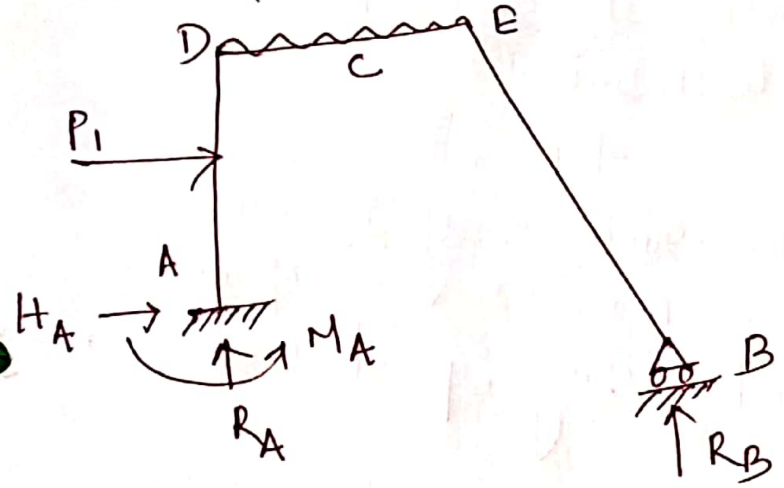
Structural Analysis   Dt: 20/4/2020
Civil Engg.

Module-2

Energy methods of Analysis :-

Application of minimum potential energy :-



Among all the geometrically compatible state of structure which satisfy deflection boundary condition and force equilibrium requirement will have final stable condition when its total potential energy is minimum.

→ Consider a portal frame as shown in figure.

→ If reactions at support A are H_A, R_A and M_A and reaction at support B is R_B (because 'A' is fixed support and B is Roller support).

Degree of indeterminacy $D_S = (4+1) - 3$
$= 1$

→ Assuming vertical reaction R_B as redundant (say R).

→ For any value of R, the reactions H_A, R_A and M_A can be calculated by the condition of equilibrium.

But true value of redundant will be when for which the total potential energy is minimum

If 'U' is total strain energy stored in frame. The total strain energy will be minimum when

$$\frac{\partial U}{\partial R} = 0 \quad (\because \text{compatibility condition})$$

It is an application of castigliano's theorem and based on principal of least work.

Algorithm for Analysis :- Steps.

(i) find degree of static indeterminacy D_S and identify redundants i.e unknown reactions or member forces.

(ii) find other reaction by using equation of equilibrium in terms of redundants.

(iii) Consider the redundants as variable and use compatibility of minimum potential energy.

→ If only one reaction is redundant (say R)

$$\frac{\partial U}{\partial R} = 0.$$

→ If two reactions are redundants (say R and M)

$$\frac{\partial U}{\partial R} = 0 \quad \frac{\partial U}{\partial M} = 0$$

(iv) Solving above compatibility conditions find true value of redundants.

(v) using all reactions, find B.M and S.F in each member and draw corresponding BMD and SFD.

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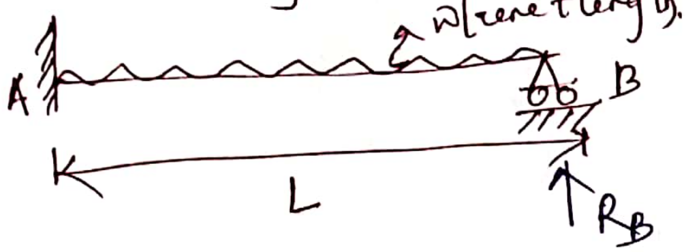
②

Q A propped cantilever of span 'L' carries a udL of w per unit length over the entire span. find the propped reaction at B. use priple of Minimum Potential energy. Consider effect of bending only.



Solution:- let R be the propped ~~cant~~ reaction at B.

