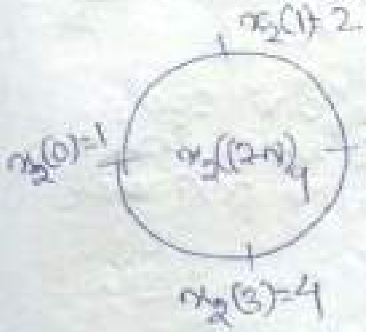
the sequence $x_2((1-n))_4$ is simply the sequence $x_2((-n))_4$ rotated counterclockwise by one unit in time

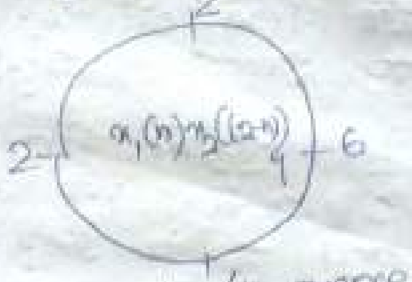
The product sequence $x_1(n)\, x_2((1-n))_4$



Finally, $x_3(1) = \underline{16}$

For m=2 $x_3(2) = \sum_{n=0}^{3} x_1(n)\, x_2((2-n))_4$

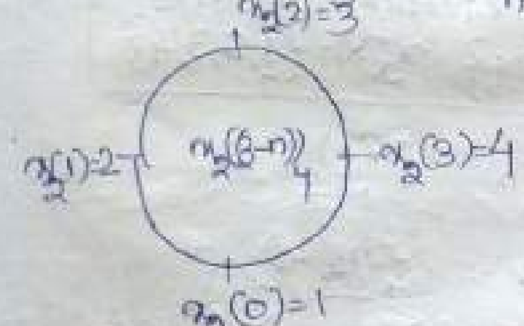


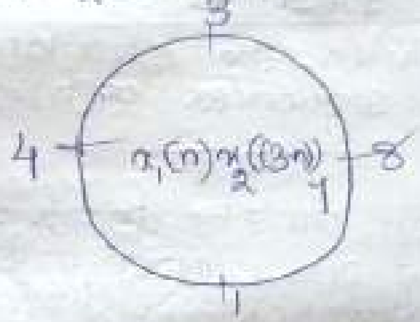


Product sequence

$x_3(2) = 14$

for m=3 $x_3(3) = \sum_{n=0}^{3} x_1(n)\, x_2((3-n))_4$





Product sequence

$x_3(3) = 16$

$x_3(n) = \{\underset{\uparrow}{14}, 16, 14, 16\}$

Methods to evaluate circular convolution of two sequences

(1) Concentric ~~matrix~~ circle Method

Given two sequences $x_1(n)$ and $x_2(n)$, the circular convolution of these two sequences $x_3 = x_1(n) \circledN x_2(n)$ can be found by using the following steps

Step 1 Graph N samples of $x_1(n)$ as equally space points around an outer circle in counter clock wise direction.

Step 2 Start the same point as $x_1(n)$ graph N samples of $x_2(n)$ as equally spaced points around an inner circle in clock wise direction.

Step-3 Multiply corresponding samples on the two circles and sum the products to produce output

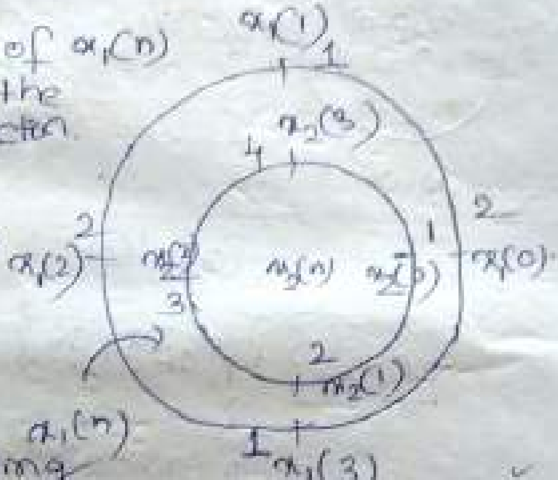
Step-4 Rotate the inner circle one sample at a time in counterclockwise direction and go to step 3 to obtain the next value of output

Step-5 Repeat step no. 4 until the inner circle first sample lines up with the first sample of the exterior circle once again

Perform the circular convolution of the two sequences $x_1(n) = \{\underset{\uparrow}{2},1,2,1\}$ $x_2(n) = \{\underset{\uparrow}{1},2,3,4\}$

Graph all the points of $x_1(n)$ on the outer circle in the counterclockwise direction.

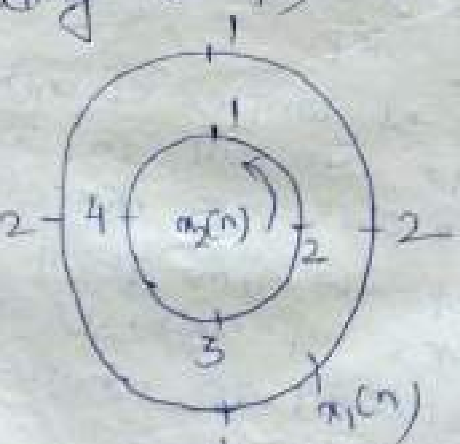
Starting the same point as $x_1(n)$ graph all points of $x_2(n)$ on the inner circle in clockwise direction.



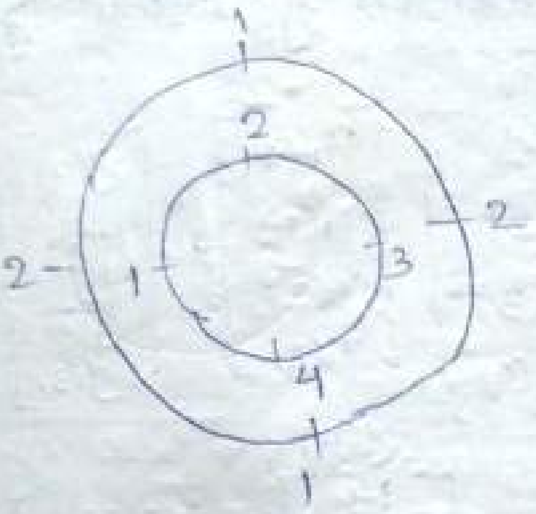
Multiply corresponding samples on the circle and add to obtain

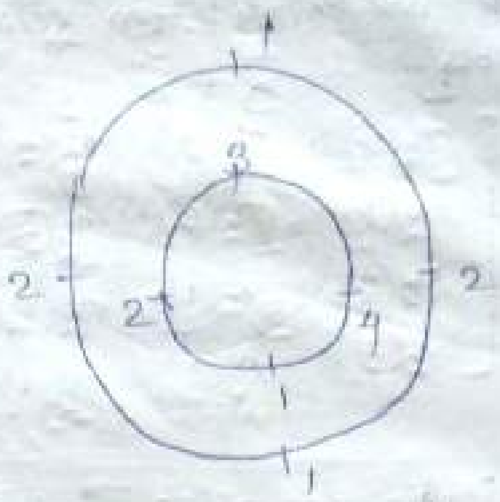
$$y(0) = 2+4+6+2 = 14$$

Rotate the inner circle in counterclockwise direction by one sample, multiply the corresponding samples to obtain $y(1)$



$$y(1) = 4+1+8+3 = 16$$





$y(2) = 6 + 2 + 2 + 4$

$= 14$

$y(3) = 8 + 3 + 4 + 1$

$= 16$

$y(n) = \{\underset{\uparrow}{14}, 16, 14, 16\}$

Matrix Multiplication Method

In this method, the circular convolution of two sequences $x_1(n)$ and $x_2(n)$ can be obtained by representing the sequences in matrix form

$$\begin{bmatrix} x_2(0) & x_2(N-1) & x_2(N-2) & \cdots & x_2(2) & x_2(1) \\ x_2(1) & x_2(0) & x_2(N-1) & & x_2(3) & x_2(2) \\ x_2(2) & x_2(1) & x_2(0) & & x_2(4) & x_2(3) \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ x_2(N-2) & x_2(N-3) & x_2(N-4) & & x_2(0) & x_2(N-1) \\ x_2(N-1) & x_2(N-2) & x_2(N-3) & & x_2(1) & x_2(0) \end{bmatrix} \begin{bmatrix} x_1(0) \\ x_1(1) \\ x_1(2) \\ \vdots \\ x_1(N-1) \end{bmatrix} = \begin{bmatrix} x_3(0) \\ x_3(1) \\ x_3(2) \\ \vdots \\ x_3(N-1) \end{bmatrix}$$

The sequence $x_2(n)$ is repeated via circular shift of samples and represented in NXN matrix form. The sequence $x_1(n)$ is represented as column matrix. The multiplication of these two matrices gives the sequence $x_3(n)$.

Perform the circular convolution of the two sequences $x_1(n) = \{2, 1, 2, 1\}$ $x_2(n) = \{1, 2, 3, 4\}$ by matrix multiplication method.

The matrix form can be written by substituting $N=4$

$$\begin{bmatrix} x_2(0) & x_2(3) & x_2(2) & x_2(1) \\ x_2(1) & x_2(0) & x_2(3) & x_2(2) \\ x_2(2) & x_2(1) & x_2(0) & x_2(3) \\ x_2(3) & x_2(2) & x_2(1) & x_2(0) \end{bmatrix} \begin{bmatrix} x_1(0) \\ x_1(1) \\ x_1(2) \\ x_1(3) \end{bmatrix} = \begin{bmatrix} y(0) \\ y(1) \\ y(2) \\ y(3) \end{bmatrix}$$

$$\begin{bmatrix} 1 & 4 & 3 & 2 \\ 2 & 1 & 4 & 3 \\ 3 & 2 & 1 & 4 \\ 4 & 3 & 2 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 2 \\ 1 \end{bmatrix} = \begin{array}{c} y(n) \\ \begin{bmatrix} 14 \\ 16 \\ 14 \\ 16 \end{bmatrix} \end{array}$$

$$y(n) = \{14, 16, 14, 16\}$$

Find the circular convolution of two finite duration sequences $x_1(n) = \{1, -1, -2, 3, -1\}$; $x_2(n) = \{1, 2, 3\}$ using (a) concentric circle method (b) matrix meth

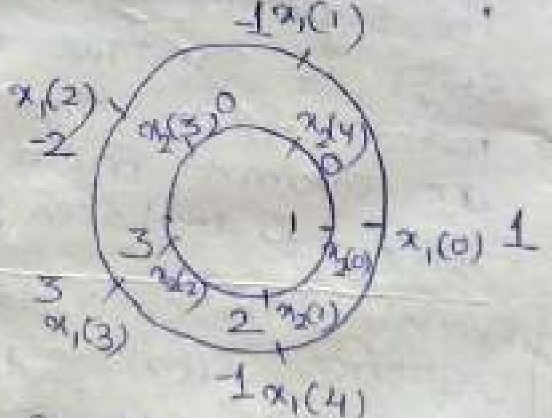
To find circular convolution, both sequences must be of same length. Therefore we append two zeros to the sequence $x_2(n)$

We have $x_1(n) = \{1, -1, -2, 3, -1\}$

$x_2(n) = \{1, 2, 3, 0, 0\}$

Concentric circle Method

Graph all the points of $x_1(n)$ on the outer circle in counter clockwise direction. Starting the same point as $x_1(n)$ graph all points of $x_2(n)$ on the inner circle in clockwise direction.



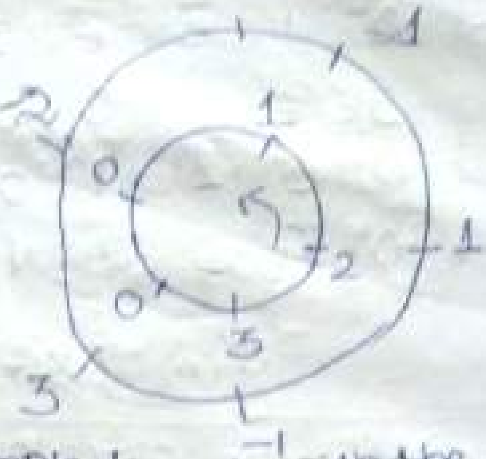
Multiply corresponding samples on the circle and add to obtain

$y(0) = 1 + 0 + 0 + \cancel{9} - 2$
$= 8$

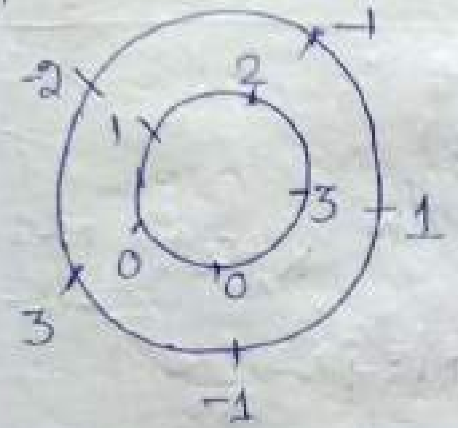
Rotate the inner circle in counterclock wise direction by one sample,

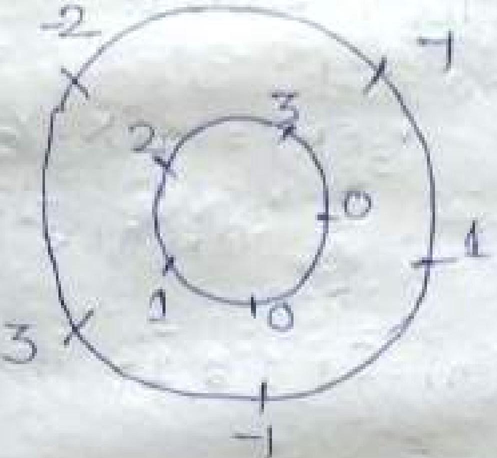
$y(1) = 2 - 1 + 0 + 0 - 3$

$= -2$



Obtain remaining samples by repeating above procedure until the inner circle first sample line up with the first sample of the exterior circle.



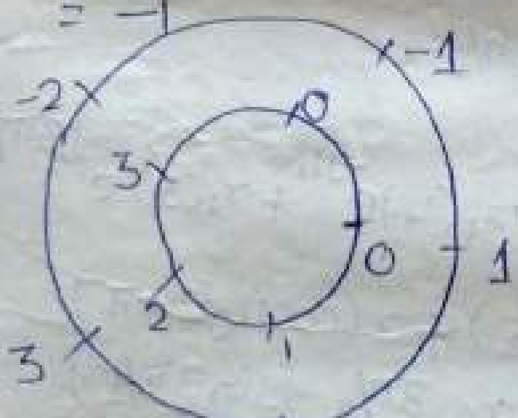


$y(2) = 3 - 2 - 2 + 0 + 0$

$= -1$

$y(3) = 0 - 3 - 4 + 3 + 0$

$= -4$



$y(4) = 0 + 0 - 6 + 6 - 1$

$= -1$

$y(n) = \{8, -2, -1, -4, -1\}$

Matrix Method

$$\begin{bmatrix} x_2(0) & x_2(4) & x_2(3) & x_2(2) & x_2(1) \\ x_2(1) & x_2(0) & x_2(4) & x_2(3) & x_2(2) \\ x_2(2) & x_2(1) & x_2(0) & x_2(4) & x_2(3) \\ x_2(3) & x_2(2) & x_2(1) & x_2(0) & x_2(4) \\ x_2(4) & x_2(3) & x_2(2) & x_2(1) & x_2(0) \end{bmatrix} \begin{bmatrix} x_1(0) \\ x_1(1) \\ x_1(2) \\ x_1(3) \\ x_1(4) \end{bmatrix} = \begin{bmatrix} y(0) \\ y(1) \\ y(2) \\ y(3) \\ y(4) \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 3 & 2 \\ 2 & 1 & 0 & 0 & 3 \\ 3 & 2 & 1 & 0 & 0 \\ 0 & 3 & 2 & 1 & 0 \\ 0 & 0 & 3 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ -2 \\ 3 \\ -1 \end{bmatrix} = \begin{bmatrix} 8 \\ -2 \\ -1 \\ -4 \\ -1 \end{bmatrix}$$

$$y(n) = \{8, -2, -1, -4, -1\}$$

Perform the circular convolution of the following sequences $x(n) = \{1, 1, 2, 1\}$ $x_2(n) = \{1, 2, 3, 4\}$ using DFT and IDFT method.

We know that

$$X_3(k) = X_1(k)\, X_2(k)$$

$$X_1(k) = \sum_{n=0}^{N-1} x_1(n)\, e^{-j2\pi kn/N} \quad k = 0, 1, \ldots N-1$$

$$x_1(n) = \{1, 1, 2, 1\} \text{ and } N = 4$$

$$X_1(0) = \sum_{n=0}^{3} x_1(n) = 1+1+2+1 = 5$$

$$X_1(1) = \sum_{n=0}^{3} x_1(n)\, e^{-j\pi n/2}$$

$$= x_1(0) + x_1(1)\, e^{-j\pi/2} + x_1(2)\, e^{-j\pi} + x_1(3)\, e^{-j3\pi/2}$$

$$= 1 + 1\,(\cos\pi/2 - j\sin\pi/2) + 2\,(\cos\pi - j\sin\pi) + 1\cdot(\cos 3\pi/2 - j\sin 3\pi/2)$$

$$= 1 - j - 2 + j = -1$$

$$X_1(2) = \sum_{n=0}^{3} x_1(n)\, e^{-j\pi n}$$

$$= x_1(0) + x_1(1)\, e^{-j\pi} + x_1(2)\, e^{-j2\pi} + x_1(3)\, e^{-j3\pi}$$

$$= 1 + 1\,(\cos\pi - j\sin\pi) + 2\,(\cos 2\pi - \sin 2\pi) + 1\,(\cos 3\pi - \sin 3\pi)$$

$$= 1 - 1 + 2 - 1 = 1$$

$$X_1(3) = \sum_{n=0}^{3} x_1(n) e^{-j3\pi n/2}$$

$$= x_1(0) + x_1(1) e^{-j3\pi/2} + x_1(2) e^{-j3\pi} + x_1(3) e^{j9\pi/2}$$

$$= 1 + 1\left(\cos 3\pi/2 - j\sin 3\pi/2\right) + 2\left(\cos 3\pi - j\sin 3\pi\right) + 1\left(\cos 9\pi/2 - j\sin 9\pi/2\right)$$

$$= 1 + j - 2 - j = -1$$

$$X_1(K) = \{5, -1, 1, -1\}$$

Similarly

$$X_2(K) = \sum_{n=0}^{N-1} x_2(n) e^{-j2\pi kn/N} \quad K = 0, 1, \ldots N-1$$

$$x_2(n) = \{1, 2, 3, 4\}$$

$$X_2(0) = \sum_{n=0}^{3} x_2(n) = 10$$

$$X_2(1) = \sum_{n=0}^{3} x_2(n) e^{-j\pi n/2}$$

$$= 1 - 2(j) - 3(1) + 4(j) = -2 + j2$$

$$X_2(2) = \sum_{n=0}^{3} x_2(n) e^{-j\pi n}$$

$$= 1 + 2(-1) + 3(1) + 4(-1) = -2$$

$$X_2(3) = \sum_{n=0}^{3} x_2(n) e^{-j3\pi n/2}$$

$$= 1 + 2(j) + 3(-1) + 4(-j)$$

$$= -2 - j2$$

$$X_2(K) = \{10, -2+j2, -2, -2-j2\}$$

$$X_3(K) = X_1(K) X_2(K) = \{50, 2-j2, -2, 2+j2\}$$

$$X_3(K) = X_1(K) X_2(K)$$

$$x_3(n) = \frac{1}{N} \sum_{K=0}^{N-1} X_3(K) e^{j2\pi nK/N} \quad n = 0, 1, \ldots N-1$$

$$x_3(0) = \frac{1}{4} \sum_{K=0}^{3} X_3(K)$$

$$= \frac{1}{4}\left(50 + \cancel{2} - \cancel{j2} - \cancel{2} + 2 + \cancel{j2}\right) = 13$$

$$x_3(1) = \frac{1}{4}\left[\sum_{K=0}^{3} X_3(K)\, e^{j\pi K/2}\right]$$

$$= \frac{1}{4}\left[X_3(0) + X_3(1)\, e^{j\pi/2} + X_3(2)\, e^{j\pi} + X_3(3)\, e^{j3\pi/2}\right]$$

$$= \frac{1}{4}\left[50 + (2-j2)\left(\cos\pi/2 + j\sin\pi/2\right) + (-2)\left(\cos\pi + j\sin\pi\right) + (2+j2)\left(\cos 3\pi/2 + j\sin 3\pi/2\right)\right]$$

$$= \frac{1}{4}\left[50 + (2-j2)j + (-2)(-1) + (2+j2)(-j)\right]$$

$$= \frac{1}{4}\left[50 + \cancel{j2} + 2 + 2 - \cancel{2j} + 2\right] = 14$$

$$x_3(2) = \frac{1}{4}\left[\sum_{K=0}^{3} X_3(K)\, e^{j\pi K}\right]$$

$$= \frac{1}{4}\left[50 + (2-j2)(-1) + (-2)(1) + (2+j2)(-1)\right]$$

$$= 11$$

$$x_3(3) = \frac{1}{4}\left[\sum_{K=0}^{3} X_3(K)\, e^{j3\pi K/2}\right]$$

$$= \frac{1}{4}\left[50 + (2-j2)(-j) + (-2)(-1) + (2+j2)(j)\right]$$

$$= 12$$

$$x_3(n) = \{13, 14, 11, 12\}$$

Additional DFT Properties $x(-n, (\text{mod } N))$

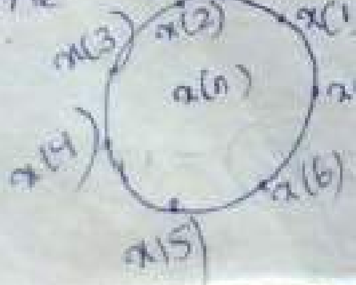
When the N-pt sequence in time is reversed, it is equivalent to reversing the DFT values.

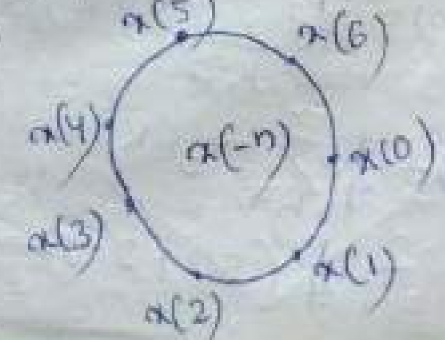
Time reversal of a sequence

If $x(n) \xleftrightarrow[n]{DFT} X(k)$

then $x((-n))_N = x(N-n) \xleftrightarrow[n]{DFT} X((-k))_N = X(N-k)$

Hence reversing the N-point sequence in time is equivalent to reversing the DFT values.





Proof:- $DFT\{x(N-n)\} = \sum_{n=0}^{N-1} x(N-n)e^{-j2\pi kn/N}$

changing the index from n to $m = N-n$

$$= \sum_{m=0}^{N-1} x(m)e^{-j2\pi k(N-m)/N}$$

let $N-n=m$
$m = N-n$
$n=0;\ m=N$
$n=N-1,\ m=1$
$m = N-N+1$

$N \to 1$
$0 \to N-1$

$$= \sum_{m=0}^{N-1} x(m)e^{j2\pi km/N}$$

$\because e^{-j2\pi k} = 1$

$$= \sum_{m=0}^{N-1} x(m)e^{-j2\pi m(N-k)/N}$$

$e^{-j2\pi m} \cdot e^{+j2\pi km/N}$

$$= X(N-k) \qquad 0 \le k \le N-1$$

We note that $X(N-k) = X((-k))_N \quad 0 \le k \le N-1$

Circular timeshift of a sequence

If $x(n) \underset{N}{\overset{DFT}{\longleftrightarrow}} X(k)$

$N=4$
$0-3$

then $x((n-l))_N \overset{DFT}{\longleftrightarrow} X(k)e^{-j2\pi kl/N}$

Proof

$$DFT\{x((n-l))_N\} = \sum_{n=0}^{N-1} x((n-l))_N e^{-j2\pi kn/N}$$

$$= \sum_{n=0}^{l-1} x((n-l))_N e^{-j2\pi kn/N} + \sum_{n=l}^{N-1} x((n-l))_N e^{-j2\pi kn/N}$$

But $\boxed{x((n-l))_N = x(N-l+n)}$

$$\sum_{n=0}^{l-1} x((n-l))_N e^{-j2\pi kn/N} = \sum_{n=0}^{l-1} x(N-l+n)e^{-j2\pi kn/N}$$

let $N-l+n=m$
$n = m-N+l$
$n=0,\ m=N-l$
$n=N-1$, $N-1+l-1=m$
$m=N-1$

$$= \sum_{m=N-l}^{N-1} x(m)e^{-j2\pi k(-N+m+l)/N}$$

$$= \sum_{m=N-l}^{N-1} x(m)e^{-j2\pi k(m+l)/N} e^{j2\pi k} \qquad e^{j2\pi k} = 1 \text{ for } k=0,1,\ldots$$

$$\sum_{n=l}^{N-1} x((n-l))_N e^{-j2\pi kn/N} = \sum_{m=0}^{N-1-l} x(m)e^{-j2\pi k(m+l)/N}$$

$N-l+n=m$
$N-l+N-1=m$
$N-l+N-1 = 2N-l-1$
$m=N-l$

$l \to N$
$l-1 \to N$
$0 \to N$

$$\sum_{m=N}^{2N-l-1} x(m)e^{-j2\pi k(m-N+l)/N}$$

$$= \sum_{m=0}^{N-l-1} x(m)^{-j2\pi k(m+l)/N}$$

$$= \sum_{m=N-l}^{N-1} x(m) e^{-j2\pi k(m+l)/N} + \sum_{m=0}^{N-1-l} x(m) e^{-j2\pi k(m+l)/N}$$

$$= e^{-j2\pi kl/N} \sum_{m=0}^{N-1} x(m) e^{-j2\pi km/N}$$

$$= e^{-j2\pi kl/N} \sum_{n=0}^{N-1} x(n) e^{-j2\pi kn/N}$$

$$= e^{-j2\pi kl/N} X(K)$$

Circular frequency shift

If $x(n) \xleftrightarrow[N]{DFT} X(K)$

then $x(n) e^{j2\pi ln/N} \xleftrightarrow[N]{DFT} X((K-l))_N$

Proof $DFT[x(n) e^{j2\pi ln/N}]$

$$= \sum_{n=0}^{N-1} x(n) e^{j2\pi ln/N} e^{-j2\pi kn/N}$$

$$= \sum_{n=0}^{N-1} x(n) e^{-j2\pi n(K-l)/N}$$

$$= \sum_{n=0}^{N-1} x(n) e^{-j2\pi n(N+K-l)/N}$$

$$= X(N+K-l)$$

$$= X((K-l))_N$$

complex conjugate Property

If $x(n) \xleftrightarrow[n]{DFT} X(K)$

$x^*(n) \xleftrightarrow[n]{DFT} X^*((-K))_N = X^*(N-K)$

Proof $DFT[x^*(n)] = \sum_{n=0}^{N-1} x^*(n) e^{-j2\pi kn/N}$

$$= \left[\sum_{n=0}^{N-1} x(n) e^{j2\pi kn/N}\right]^*$$

$$= \left[\sum_{n=0}^{N-1} x(n) e^{-j2\pi n(N-K)/N} \right]^*$$

$$= X^*(N-K)$$

$$DFT\{x^*(N-n)\} = X^*(K)$$

<u>proof</u>

$$IDFT\{X^*(K)\} = \frac{1}{N} \sum_{K=0}^{N-1} X^*(K) e^{j2\pi Kn/N}$$

$$= \frac{1}{N} \left[\sum_{K=0}^{N-1} X(K) e^{-j2\pi Kn/N} \right]^*$$

$$= \frac{1}{N} \left[\sum_{K=0}^{N-1} X(K) e^{j2\pi K(N-n)/N} \right]^*$$

$$= x^*(N-n)$$

Multiplication of two sequences

If $x_1(n) \xleftrightarrow[N]{DFT} X_1(K)$

and $x_2(n) \xleftrightarrow[N]{DFT} X_2(K)$

then $x_1(n)\, x_2(n) \xleftrightarrow[N]{DFT} \frac{1}{N} X_1(K) \; N \; X_2(K)$

Circular convolution

If $x_1(n) \xleftrightarrow[N]{DFT} X_1(K)$

and $x_2(n) \xleftrightarrow[N]{DFT} X_2(K)$

then $x_1(n) \; N \; x_2(n) \xleftrightarrow[N]{DFT} X_1(K)\, X_2(K)$

Linear Filtering Methods based on DFT

A system with frequency response $H(\omega)$, when excited with an input signal that has a spectrum $X(\omega)$, possesses an output spectrum waveform $Y(\omega) = X(\omega) H(\omega)$. The output sequence $y(n)$ is determined from its spectrum via the inverse Fourier transform. computation the problem with this frequency domain approach is that $X(\omega)$, $H(\omega)$ and $Y(\omega)$ are functions of the continuous variable ω. As a consequence, the computations can't be done on a digital computer, since the computer can only store and perform computations on quantities at discrete frequencies.

On the other hand, the DFT does lend itself to computation on a digital computer. We describe how DFT can be used to perform linear filtering in the frequency domain. In particular, we represent a computational procedure that serves as an alternative to time-domain computation. In fact, the frequency domain approach based on the DFT, is computationally more efficient than time-domain convolution due to existance of efficient algorithms for computing DFT.

These algorithms are collectively called Fast fourier transform (FFT) algorithms.

Use of the DFT in linear Filtering

It was demonstrated that the product of two DFTs is equivalent to the circular convolution of the corresponding time-domain sequences. Unfortunately, circular convolution is of no use to us if our objective is to determine the output of a linear filter to given input sequence. In this case we seek a frequency-domain methodology equivalent to linear convolution.

Suppose we have a finite-duration sequence $x(n)$ of length L which excites an FIR filter of length M.

$x(n) = 0, \quad n<0 \text{ \& } n \geq L$

$h(n) = 0, \quad n<0 \text{ and } n>M$

where $h(n)$ is the impulse response of the FIR filter.