The B.M at a section x from B is given by

$$\boxed{M_x = R_x - \frac{wx^2}{2}}$$

Total strain energy stored in beam

$$U = \int \frac{M_x^2\,dx}{2EI} = \int_0^L \frac{\left(R_x - \frac{wx^2}{2}\right)^2 dx}{2EI}$$

The true value of redundant will be when, the total strain energy stored in beam is minimum

$$\frac{\partial U}{\partial R} = 0.$$

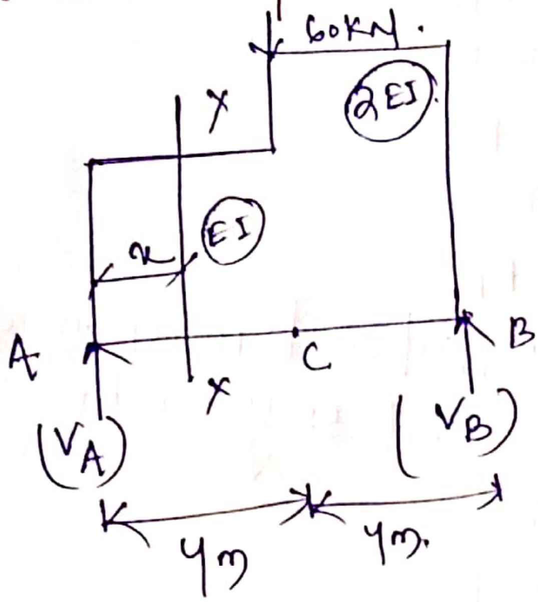
$$\Rightarrow \frac{1}{2EI}\int_0^L \left(R_x - \frac{wx^3}{2}\right) x\,dx = 0$$

$$\Rightarrow \int_0^L R_x^2\,dx - \int_0^L \frac{wx^4}{8}\Bigg]_0^L = 0$$

$$\Rightarrow \frac{RL^3}{3} - \frac{WL^4}{8} = 0 \quad \Rightarrow \boxed{R = \frac{3}{8}WL} \uparrow$$

* Strain energy method:-

using strain energy method determine the deflection under 60KN Load in the beam as shown in figure.



Solution

Σ upward forces = Σ downward forces

$\Rightarrow V_A + V_B = 60$ ——— (1)

$\Sigma M_A = 0.$

$\Rightarrow V_A \times 0 - V_B \times 8 + 60 \times 4 = 0$

$\Rightarrow \boxed{V_B = 30\,KN}$ $\quad V_A + \cancel{V_B}^{30} = 60$

$\boxed{V_A = 30\,KN}$

Consider a section x-x at a distance x from A.

Moment about section x-x

$M_x = 30 \times x = 30x$

Strain energy stored in a beam $W_1 = \int_0^L \frac{M^2}{2EI}\,dx$

(3)

$$= \int_0^4 \frac{(30x)^2}{2EI} dx + \int_0^4 \frac{(30x)^2 dx}{2(2EI)}$$

$$= \int_0^4 \frac{(30x)^2}{2EI}\left(1+\frac{1}{2}\right)$$

$$= \frac{3}{2} \times \frac{900}{2EI} \int_0^4 x^2 dx$$

$$= \frac{14400}{EI}$$

- y_c is the deflection at centre.

Workdone = Avg load × deflection

$$= \frac{0+60}{2} \times y_c$$

$$= 30 y_c.$$

$$\Rightarrow 30 y_c = \frac{14400}{EI}$$

- $\Rightarrow \boxed{y_c = \frac{480}{EI}}$

Comparison between force and displacement method

Force Method:- (FFC method)

In this method, the redundant forces are chosen as unknowns

→ It is called FFC (Force, flexibility and compatibility condition).

→ Force method

→ Flexibility Matrix used

→ Compatibility condition used.

→ In force method the unknowns are taken as internal member forces or reactions.

→ In force method compatibility conditions are used. So No. of compatibility conditions needed is equal to No. of redundant forces. is equal to static indeterminacy

→ This method is suitable when $\boxed{D_s < D_k}$

Some examples of force method are

→ Strain energy method

→ method of minimum potential energy

→ castigliano's theorem

→ unit load method.

→ 3 moment method.

→ virtual work method

→ column analogy method.

→ Flexibility matrix method.

→ strain energy method.

Displacement method (DSE)

→ It is otherwisely called as stiffness method or equilibrium method.

→ In force method forces are unknown where as in displacement method displacement are unknown.

→ To find unknown displacement joint equilibrium conditions are written.

→ Joint equilibrium condition needed = degree of Kinematic indeterminacy.

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Pg (4) Displacement method is suitable when ($D_K > D_S$)

Some examples of Displacement method are

→ moment distribution method

→ slope deflection method

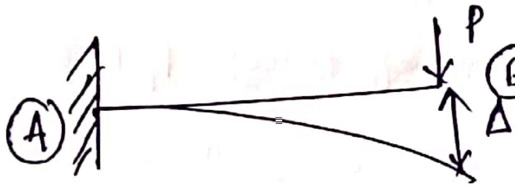
→ Kani's method

→ Stiffness matrix method.

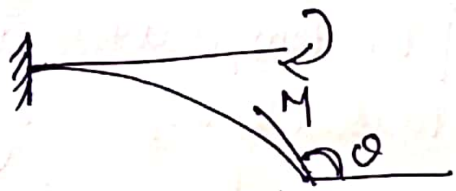
Castigliano's theorem :-

Castigliano's theorem (1st) :-

In any structure, the material of which is linear elastic temperature is constant. support are unyielding then the first partial derivative of total strain energy u with respect to any displacement component is equal to external force applied at that point in the direction of that displacement.



$$\boxed{\frac{\partial U}{\partial \Delta} = P}$$



$$\boxed{\frac{\partial U}{\partial \theta} = M}$$

Castigliano's 2nd theorem :-

In any structure, the material of which is linear elastic temperature is constant, supports are unyielding then the first partial derivative of total strain energy u with respect to any force is equal to deflection at that point in the direction of that force.

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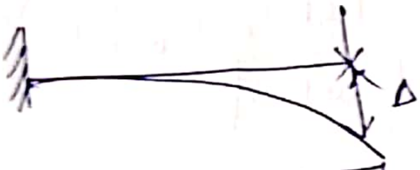
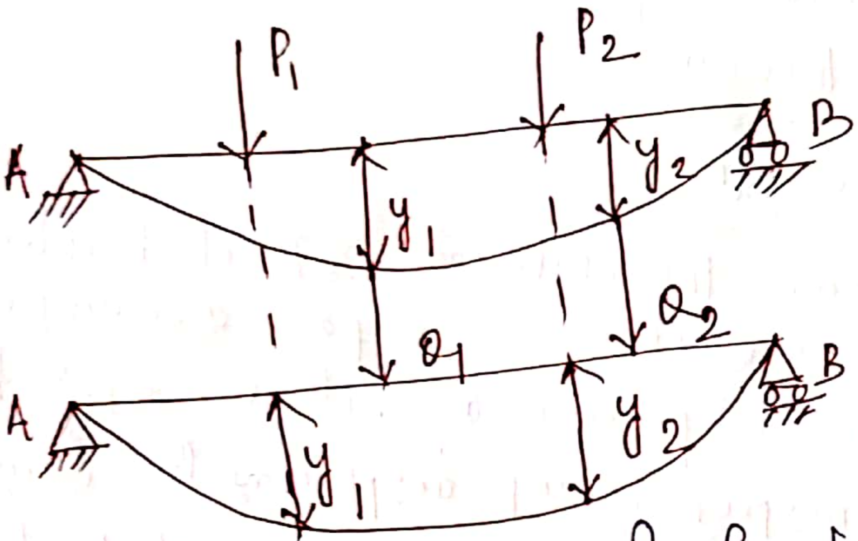


$$\boxed{\frac{\partial U}{\partial P} = \Delta}$$

$$\boxed{\frac{\partial U}{\partial M} = \theta}$$

Betti's Law



In any structure the material of which is linear elastic, and follow Hooke's Law, supports are unyielding and the temperature is constant, the external virtual workdone by P-system of forces P_1, P_2 and P_3 during the distortion caused by Q system of forces Q_1, Q_2 and Q_3 during the distortion caused by the P-system of forces P_1, P_2 & P_3 . . . .

Virtual workdone by P-system of forces

$$\boxed{(W_e)_P = P_1 y_1 + P_2 y_2}$$

and Virtual workdone by Q system of forces

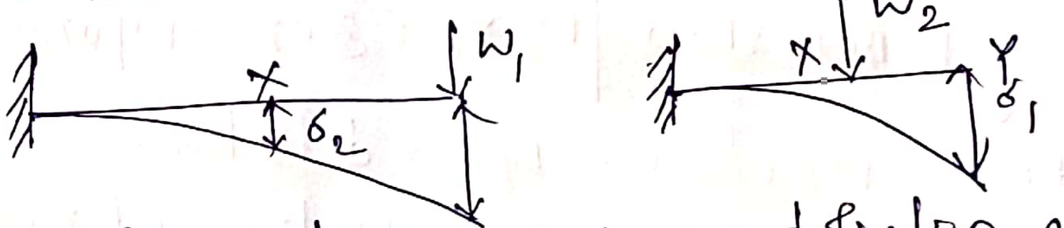
$(W_e)_Q = Q_1 y_1 + Q_2 y_2$ ⇒ Virtual work done by Q-system of forces