The output sequence $y(n)$ of the FIR filter can be expressed in the time domain as the convolution of $x(n)$ and $h(n)$, i.e

$$\boxed{y(n) = \sum_{K=0}^{M-1} h(K)\, x(n-K)}$$

Since $h(n)$ and $x(n)$ are finite-duration sequences, their convolution is also finite in duration. In fact, the duration of $y(n)$ is $L+M-1$

The frequency-domain

$$Y(\omega) = X(\omega) H(\omega)$$

Now if

$$Y(K) \equiv Y(\omega)\big|_{\omega = 2\pi K/N} \qquad K=0,1,\dots N-1$$

$$= X(\omega) H(\omega)\big|_{\omega = 2\pi K/N} \qquad K=0,1,\dots N-1$$

then $\boxed{Y(K) = X(K) H(K)} \qquad K=0,1,\dots,N-1$

where $\{X(K)\}$ and $\{H(K)\}$ are the N-point DFTs of the sequences $x(n)$ and $h(n)$ respectively. Since the sequences $x(n)$ and $h(n)$ have a duration less than N, we simply pad these sequences with zeros to increase their length to N. This increase in the size of the sequences does not alter their spectra $X(\omega)$ and $H(\omega)$, which are continuous spectra, since the sequences are aperiodic.

Since the $N = L+M-1$ point DFT of the output sequence $y(n)$ is sufficient to represent $y(n)$ in the frequency domain, it follows that the multiplication of the N-point DFTs $X(K)$ and $H(K)$, followed by the computation of N-point IDFT, must yield the sequence $\{y(n)\}$. In turn, this implies that the N-point circular convolution of $x(n)$ with $h(n)$ must be equivalent to the linear convolution of $x(n)$ with $h(n)$. In other words by increasing the length of the sequence $x(n)$ and $h(n)$ to N points (by appending zeros) and circularly convolving the resulting sequence we obtain the same result as would have been obtained with linear convolution. Thus with zero padding, the DFT can be used to perform linear filtering.

By means of DFT and IDFT, determine the response of the FIR filter with impulse response $h(n) = \{\underset{\uparrow}{1}, 2, 3\}$ to the input sequence $x(n) = \{\underset{\uparrow}{1}, 2, 2, 1\}$

The input sequence has length L=4 and the impulse response has length M=3. Linear convolution of these two sequences produces a sequence of length N=6. Consequently the size of the DFTs must be atleast 6.

For simplicity we compute eight-point DFTs. We should also mention that the efficient computation of DFT via fast Fourier transform (FFT) algorithm is usually performed for a length N that is a power of 2. Hence the eight-point DFT of $x(n)$ is

$$X(k) = \sum_{n=0}^{7} x(n) e^{-j2\pi kn/8}$$

$$= 1 + 2e^{-j\pi k/4} + 2e^{-j\pi k/2} + e^{-j3\pi k/4} \quad k = 0, 1, 2 \cdots 7$$

this computation yields

$X(0) = 6$ $\quad X(1) = \frac{2+\sqrt{2}}{2} - j\left(\frac{4+3\sqrt{2}}{2}\right)$

$X(2) = -1-j$ $\quad X(3) = \frac{2-\sqrt{2}}{2} + j\left(\frac{4-3\sqrt{2}}{2}\right)$

$X(4) = 0$ $\quad X(5) = \frac{2-\sqrt{2}}{2} - j\,\frac{4-3\sqrt{2}}{2}$

$X(6) = -1+j$ $\quad X(7) = \frac{2+\sqrt{2}}{2} + j\left(\frac{4+3\sqrt{2}}{2}\right)$

The eight point DFT of h(n) is

$$H(K) = \sum_{n=0}^{7} h(n) e^{-j2\pi kn/8}$$

$$= 1 + 2e^{-j\pi k/4} + 3e^{-j\pi k/2}$$

Here,

$H(0) = 6$ $\quad H(1) = 1+\sqrt{2} - j(3+\sqrt{2})$ $\quad H(2) = -2-j2$

$H(3) = 1-\sqrt{2} + j(3-\sqrt{2})$, $\quad H(4) = 2$

$H(5) = 1-\sqrt{2} - j(3-\sqrt{2})$ $\quad H(6) = -2+j2$

$H(7) = 1+\sqrt{2} + j(3+\sqrt{2})$

The product of these two DFTs yields Y(k), which is

$Y(0) = 36$, $\; Y(1) = -14.07 - j17.48$, $\; Y(2) = j4$

$Y(3) = 0.07 + j0.515$, $\; Y(4) = 0$, $\; Y(5) = 0.07 - j0.515$,

$Y(6) = -j4$, $\; Y(7) = -14.07 + j17.48$

Finally the eight point IDFT is

$$y(n) = \frac{1}{8} \sum_{k=0}^{7} Y(k) e^{j2\pi kn/8} \quad n = 0,1,\cdots 7$$

This computation yields the result

$$y(n) = \{\underset{\uparrow}{1}, 4, 9, 11, 8, 3, 0, 0\}$$

Determine the output response y(n) if $h(n) = \{1,1,1\}$; $x(n) = \{1,2,3,1\}$ by using (i) linear convolution (ii) circular convolution (iii) circular convolution with zero padding

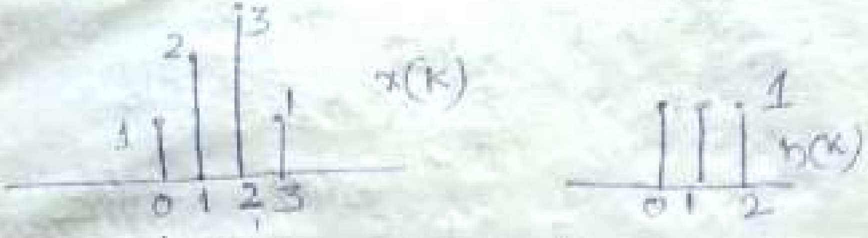
Given $x(n) = \{1,2,3,1\}$

$h(n) = \{1,1,1\}$

We know that

$$y(n) = \sum_{k=-\infty}^{\infty} x(k) h(n-k)$$

$y(n) = \{1, 5, 6, 6, 4, 1\}$



$$y(0)=\sum_{k=-\infty}^{\infty} x(k)h(-k) = 1$$

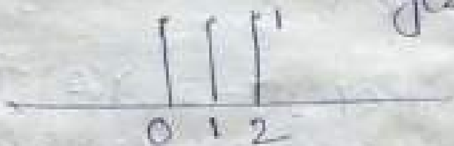
n=0; h(-k)

$$y(1)=\sum_{k=-\infty}^{\infty} x(k)h(1-k) = 3$$

n=1; h(1-k)

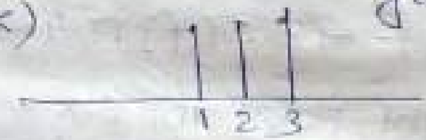
$$y(2)=\sum_{k=-\infty}^{\infty} x(k)h(2-k) = 6$$

n=2; h(2-k)



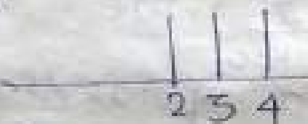
$$y(3)=\sum_{k=-\infty}^{\infty} x(k)h(3-k) = 6$$

n=3; h(3-k)



$$y(4)=\sum_{k=-\infty}^{\infty} x(k)h(4-k) = 4$$

n=4; h(4-k)



$$y(5)=\sum_{k=-\infty}^{\infty} x(k)h(5-k) = 1$$

n=5; h(5-k)

$y(n) = \{1, 3, 6, 6, 4, 1\}$

$L+M-1 = 4+3-1 = \underline{6}$

(ii) Circular convolution $x(n) = \{1, 2, 3, 1\}$

matrix approach $h(n) = \{1, 1, 1, 0\}$

$$\begin{bmatrix} 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 4 \\ 6 \\ 6 \end{bmatrix}$$

$y(n) = x(n) \; N \; h(n)$

$= \{5, 4, 6, 6\}$

(ii) Circular convolution with Zero padding

$x(n) = \{1, 2, 3, 1, \underbrace{0, 0}_{(M-1)\text{ Zeros appended}}\}$

$h(n) = \{1, 1, 1, \underbrace{0, 0, 0}_{(L-1)\text{ Zeros appended}}\}$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 6 \\ 6 \\ 4 \\ 1 \end{bmatrix} \quad N = 6$$

$y(n) = \{1, 3, 6, 6, 4, 1\}$ $\boxed{N = L + M - 1}$

Filtering of Long Data Sequences

In practical applications involving linear filtering of signals, the input sequence $x(n)$ is often a very long sequence.

Since linear filtering performed via the DFT involves operations on a block of data, which must be limited in size due to limited memory of a digital computer.

A long input signal sequence must be segmented to fixed-size blocks prior to processing.

Since filtering is linear, successive blocks can be processed one at a time via the DFT and output blocks are fitted together to form the overall output signal sequence.

There are two methods for linear FIR filtering a long sequence on a block-by-block basis using the DFT.

The input sequence is segmented into blocks and each block is processed via the DFT and IDFT to produce block of output data.

The output blocks are fitted together to form an overall output sequence which are identical to the sequence obtained if the long block had been processed via time-domain convolution.

The two methods are called the overlap-save method and over-lap add method. For both methods we assume that the FIR filter has duration M. The input data sequence is segmented into blocks of L points, where by assumption L>>M

Overlap-save method

In this method the size of the input data blocks is N=L+M-1 and size of the DFTs & IDFTs are of length N.

Each data block consists of the last M-1 data points of previous data block followed by L new data points to form a data sequence of length N=L+M-1.

An N-point DFT is computed for each data block. The impulse response of the FIR filter is increased in length by appending L-1 zeros & an N-point DFT of the sequence is computed once and stored.

(2) To begin the processing, the first M-1 points of the first record are set to zero. Thus the blocks of data sequences are

$$x_1(n)=\{\underbrace{0,0,\ldots,0}_{M-1 \text{ points}}, x(0), x(1), \ldots, x(L-1)\}$$

$$x_2(n)=\{\underbrace{x(L-M+1), \ldots x(L-1)}_{\substack{M-1 \text{ data points} \\ \text{of } x_1(n)}}, \underbrace{x(L), \ldots x(2L-1)}_{L \text{ new data points}}\}$$

$$x_3(n)=\{\underbrace{x(2L-M+1), \ldots x(2L-1)}_{\substack{M-1 \text{ data points} \\ \text{from } x_2(n)}}, \underbrace{x(2L), \ldots x(3L-1)}_{L \text{ new data points}}\}$$

and so forth.

The resulting data sequences from IDFT where first M-1 points are discarded due to aliasing and the remaining L points constitute the desired data result from linear convolution.

Overlap-add Method

In this method the size of the input data block is L points and the size of the DFTs and IDFT is $N = L+M-1$. To each data block we append $M-1$ zeros and compute the N point DFT.

(3) Thus the data blocks may be represented as

$$x_1(n) = \{x(0), x(1), \dots, x(L-1), \underbrace{0, 0, \dots 0}_{M-1 \text{ zeros}}\}$$

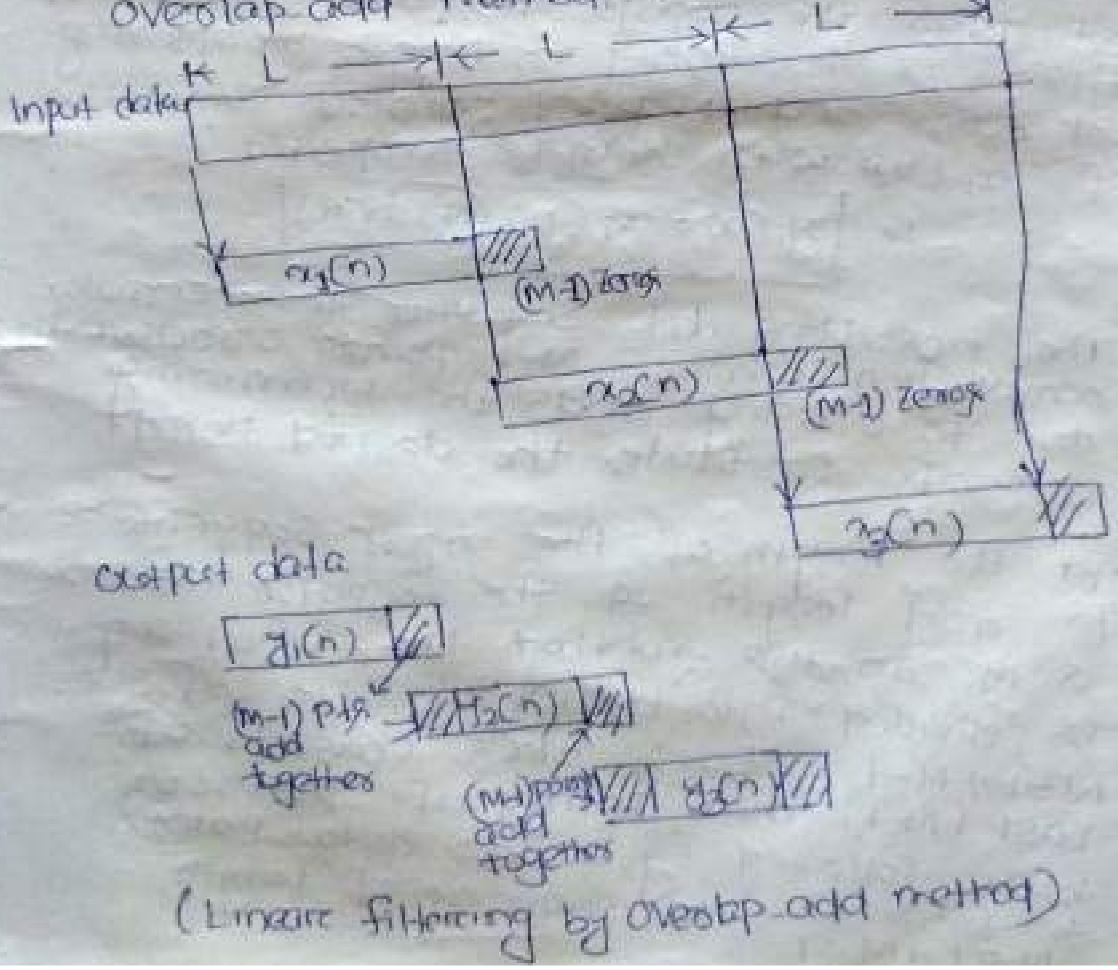
$$x_2(n) = \{x(L), x(L+1), \dots, x(2L-1), \underbrace{0, 0, \dots 0}_{M-1 \text{ zeros}}\}$$

$$x_3(n) = \{x(2L), \dots x(3L-1), \underbrace{0, 0, \dots 0}_{M-1 \text{ zeros}}\}$$