According to Betti's Law

$$\boxed{P_1 y_1 + P_2 y_2 = Q_1 y_1 + Q_2 y_2}$$

Example

Q:- In the cantilever beam shown in given figure, δ_2 is the deflection under X due to load W_1 and Y and δ_1 is the deflection under Y due to load W_2 at X. The ratio of δ_1/δ_2 is ?



Virtual work done by W_1 during deflection caused by W_2 is given by

$$\boxed{(W_e)_{W_1} = W_1 \times \delta_1}$$

Virtual work done by W_2 during deflection caused by W_1 is given by

$$\boxed{(W_e)_{W_2} = W_2 \times \delta_2}$$

According to Betti's Law

$$W_1 \times \delta_1 = W_2 \times \delta_2$$

$$\Rightarrow \boxed{\frac{\delta_1}{\delta_2} = \frac{W_2}{W_1}}$$

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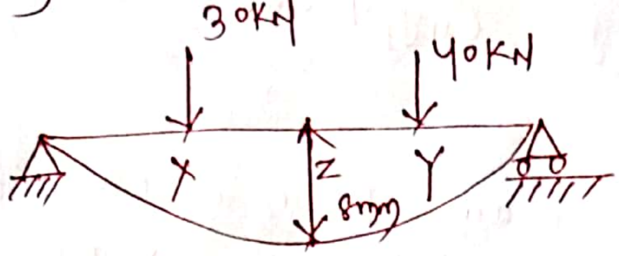
Q The beam shown in figure carries Load of 30KN and 40KN at points X and Y respectively and produces a deflection of 8mm at Z ?



To produce 8mm and 5mm at X & Y the Load required at Z ?

In question given that due to load of 30KN and 40KN at point X and Y the deflection at point Z is 8mm. but we have to calculate Load and Z point

According to Betti's Law $\boxed{W_Z \times 8 = 30 \times 8 + 40 \times 5}$

we know workdone = Load x deflection.

To produce 8mm and 5mm deflection at X and Y the Load required at Z is ?

$$\Rightarrow W_Z \times 8 = 30 \times 8 + 40 \times 5$$

$$\Rightarrow W_Z = \frac{(30 \times 8) + (40 \times 5)}{8}$$

$$\Rightarrow W_Z = 55 KN$$

$\Rightarrow$ Load required at Z is 55KN. & deflection 8mm

At X Load is 30KN and deflection 8mm

at Y Load is 40KN and deflection is 5mm.

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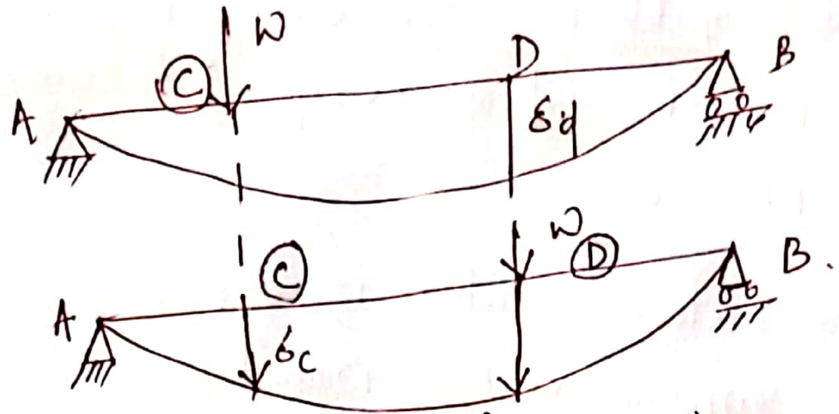
Maxwell's Reciprocal Theorem :-



- It is a special case of Betti's Law in which (P=Q) for example in a beam or structure, the deflection at any point D due to Load W at any other point C is the same as the deflection at C due to the same Load applied at D i.e

$\boxed{\delta_c = \Delta_d}$ or $\boxed{\delta_c = \delta_d}$

Virtual work done by W when applied at C

$= \boxed{W \times \delta_c}$

- Virtual work done by W when applied at D

$= \boxed{W \times \delta_d}$

As per Betti's Law $W \times \delta_c = W \times \delta_d$

$\Rightarrow \boxed{\delta_c = \delta_d}$.