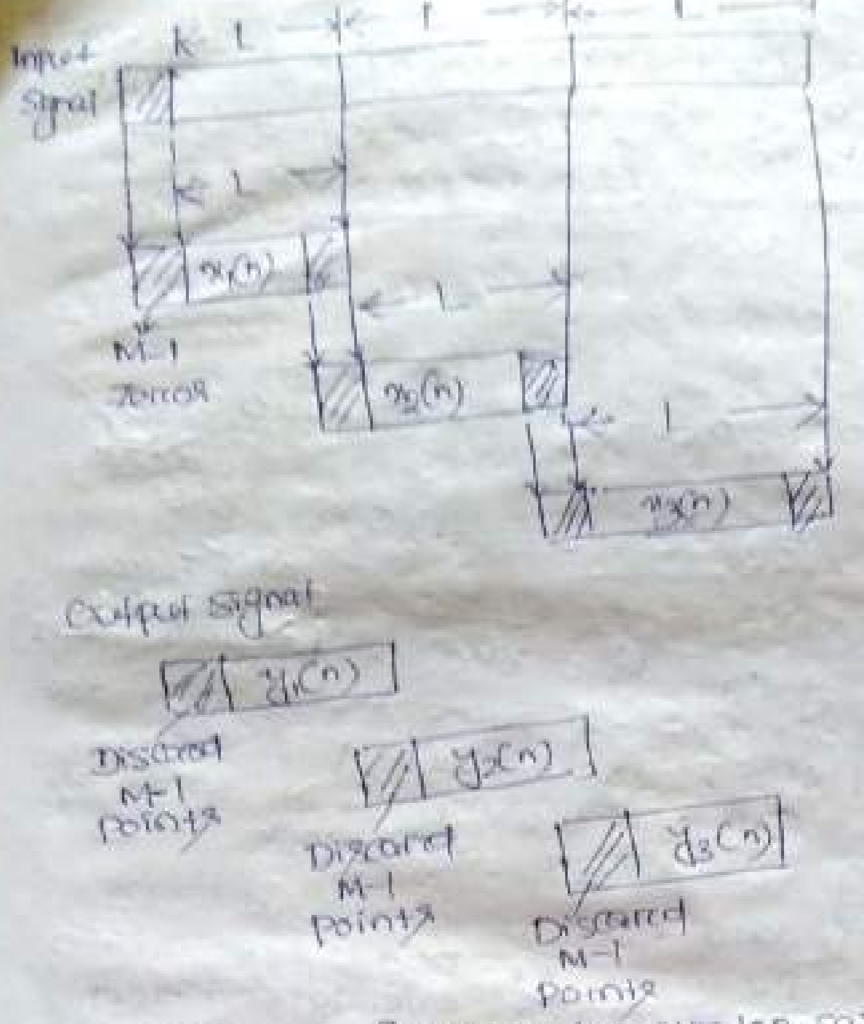
and so on.

The IDFT yields data block of length N that are free of aliasing since the size of the DFTs and IDFT is $N = L+M-1$ and the sequences are increased to N-points by appending zeros to each block.

(4) Since each data block is terminated with $M-1$ zeros, the last $M-1$ points from each output block must be overlapped and added to the first $M-1$ points of succeeding block. Hence this method is called the overlap-add method.



(Linear filtering by overlap-add method)



(linear filtering by overlap-save method)

overlap-save method

③ contd. Now the impulse response of the FIR filter is increased in length by appending L-1 zeros and N point circular convolution of $x_i(n)$ with $h(n)$ is computed.

i.e $\boxed{y_i(n) = x_i(n) \circledN h(n)}$.

The resulting data sequence from circular convolution where M-1 points are discarded due to aliasing and the remaining L points constitute the desired result.

① let the length of the input sequence be L and length of the impulse response is M. In this method the input sequence is divided into blocks of data of size $N = L+M-1$. Each block consists of last (M-1) data points of previous block followed by L new data points to form a data sequence of length $N = L+M-1$

Overlap-Add method

(1) let the length of the sequence be L and length of the impulse response is M. The sequence is divided into blocks of data size having length L and M-1 zeros are appended to it to make the data size of L+M-1

(2) Now L-1 zeros are added to the impulse response h(n) & N-point circular convolution is performed.

(3) The output blocks are of the form

$y_1(n) = \{y_1(0), y_1(1), \cdots, y_1(L-1), y_1(L), \cdots y_1(N-1)\}$

$y_2(n) = \{y_2(0), y_2(1), \cdots\cdots y_2(L-1), y_2(L), \cdots y_2(N-1)\}$

$y_3(n) = \{y_3(0), y_3(1), \cdots y_3(L-1), y_3(L), \cdots y_3(N-1)\}$

The output sequence is

$y(n) = \{y_1(0), y_1(1), \cdots, y_1(L-1), y_1(L) + y_2(0), y_1(L+1) + y_2(1) \cdots y_1(N-1) + y_2(M-1), y_2(M) \cdots\}$

Find the output $y(n)$ of a filter whose impulse response is $h(n)=\{1,1,1\}$ and input signal $x(n)=\{3,-1,0,1,3,2,0,1,2,1\}$ using (i) overlap-save method (ii) overlap-add method

(i) Overlap-save method

let the length of the data block be 3

The input sequence can be divided into blocks of data as follows

$$x_1(n)=\{\underbrace{0,0,}_{M-1=2\text{ zeros}}\ \underbrace{3,-1,0}_{L=3\text{ data points}}\}$$

$$x_2(n)=\{\underbrace{-1,0,}_{\text{two datas from prev. block}}\ \underbrace{1,3,2}_{3\text{ new data points}}\}$$

$$x_3(n)=\{3,2,0,1,2\}$$

$$x_4(n)=\{1,2,1,0,0\}$$

given $h(n)=\{1,1,1\}$

increase the length of the sequence $L+M-1=5$ by adding two zeros

ie $h(n)=\{1,1,1,0,0\}$

$$y_1(n)=x_1(n) \text{ (N) } h(n) = \{-1,0,3,2,2\}$$

$$\begin{bmatrix} 0 & 0 & -1 & 3 & 0 \\ 0 & 0 & 0 & -1 & 3 \\ 3 & 0 & 0 & 0 & -1 \\ -1 & 3 & 0 & 0 & 0 \\ 0 & -1 & 3 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 3 \\ 2 \\ 2 \end{bmatrix}$$

$$y_2(n)=x_2(n) \text{ (N) } h(n) = \{4,1,0,4,6\}$$

$$\begin{bmatrix} -1 & 2 & 3 & 1 & 0 \\ 0 & -1 & 2 & 3 & 1 \\ 1 & 0 & -1 & 2 & 3 \\ 3 & 1 & 0 & -1 & 2 \\ 2 & 3 & 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \\ 0 \\ 4 \\ 6 \end{bmatrix}$$

$y_3(n) = x_3(n) \, \text{N} \, h(n) = \{6, 7, 5, 3, 3\}$

$$\begin{bmatrix} 3 & 2 & 1 & 0 & 2 \\ 2 & 3 & 2 & 1 & 0 \\ 0 & 2 & 3 & 2 & 1 \\ 1 & 0 & 2 & 3 & 2 \\ 2 & 1 & 0 & 2 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 6 \\ 7 \\ 5 \\ 3 \\ 3 \end{bmatrix}$$

$y_4(n) = x_4(n) \, \text{N} \, h(n) = \{1, 3, 4, 3, 1\}$

$$\begin{bmatrix} 1 & 0 & 0 & 1 & 2 \\ 2 & 1 & 0 & 0 & 1 \\ 1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 4 \\ 3 \\ 1 \end{bmatrix}$$

$\boxed{\underbrace{-1, 0}_{\text{discard}}, 5, 2, 2}$

$\boxed{\underbrace{4, 1}_{\text{discard}}, 0, 4, 6}$

$\boxed{\underbrace{6, 7}_{\text{discard}}, 5, 3, 3}$

$\boxed{\underbrace{1, 3}_{\text{discard}}, 4, 3, 1}$

$y(n) = \{3, 2, 2, 0, 4, 6, 5, 3, 3, 4, 3, 1\}$

Scanned with CamScanner

(ii) Overlap Add method

let the length of the data block be 3

two zeros are added to bring the length to five

ie $L+M-1=5$

$x_1(n)=\{3,-1,0,0,0\}$

$x_2(n)=\{1,3,2,0,0\}$

$x_3(n)=\{0,1,2,0,0\}$

$x_4(n)=\{1,0,0,0,0\}$

$y_1(n)=x_1(n) \; (N) \; h(n)=\{3,2,2,-1,0\}$

$$\begin{bmatrix} 3 & 0 & 0 & 0 & -1 \\ -1 & 3 & 0 & 0 & 0 \\ 0 & -1 & 3 & 0 & 0 \\ 0 & 0 & -1 & 3 & 0 \\ 0 & 0 & 0 & -1 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ 2 \\ -1 \\ 0 \end{bmatrix}$$

$y_2(n)=x_2(n) \; (N) \; h(n)=\{1,4,6,5,2\}$

$$\begin{bmatrix} 1 & 0 & 0 & 2 & 3 \\ 3 & 1 & 0 & 0 & 2 \\ 2 & 3 & 1 & 0 & 0 \\ 0 & 2 & 3 & 1 & 0 \\ 0 & 0 & 2 & 3 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \\ 6 \\ 5 \\ 2 \end{bmatrix}$$

$y_3(n)=x_3(n) \; (N) \; h(n)=\{0,1,3,3,2\}$

$$\begin{bmatrix} 0 & 0 & 0 & 2 & 1 \\ 1 & 0 & 0 & 0 & 2 \\ 2 & 1 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & 2 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 3 \\ 3 \\ 2 \end{bmatrix}$$

$y_4(n)=x_4(n) \; (N) \; h(n)=\{1,1,1,0,0\}$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

$\boxed{3, 2, 2, -1, 0}$

add

$\boxed{1, 4, 6, 5, 2}$

add

$\boxed{0, 1, 3, 3, 2}$

$\boxed{1, 1, 1, 0, 0}$

$y(n) = \{3, 2, 2, 0, 4, 6, 5, 3, 3, 4, 3, 1, \cancel{0, 0}\}$

~~Frequency Analysis of signals using DFT~~

Implementation of Discrete-time Systems

Structures for the Realization of Discrete-time system

let us consider the linear time-invariant discrete-time system characterized by the general linear constant-coefficient difference equation

$$\boxed{y(n) = -\sum_{k=1}^{N} a_k y(n-k) + \sum_{k=0}^{M} b_k x(n-k)}$$

By means of z-transform, the linear time-invariant discrete-time systems are also characterized by the rational system function

$$\boxed{H(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{1 + \sum_{k=1}^{N} a_k z^{-k}}}$$

which is a ratio of two polynomials in z^{-1}. The system parameters $\{b_k\}$ and $\{a_k\}$ determine the frequency response characteristics of the system.

Structures for FIR Systems

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In general, an FIR system is described by the difference equation

$$\boxed{y(n) = \sum_{k=0}^{M-1} b_k x(n-k)}$$

or equivalently, by the system function

$$\boxed{H(z) = \sum_{k=0}^{M-1} b_k z^{-k}}$$

the unit sample response of the FIR system is identical to the coefficients $\{b_k\}$ that is

$$h(n) = \begin{cases} b_n, & 0 \le n \le M-1 \\ 0, & \text{otherwise} \end{cases}$$

where the length of the FIR filter is selected as M.

A FIR system can be implemented in several forms

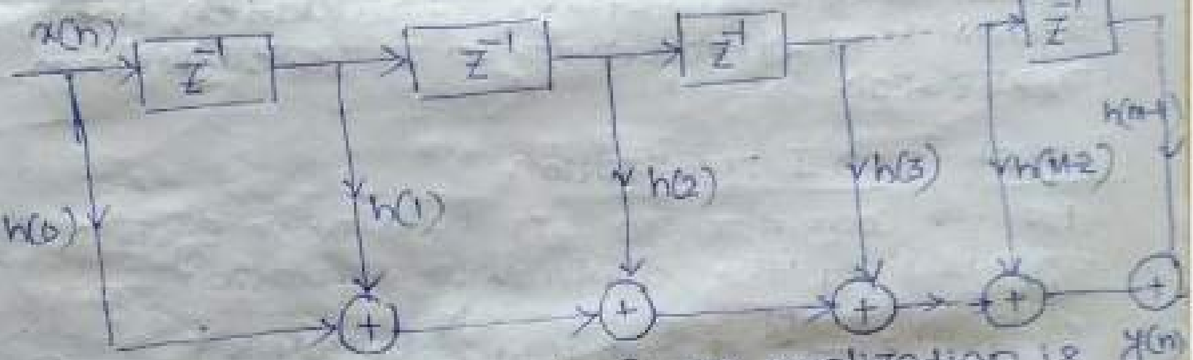
Direct-Form Structure

The convolution sum provides the system response as

$$\boxed{y(n) = \sum_{k=0}^{M-1} h(k)\,x(n-k)}$$

here $y(n)$ and $x(n)$ are the output and input sequences respectively. This equation provides the input-output relation for the FIR system in time-domain.

$$y(n) = h(0)x(n) + h(1)x(n-1) + h(2)x(n-2) + \dots h(M-2)x(n-(M-2)) + h(M-1)x(n-(M-1))$$



Consequently the direct-form realization is often called a transversal or tapped-delay-line filter.

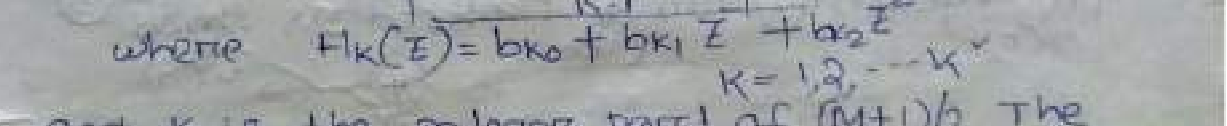
Cascade-Form Structure

The cascade realization follows naturally from the system function. It is a simple matter to factor $H(Z)$ into second order FIR systems so that

$$\boxed{H(Z) = \prod_{k=1}^{K} H_k(Z)}$$

where $H_k(Z) = b_{k0} + b_{k1}Z^{-1} + b_{k2}Z^{-2}$, $k = 1, 2, \dots K$

and K is the integer part of $(M+1)/2$. The filter parameter b_0 may be equally distributed along the K filter sections, such as $b_0 = b_{10}b_{20}\dots b_{K0}$ or it may be assigned to a single filter section.



Direct-form realization

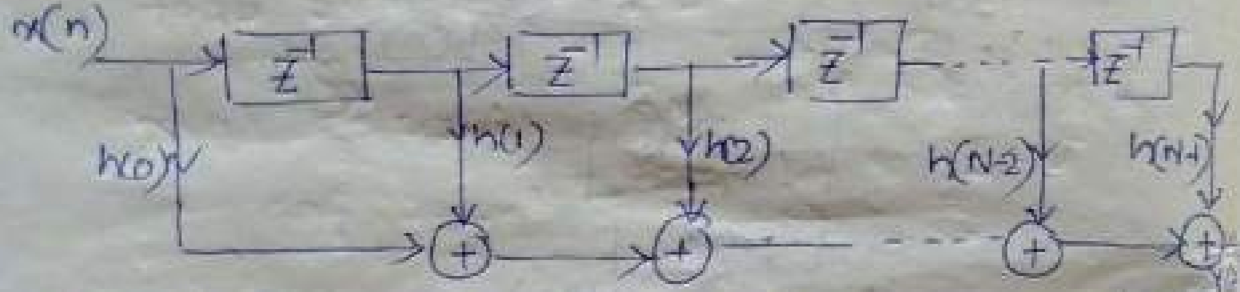
The system function of an FIR filter can be written as

$$H(Z) = \sum_{n=0}^{N-1} h(n) Z^{-n}$$

$$= h(0) + h(1) Z^{-1} + h(2) Z^{-2} + \cdots + h(N-1) Z^{-(N-1)}$$

$$Y(Z) = h(0)X(Z) + h(1) Z^{-1} X(Z) + h(2) Z^{-2} X(Z) + \cdots + h(N-1) Z^{-(N-1)} X(Z)$$

$$y(n) = h(0)x(n) + h(1)x(n-1) + h(2)x(n-2) + \cdots + h(N-1)\, x(n-(N-1))$$



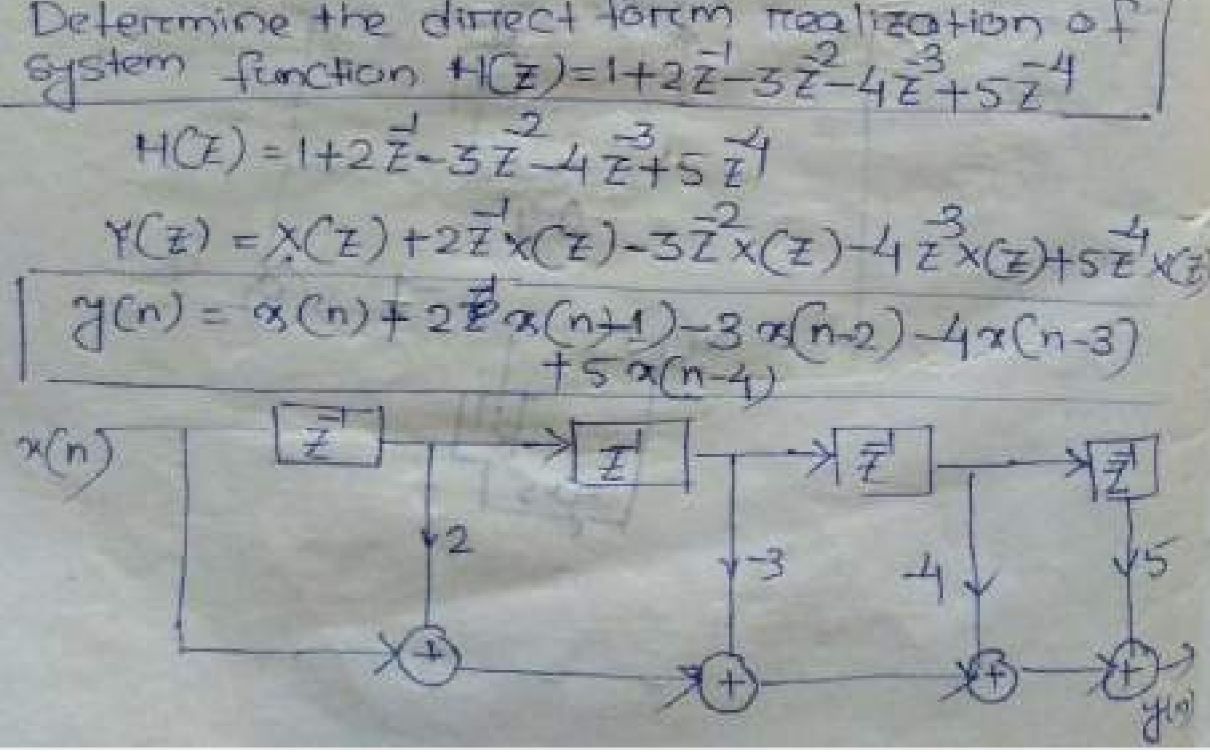
This structure is known as transverse structure or direct form realization. The transversal structure requires N multipliers, N-1 adders and N-1 delay elements.

Determine the direct form realization of system function $H(Z) = 1 + 2Z^{-1} - 3Z^{-2} - 4Z^{-3} + 5Z^{-4}$

$$H(Z) = 1 + 2Z^{-1} - 3Z^{-2} - 4Z^{-3} + 5Z^{-4}$$

$$Y(Z) = X(Z) + 2Z^{-1}X(Z) - 3Z^{-2}X(Z) - 4Z^{-3}X(Z) + 5Z^{-4}X(Z)$$

$$y(n) = x(n) + 2x(n-1) - 3x(n-2) - 4x(n-3) + 5x(n-4)$$



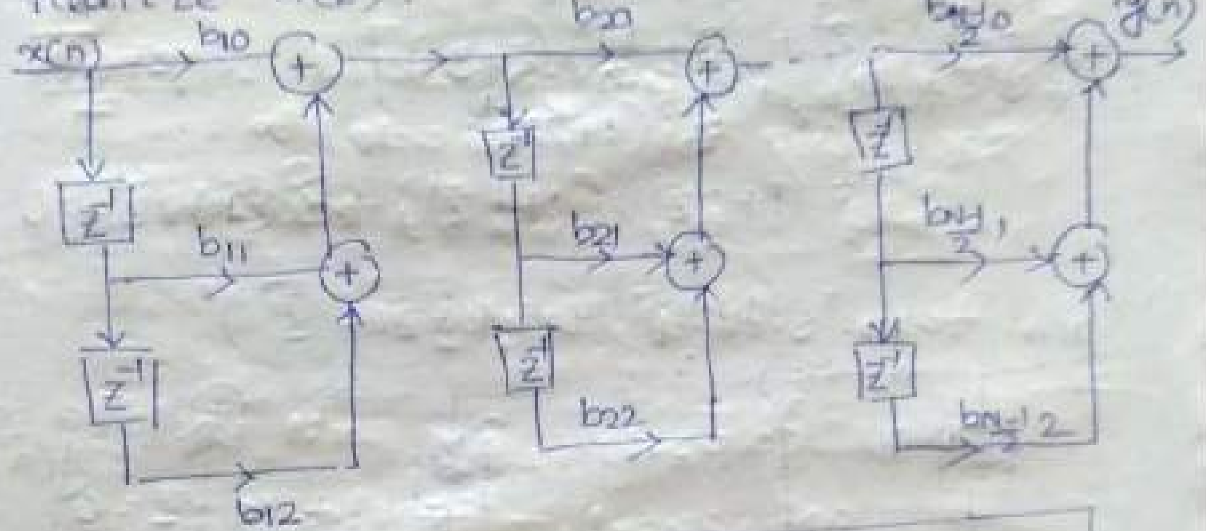
Cascade realization

$$H(Z) = \prod_{k=1}^{\frac{N-1}{2}} \left(b_{k0} + b_{k1} Z^{-1} + b_{k2} Z^{-2}\right)$$

$$= \left(b_{10} + b_{11} Z^{-1} + b_{12} Z^{-2}\right)\left(b_{20} + b_{21} Z^{-1} + b_{22} Z^{-2}\right) \cdots$$

$$\left(b_{\left(\frac{N-1}{2}\right)0} + b_{\left(\frac{N-1}{2}\right)1} Z^{-1} + b_{\left(\frac{N-1}{2}\right)2} Z^{-2}\right)$$

Each second order factored form of H(Z) is realized in direct form and is cascaded to realize H(Z).



Obtain the cascade realization of system function $H(Z) = (1 + 2Z^{-1} - Z^{-2})(1 + Z^{-1} - Z^{-2})$

$$H(Z) = H_1(Z) H_2(Z)$$

$$H_1(Z) = 1 + 2Z^{-1} - Z^{-2}$$

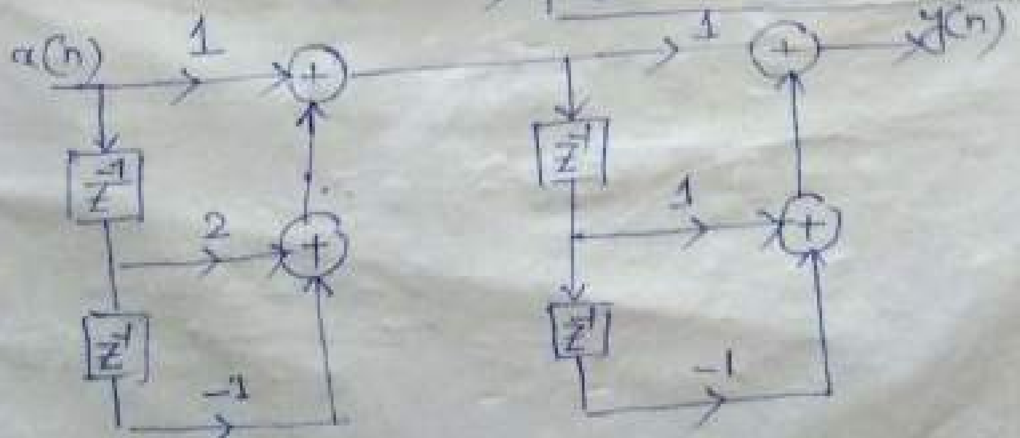
$$H_1(Z) = \frac{Y_1(Z)}{X_1(Z)} \Rightarrow Y_1(Z) = X_1(Z) + 2Z^{-1} X_1(Z) - Z^{-2} X_1(Z)$$

$$\boxed{y_1(n) = x_1(n) + 2x_1(n-1) - x_1(n-2)}$$

$$H_2(Z) = 1 + Z^{-1} - Z^{-2}$$

$$H_2(Z) = \frac{Y_2(Z)}{X_2(Z)} \Rightarrow Y_2(Z) = X_2(Z) + Z^{-1} X_2(Z) - Z^{-2} X_2(Z)$$

$$\Rightarrow \boxed{y_2(n) = x_2(n) + x_2(n-1) - x_2(n-2)}$$



Obtain the cascade realization of the system function $H(Z) = 1 + 5/2\, Z^{-1} + 2Z^{-2} + 2Z^{-3}$

6.105 $H(Z) = 1 + \frac{5}{2} Z^{-1} + 2Z^{-2} + 2Z^{-3}$

$= (Z+2)(Z+\frac{1}{2})$

$= (1+2Z^{-1})(1+\frac{1}{2}Z^{-1}) + 2Z^{-3} + Z^{-2}$

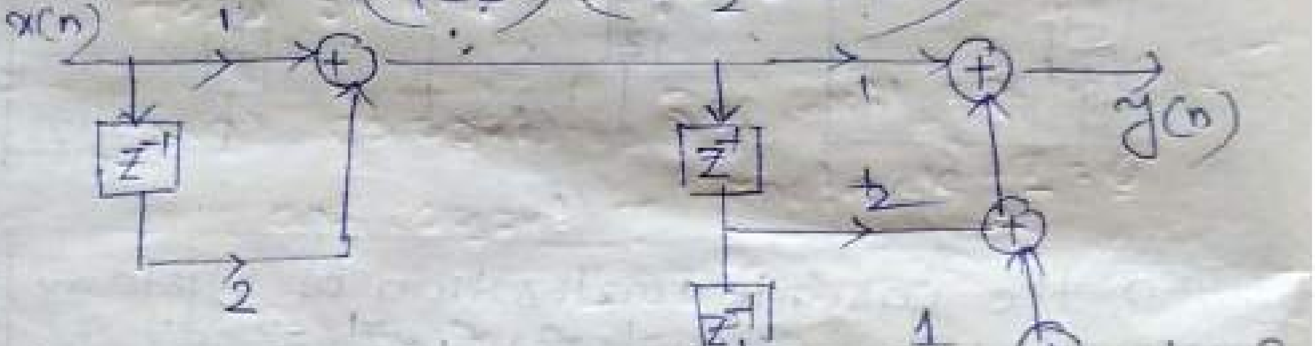
$Z^{-3}\left[Z^3 + \frac{5}{2}Z^2 + 2Z + 2\right]$ $\qquad Z = \frac{-2 \pm \sqrt{4 - 4\frac{5}{2}}}{2}$

$1 + \frac{1}{2}Z^{-1} + 2Z^{-1} + Z^{-2} + Z^{-2} + 2Z^{-3}$ $\quad \left[Z^3 + (Z+\frac{1}{2})(Z+3)\right]$ $\qquad \frac{-2 \pm \sqrt{-16}}{2}$

$Z^{-2}(1+2Z^{-1})$

$= (1+2Z^{-1})(1+\frac{1}{2}Z^{-1}) + Z^{-2}(1+2Z^{-1})$

$= (1+2Z^{-1})(1+\frac{1}{2}Z^{-1} + Z^{-2})$



$H_2(Z) = 1 + \frac{1}{2}Z^{-1} + Z^{-2}$

$H_1(Z) = 1 + 2Z^{-1}$

$Y_1(Z) = X_1(Z) + 2Z^{-1}X_1(Z)$

$\boxed{y_1(n) = x_1(n) + 2x_1(n-1)}$

$Y_2(Z) = X_2(Z) + \frac{1}{2}Z^{-1}X_2(Z) + Z^{-2}X_2(Z)$

$\boxed{y_2(n) = x_2(n) + \frac{1}{2}x_2(n-1) + x_2(n-2)}$

Structures of IIR system

There are several types of structures or realizations including direct-form structures, cascade-form structures, lattice structures and lattice-ladder structures. In addition, IIR system lend themselves to a parallel form realization.

Direct form I realization

Let us consider an LTI recursive system described by the difference equation

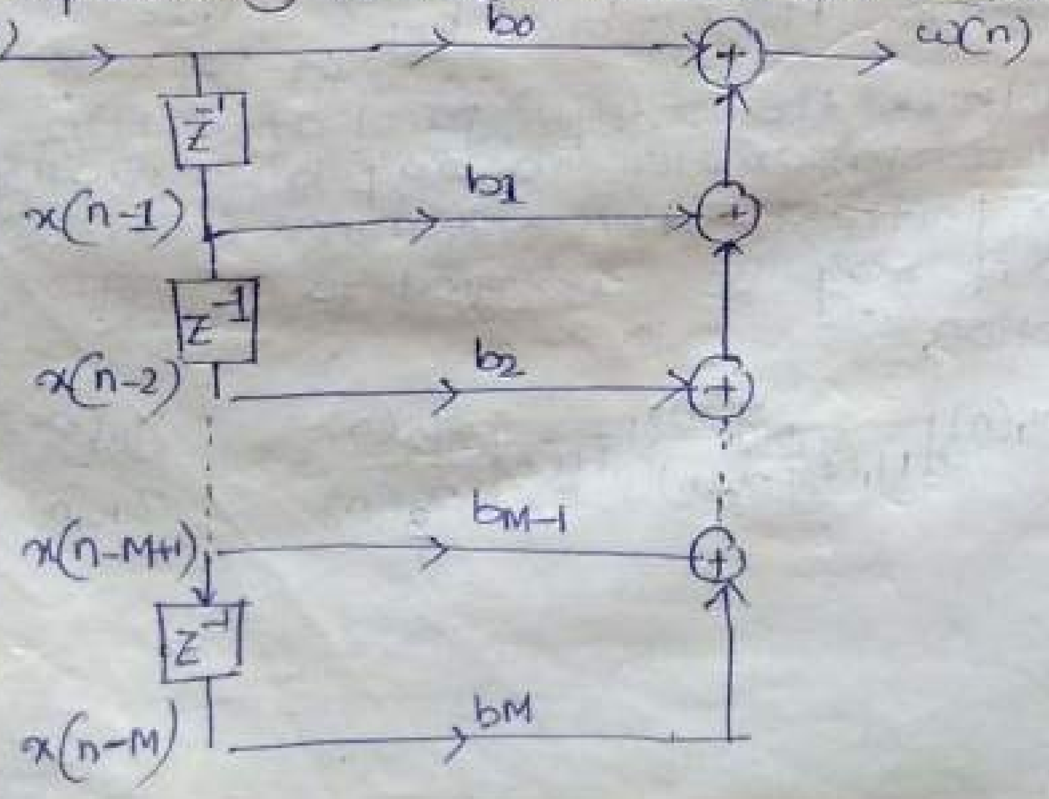
$$\boxed{y(n) = -\sum_{K=1}^{N} a_K y(n-K) + \sum_{K=0}^{M} b_K x(n-K)} \quad \text{--- (1)}$$

$$y(n) = -a_1 y(n-1) - a_2 y(n-2) - \cdots - a_{N-1} y(n-N+1)$$
$$- a_N y(n-N) + b_0 x(n) + b_1 x(n-1) + \cdots + b_M x(n-M)$$

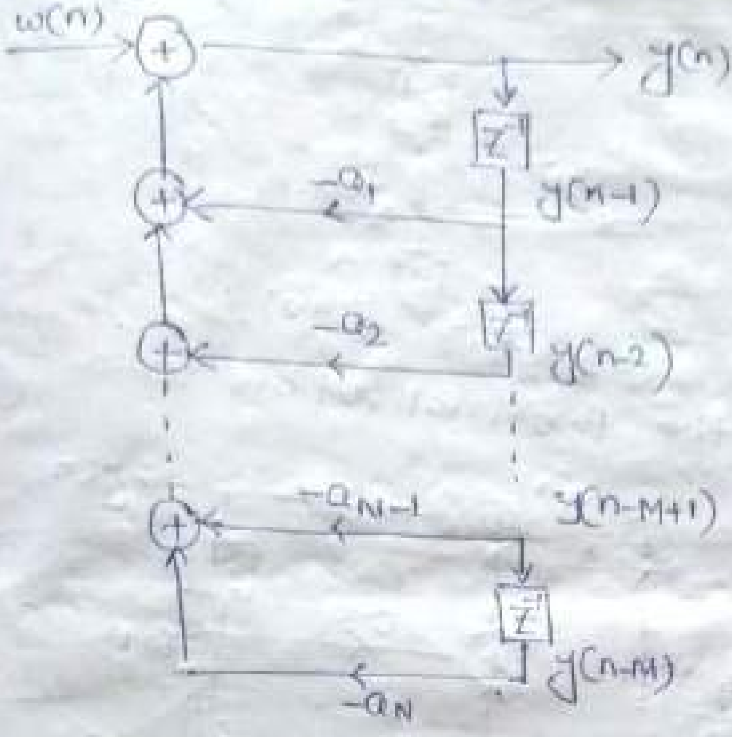
let $b_0 x(n) + b_1 x(n-1) + \cdots + b_M x(n-M) = \omega(n)$ --- (2)

then $y(n) = -a_1 y(n-1) - a_2 y(n-2) \cdots - a_N y(n-N) + \omega(n)$ --- (3)

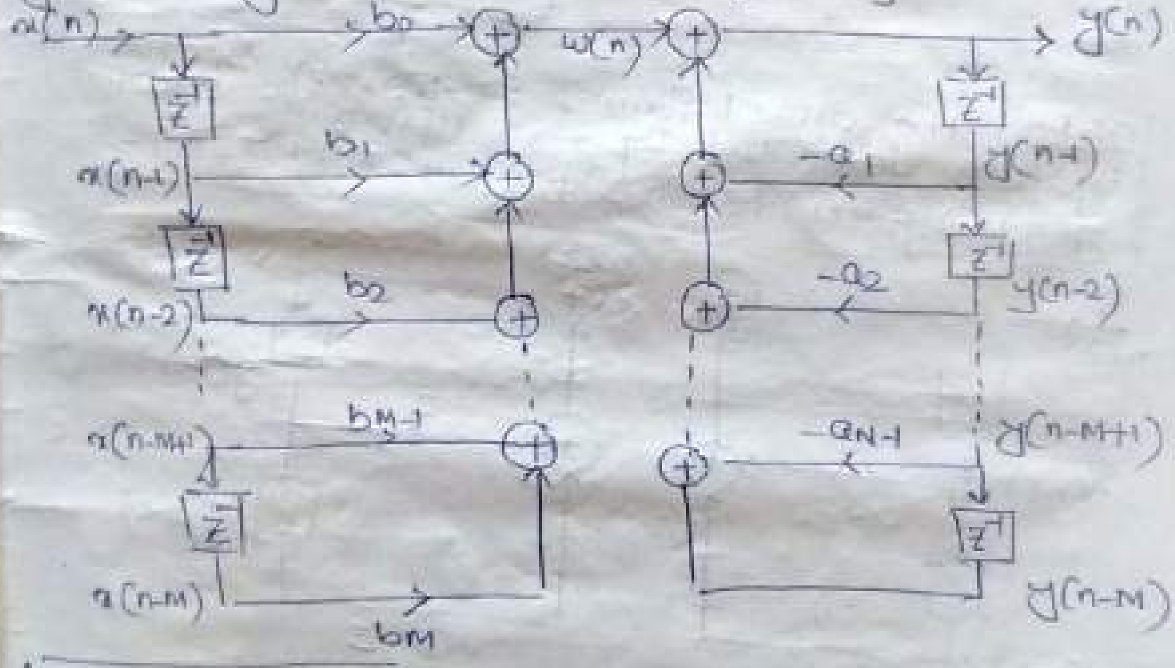
The equation (2) can be realized as shown



Equation (3) can be realized as shown



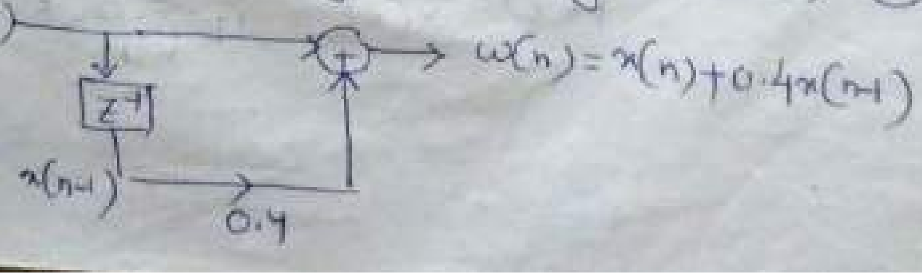
By combining the two structures we get

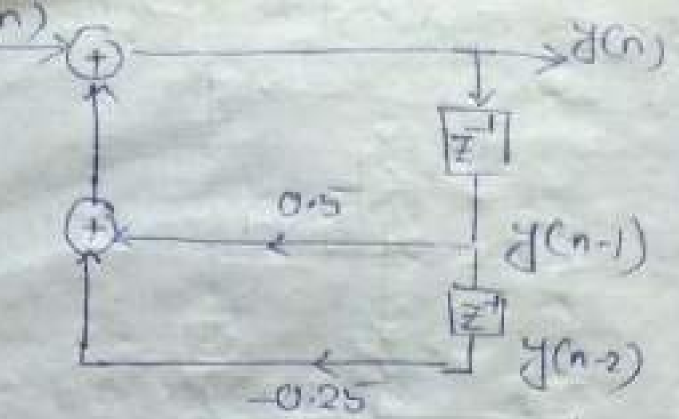


$\boxed{\text{Obtain the direct form-I realization for the system described by difference equation } y(n) = 0.5y(n-1) - 0.25y(n-2) + x(n) + 0.4x(n-1)}$

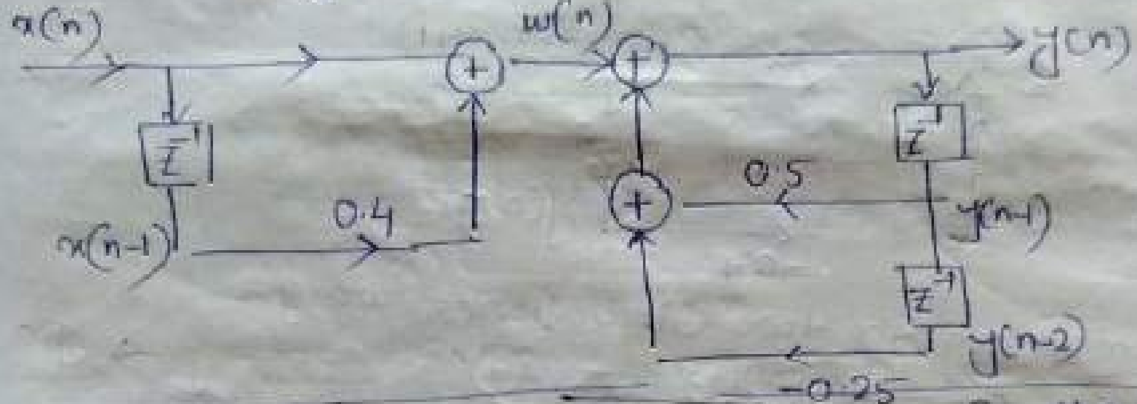
Let $x(n) + 0.4x(n-1) = w(n)$ —— (1)

then $y(n) = 0.5y(n-1) - 0.25y(n-2) + w(n)$ —— (2)

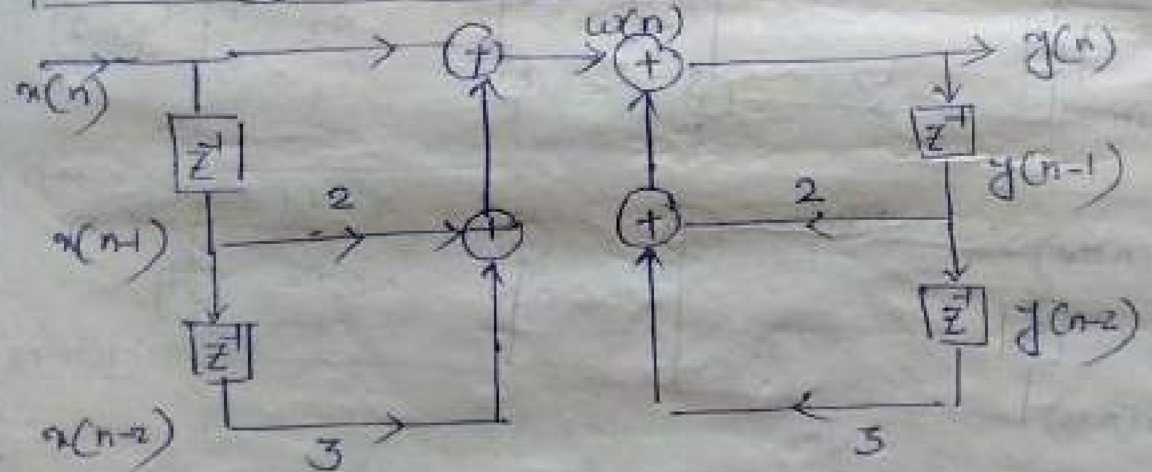




combining the two realizations



Obtain the direct form-I realization for the systems described by the difference equation
$y(n) = 2y(n-1) + 3y(n-2) + x(n) + 2x(n-1) + 3x(n-2)$



Direct-II realization

consider the difference equation of the form $y(n) = -\sum_{k=1}^{N} a_k y(n-k) + \sum_{k=0}^{M} b_k x(n-k)$

the system function

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{M} b_k z^{-k}}{1 + \sum_{k=1}^{N} a_k z^{-k}}$$

let $\frac{Y(Z)}{X(Z)} = \frac{Y(Z)}{W(Z)} \cdot \frac{W(Z)}{X(Z)}$

where $\frac{W(Z)}{X(Z)} = \frac{1}{1+\sum_{K=1}^{N} a_K Z^{-K}}$

which gives as

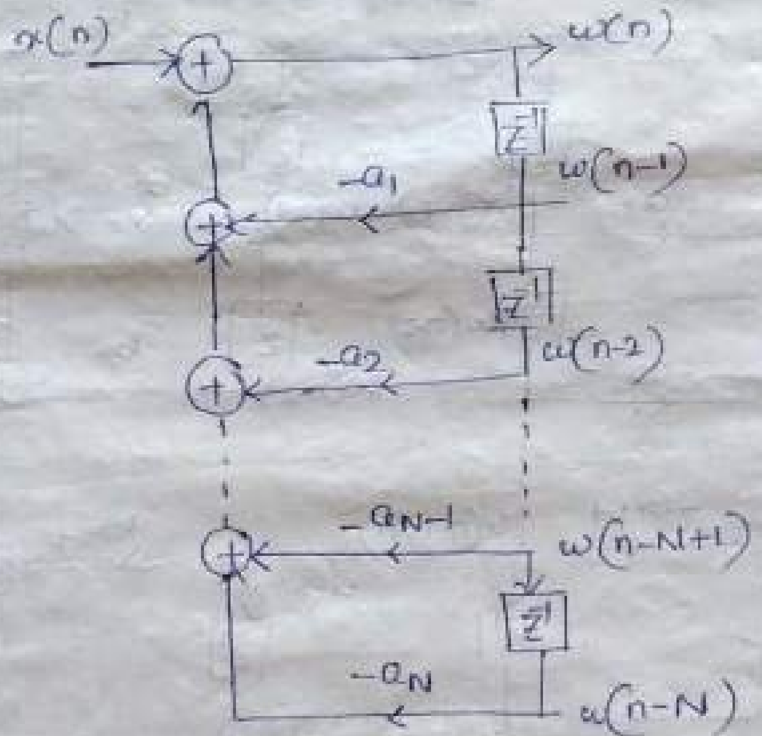
$$W(Z) = \frac{X(Z)}{1+a_1 Z^{-1} + a_2 Z^{-2} + \cdots + a_N Z^{-N}}$$

$$W(Z) + a_1 Z^{-1} W(Z) + a_2 Z^{-2} W(Z) + \cdots a_N Z^{-N} W(Z) - X(Z) = 0$$

$$W(Z) = X(Z) - a_1 Z^{-1} W(Z) - a_2 Z^{-2} W(Z) - \cdots - a_N Z^{-N} W(Z)$$

in difference equation form:

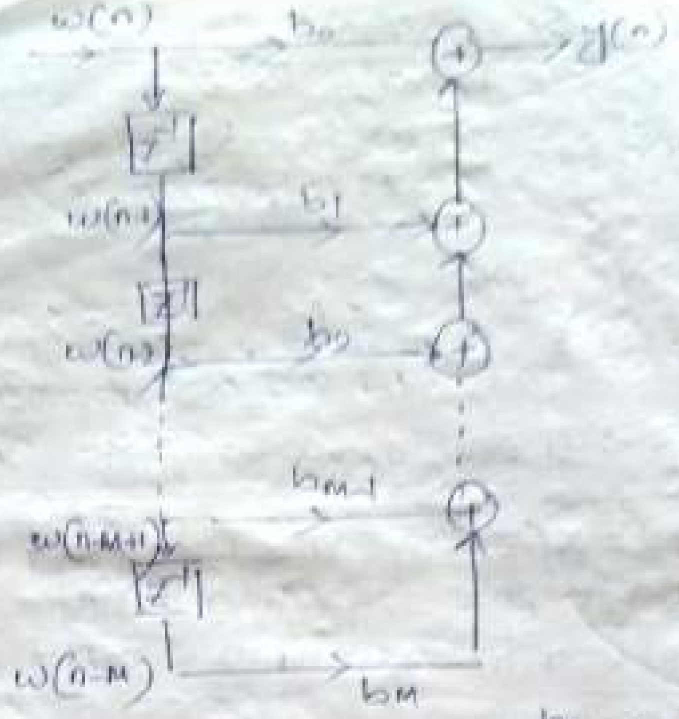
$$\boxed{w(n) = x(n) - a_1 w(n-1) - a_2 w(n-2) - \cdots - a_N w(n-N)}$$

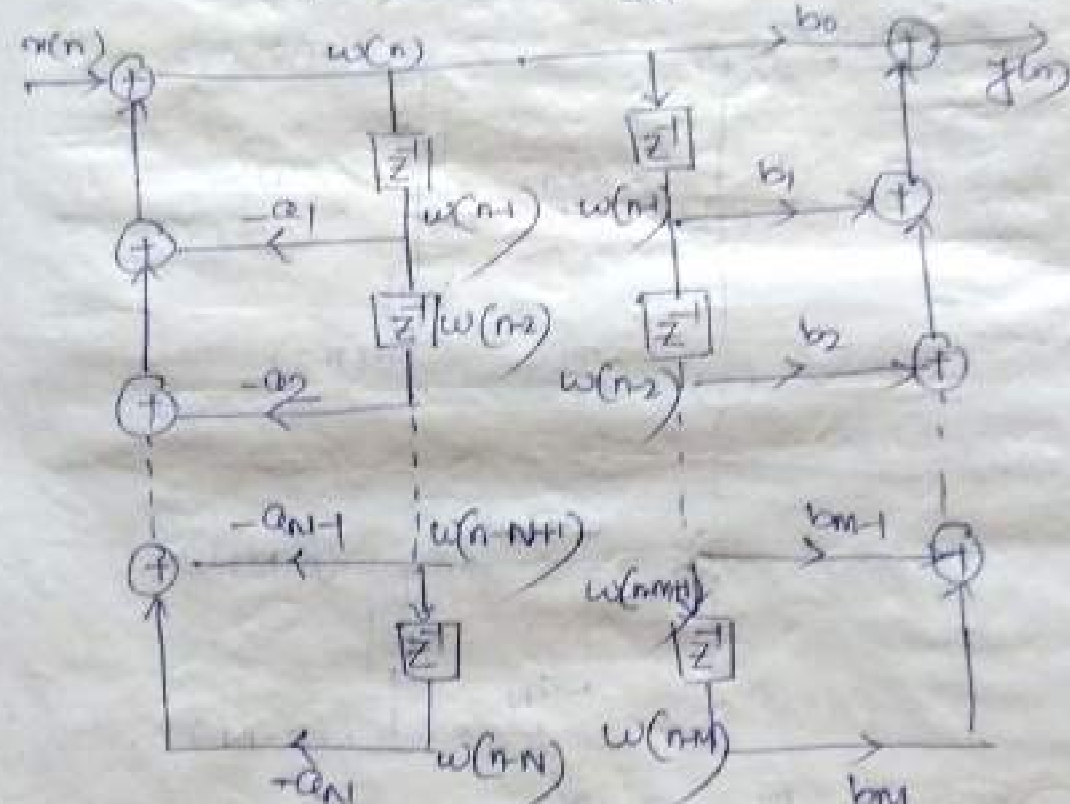


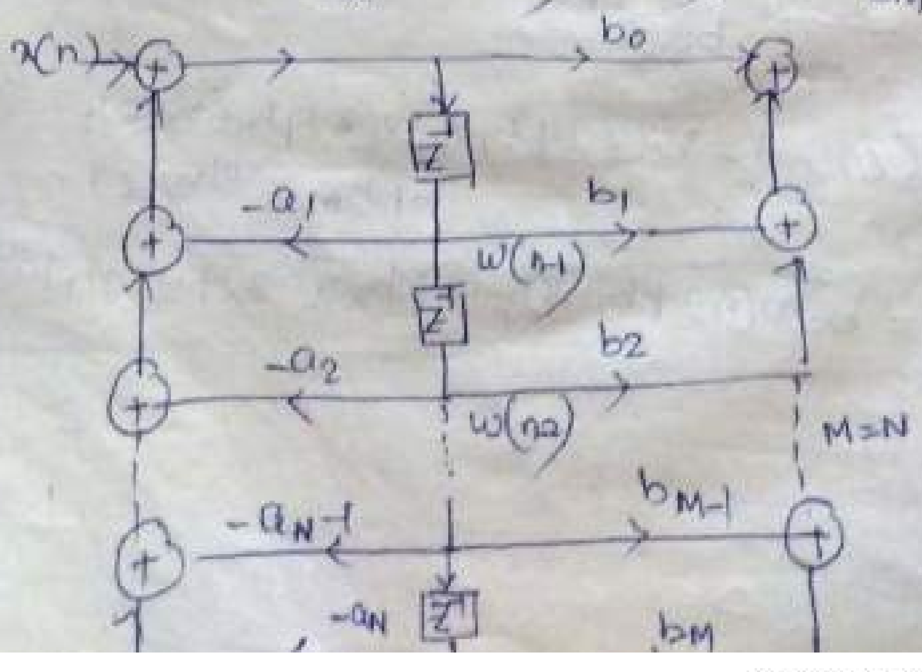
$$\frac{Y(Z)}{W(Z)} = \sum_{K=0}^{M} b_K Z^{-K}$$

$$Y(Z) = b_0 W(Z) + b_1 Z^{-1} W(Z) + b_2 Z^{-2} W(Z) + \cdots + b_M Z^{-M} W(Z)$$

$$\boxed{y(n) = b_0 w(n) + b_1 w(n-1) + b_2 w(n-2) + \cdots + b_M w(n-M)}$$


w(n)
b0
y(n)
z^-1
w(n-1)
b1
w(n-2)
b2
bM-1
w(n-M+1)
w(n-M)
bM


x(n)
w(n)
b0
y(n)
z^-1
-a1
w(n-1)
b1
w(n-2)
-a2
b2
-aN-1
w(n-N+1)
bM-1
w(n-M+1)
w(n-N)
-aN
w(n-M)
bM


x(n)
b0
z^-1
-a1
b1
w(n-1)
-a2
b2
w(n-2)
M=N
-aN-1
bM-1
-aN
bM

Determine the directform II realization for the following system $y(n) = -0.1y(n-1) + 0.72y(n-2) + 0.7x(n) - 0.25x(n-2)$

$$y(n) = -0.1y(n-1) + 0.72y(n-2) + 0.7x(n) - 0.25x(n-2)$$

The system function is given by

$$\frac{Y(Z)}{X(Z)} = \frac{0.7 - 0.25Z^{-2}}{1+0.1Z^{-1} + 0.72Z^{-2}}$$

let $\frac{Y(Z)}{W(Z)} = 0.7 - 0.25Z^{-2}$

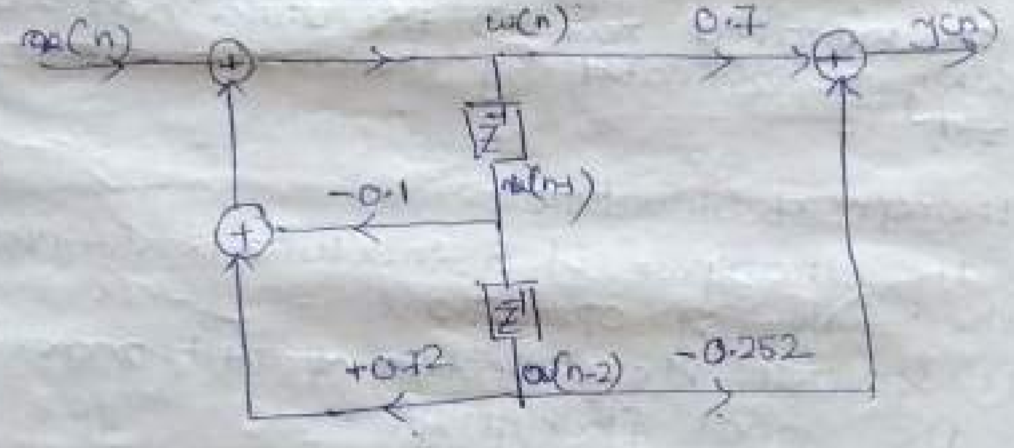
$$Y(Z) = 0.7W(Z) - 0.25Z^{-2}W(Z)$$

$$\boxed{y(n) = 0.7w(n) - 0.25w(n-2)}$$

$$\frac{W(Z)}{X(Z)} = \frac{1}{1+0.1Z^{-1} - 0.72Z^{-2}}$$

$$W(Z) = X(Z) - 0.1Z^{-1}W(Z) + 0.72Z^{-2}W(Z)$$

$$\boxed{w(n) = x(n) - 0.1w(n-1) + 0.72w(n-2)}$$

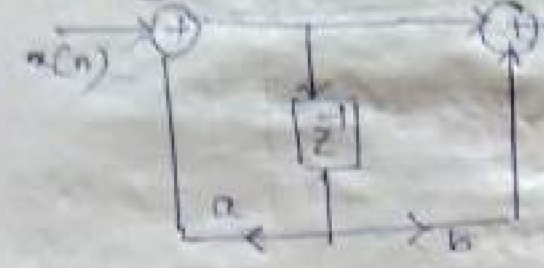


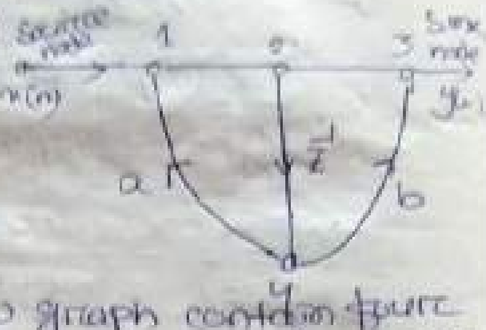
Signal flow Graph

A signal flowgraph is a graphical represen of relationship between the variables of a set of linear difference equation. The basic elements of signal flow graph are branches and nodes. A signal flow graph is basically a set of directed branches that connect at nodes. A node represents a system variable, which is equal to the sum of incoming signals from all branches connecting to the node. There are two types of nodes Source nodes are nodes that have no entering branches. Sink nodes are nodes that have only entering branch

A signal travels along a branch from one node to another node. The arrow head shows the direction of the branch and the branch gain is indicated next to the arrow head. The delay is indicated by the branch transmittance z^{-1}.

Let us consider a block diagram representation of a first order digital filter. The system block diagram can be converted to the signal flow graph shown.





We find that the signal flow graph contain four nodes out of which two (1&3) are summing nodes while the other two nodes represent the (2&4) branching points. Branch transmittance are indicated next to the arrowhead and delay is indicated by the branch transmittance z^{-1}.

Transposed Structure

The transpose of a structure is defined by the following operations.

(i) Reverse the direction of all branches in the signal flow graph.

(ii) Interchange the inputs and outputs

(iii) Change summing points to branching points

(iv) Change branching points to summing points

Determine the direct form II and transposed direct form II for the given system

$y(n) = \frac{1}{2} y(n-1) - \frac{1}{4} y(n-2) + x(n) + x(n-1)$

$$y(n) = \frac{1}{2} y(n-1) - \frac{1}{4} y(n-2) + x(n) + x(n-1)$$

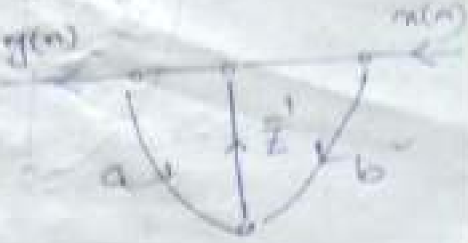
$$Y(z) = \frac{1}{2} z^{-1} Y(z) - \frac{1}{4} z^{-2} Y(z) + X(z) + z^{-1} X(z)$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1+z^{-1}}{1-0.5z^{-1}+0.25z^{-2}}$$

$$\frac{Y(z)}{W(z)} = 1+z^{-1}$$

$$Y(z) = W(z)+z^{-1}W(z)$$

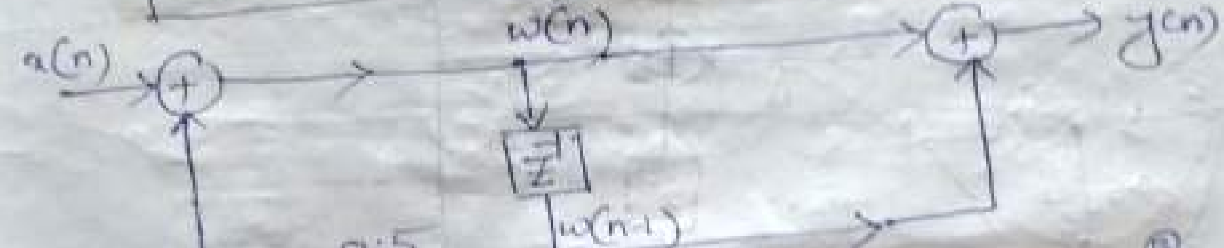
$$\boxed{y(n) = w(n)+w(n-1)}$$

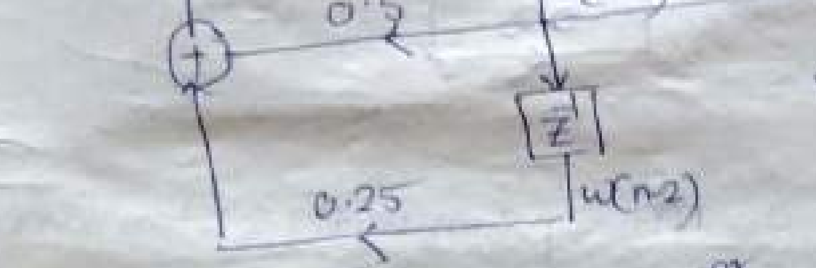


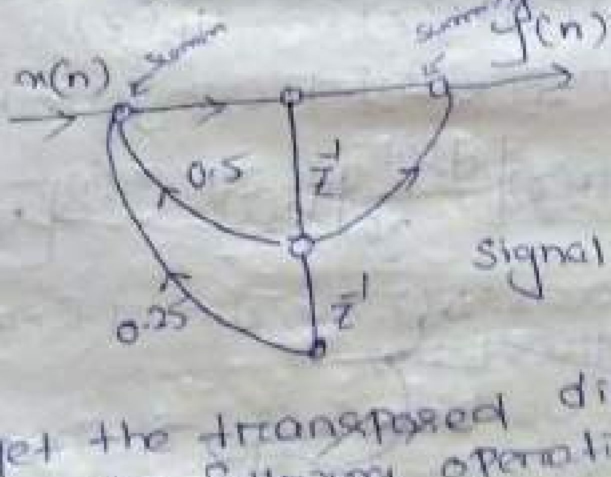
$$\frac{W(z)}{X(z)} = \frac{1}{1-0.5z^{-1}+0.25z^{-2}}$$

$$W(z) = X(z)+0.5z^{-1}W(z)+0.25z^{-2}W(z)$$

$$\boxed{w(n) = x(n)+0.5w(n-1)+0.25w(n-2)}$$



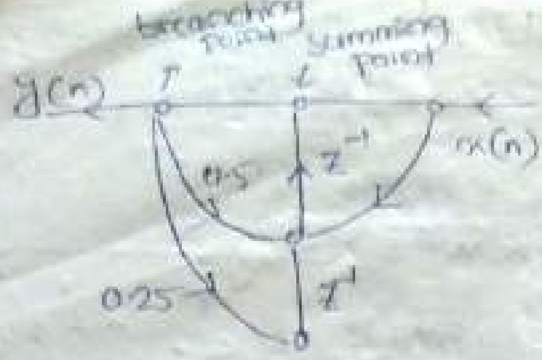


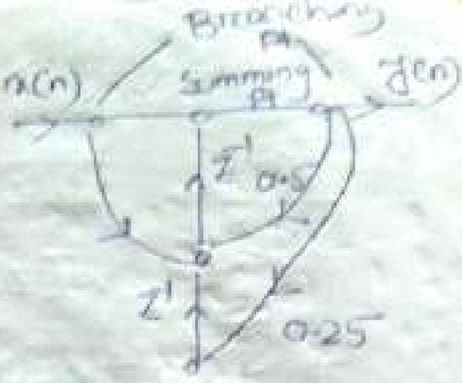


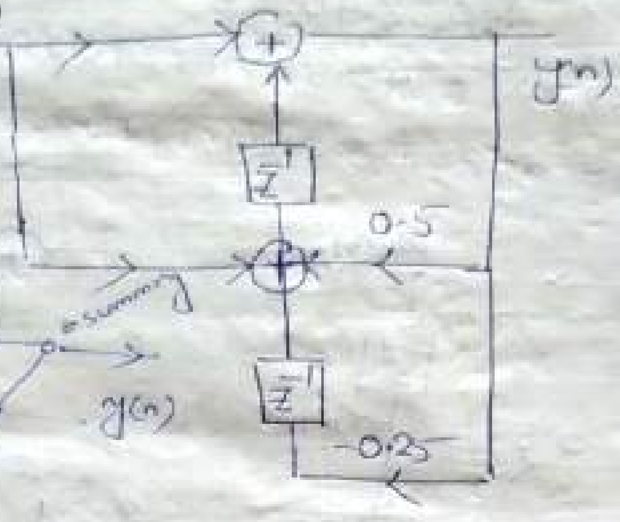
Signal flow graph

To get the transposed direct form II we do the following operations

1. change the direction of all branches
2. Interchange the input and output
3. change the summing point to branching point and vice versa.







Cascade form

Let us consider an IIR system with system func

$$H(Z) = H_1(Z)\, H_2(Z) \cdots H_k(Z)$$

This can be represented using block diagram

Now realize each $H_k(Z)$ in direct form II and cascade all structures.

For example let us take a system whose transfer function

$$H(Z) = \frac{(b_{k0} + b_{k1} Z^{-1} + b_{k2} Z^{-2})(b_{m0} + b_{m1} Z^{-1} + b_{m2} Z^{-2})}{(1 + a_{k1} Z^{-1} + a_{k2} Z^{-2})(1 + a_{m1} Z^{-1} + a_{m2} Z^{-2})}$$

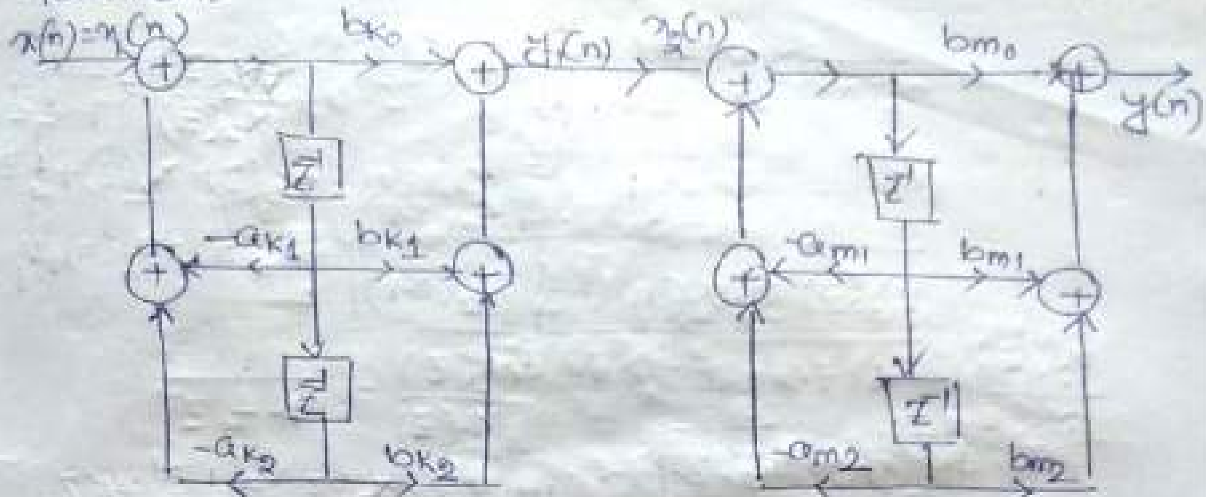
$$= H_1(Z)\, H_2(Z)$$

where

$$H(Z) = \frac{b_{k0} + b_{k1} Z^{-1} + b_{k2} Z^{-2}}{1 + a_{k1} Z^{-1} + a_{k2} Z^{-2}}$$

$$H_2(Z) = \frac{b_{m0} + b_{m1} Z^{-1} + b_{m2} Z^{-2}}{1 + a_{m1} Z^{-1} + a_{m2} Z^{-2}}$$

Realizing $H_1(Z)$ and $H_2(Z)$ in direct form II and cascading we obtain cascade form of the system function.



Realize the system with difference equation $y(n) = \frac{3}{4}y(n-1) - \frac{1}{8}y(n-2) + \frac{1}{3}x(n-1)$ in cascade form.

$$Y(Z) = \frac{3}{4}Z^{-1}Y(Z) - \frac{1}{8}Z^{-2}Y(Z) + \frac{1}{3}Z^{-1}X(Z)$$

$$H(Z) = \frac{Y(Z)}{X(Z)} = \frac{1+\frac{1}{3}Z^{-1}}{1-\frac{3}{4}Z^{-1}+\frac{1}{8}Z^{-2}} = H_1(Z)H_2(Z)$$

$$H(Z) = \frac{1+\frac{1}{3}Z^{-1}}{\left(1-\frac{1}{2}Z^{-1}\right)\left(1-\frac{1}{4}Z^{-1}\right)}$$

$$H_1(Z) = \frac{1+\frac{1}{3}Z^{-1}}{1-\frac{1}{2}Z^{-1}} \qquad H_2(Z) = \frac{1}{1-\frac{1}{4}Z^{-1}}$$

$H_1(Z)$ in direct form II

$$\frac{Y_1(Z)}{W_1(Z)} = 1+\frac{1}{3}Z^{-1}$$

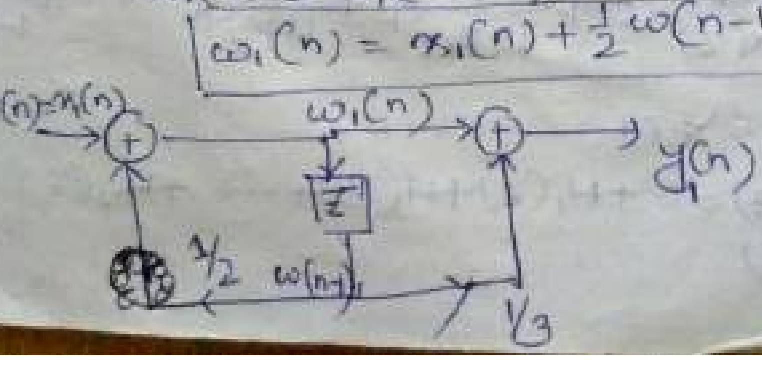
$$Y_1(Z) = W_1(Z) + W_1(Z)\frac{1}{3}Z^{-1}$$

$$\boxed{y_1(n) = \omega_1(n) + \frac{1}{3}\omega_1(n-1)}$$

$$\frac{W_1(Z)}{X_1(Z)} = \frac{1}{1-\frac{1}{2}Z^{-1}}$$

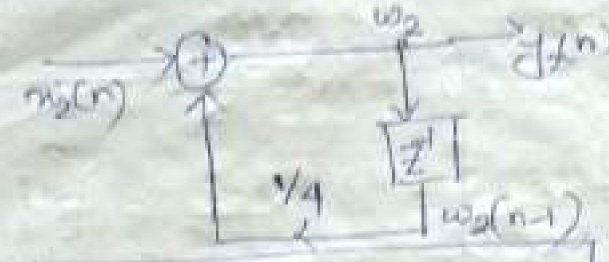
$$W_1(Z) = X_1(Z) + \frac{1}{2}Z^{-1}W(Z)$$

$$\boxed{\omega_1(n) = x_1(n) + \frac{1}{2}\omega(n-1)}$$



$H_2(Z)$ in direct form II

$$H_2(Z) = \frac{1}{1-\frac{1}{4}Z^{-1}}$$



$$\frac{W_2(Z)}{X_2(Z)} = \frac{1}{1-\frac{1}{4}Z^{-1}}$$

$$W_2(Z) = X_2(Z) + \frac{1}{4}Z^{-1}W_2$$

$$\boxed{w_2(n) = x_2(n) + \frac{1}{4}w_2(n-1)}$$

$$\boxed{w_2(n) = x_2(n) + \frac{1}{4}w_2(n-1)}$$

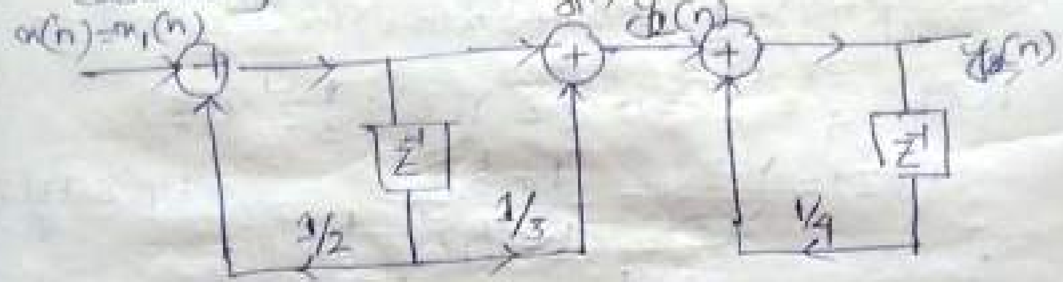
$$\frac{Y_2(Z)}{W_2(Z)} = 1$$

$$\boxed{y_2(n) = w_2(n)}$$

$$\frac{W_2(Z)}{X_2(Z)} = \frac{1}{1-\frac{1}{4}Z^{-1}}$$

$$W_2(Z) = X_2(Z) + \frac{1}{4}Z^{-1}W(Z)$$

cascading the realization of $H_1(Z)$ & $H_2(Z)$



Parallel form Structure

A parallel-form realization of an IIR system can be obtained by performing a partial fraction expansion of $H(Z)$

assume that $N \geq M$

Then by performing a partial-fraction expansion of $H(Z)$, we obtain the result

$$\boxed{H(Z) = C + \sum_{k=1}^{N} \frac{A_k}{1-P_k Z^{-1}}}$$

where $\{P_k\}$ are the poles, $\{A_k\}$ are the coefficients in the partial fraction expansion and constant C is defined as $C = b_N/a_N$

$$H(Z) = \frac{Y(Z)}{X(Z)} = C + \frac{A_1}{1-P_1 Z^{-1}} + \frac{A_2}{1-P_2 Z^{-1}} + \cdots + \frac{A_N}{1-P_N Z^{-1}}$$

$\cancel{H(Z) = C\,X(Z)}$

$$\frac{Y(Z)}{X(Z)} = C + H_1(Z) + H_2(Z) + \cdots + H_N(Z)$$

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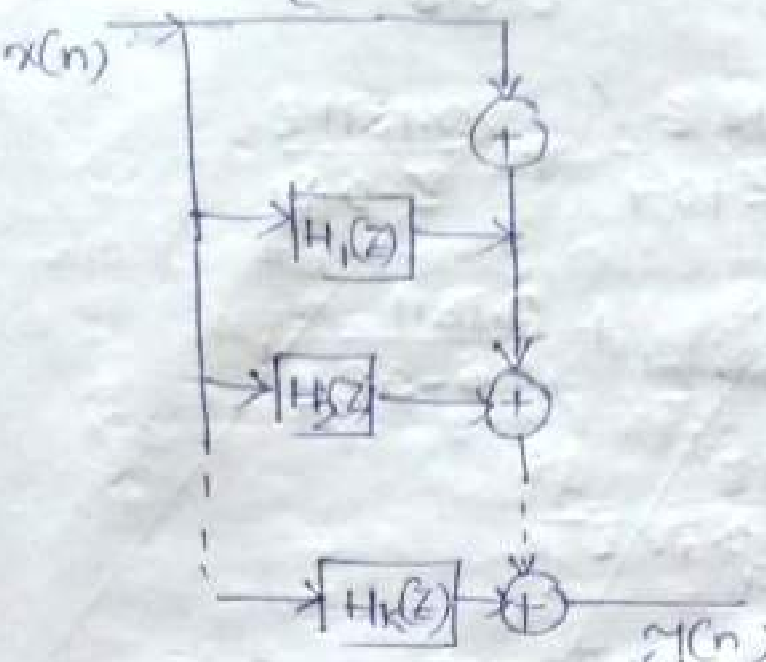
Now $Y(Z) = C\,X(Z) + H_1(Z)X(Z) + H_2(Z)X(Z) + \cdots + H_N(Z)X(Z)$



> Realize the system given by difference equation
> $y(n) = -0.1\,y(n-1) + 0.72\,y(n-2) + 0.7x(n) - 0.252x(n-2)$
> in parallel form.

$$y(n) = -0.1\,y(n-1) + 0.72\,y(n-2) + 0.7x(n) - 0.252x(n-2)$$

$$Y(Z) = -0.1\,Z^{-1}Y(Z) + 0.72\,Z^{-2}Y(Z) + 0.7X(Z) - 0.252\,Z^{-2}X(Z)$$

$$\boxed{H(Z) = \frac{Y(Z)}{X(Z)} = \frac{0.7 - 0.252Z^{-2}}{1 + 0.1Z^{-1} - 0.72Z^{-2}}}$$

$$= \frac{0.7Z^2 - 0.252Z}{Z^2 + 0.1Z - 0.72} = \frac{Z(0.7Z - 0.252)}{Z^2 + 0.1Z - 0.72}$$

$$= \frac{Z(0.7Z - 0.252)}{(Z+0.9)(Z-0.8)}$$