Note:- This ~~law~~ method can be applied for both determinate and indeterminate beams

→ Its special utility is seen in cantileverbeams .

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Principle of virtual work:-

Case-1:- For deformable bodies or elastic bodies.

The total virtual work done is equal to zero.

$$\boxed{W_e + W_i = 0}$$

Where W_e = External virtual work done.

W_i = Internal virtual work done.

The internal virtual work done (W_i) by internal resistance (Such as stress) is negative.

Hence $|W_e| = |W_i|$.

Case-2:- For rigid bodies

In rigid bodies there is no internal deformation. Hence internal virtual work done is always zero. Therefore external virtual work done is also zero.

$$\boxed{W_e = 0}$$

Example:-

For the truss as shown in figure calculate the horizontal component of deflection at joint E due to following movement of supports.

(a) Support A moves horizontally by 0.0050 m from right to left and 0.0075 m vertically down.

(b) Support C moves vertically down by 0.0250 m

Assume there is no change in length of any member

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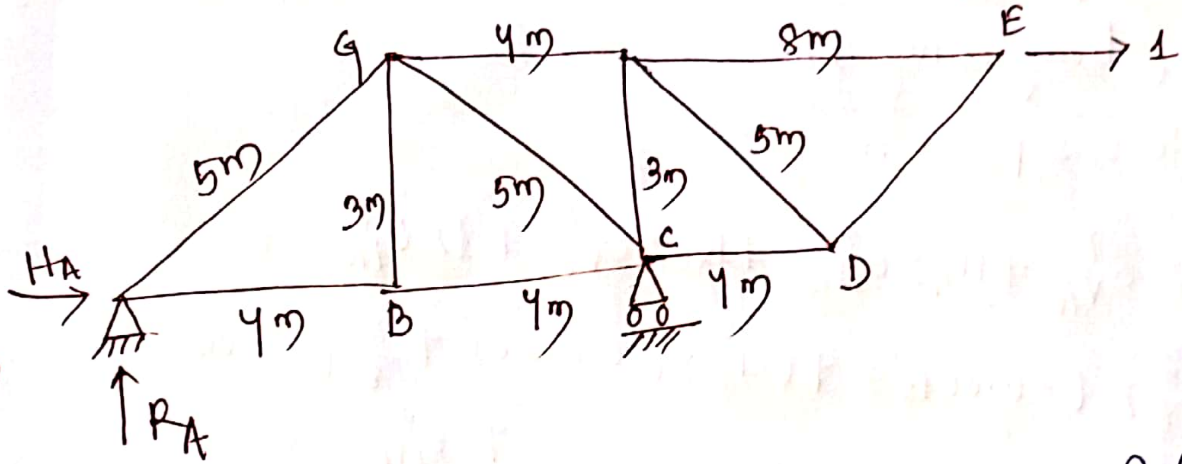
(7)

Assume there is no change in length of any member because all members are rigid and joint displacement is due to movement of support only.



Solution :- Let us apply unit Load in horizontal direction at E. Let R_A, H_A and R_C are reactions developed at A and C as shown in figure due to unit Load at E.

$\Sigma F_x = 0$

$\Rightarrow H_A + 1 = 0$

$\Rightarrow H_A = -1$

$\Sigma F_y = 0$

$\Rightarrow R_A + R_C = 0$ —— (1)

$\Sigma M_A = 0$

$\Rightarrow R_C \times 8 - 1 \times 3 = 0$

$\Rightarrow \boxed{R_C = 3/8 (\uparrow)}$

→ If we put value of R_C in equation (1) then we get $R_A + R_C = 0$

$\Rightarrow \boxed{R_A = -3/8 (\downarrow)}$

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since displacement of support A and C is due to some other agency.

→ Hence total external virtual work done should be zero (body is rigid)

$$\boxed{W_e = \Sigma P\Delta^* = 0}$$

$$\Rightarrow R_A \times \delta^*_{VA} + H_A \times \delta^*_{HA} + R_C \times \delta^*_{VC} + 1 \times \delta^*_{HE} = 0$$

$$\Rightarrow -\frac{3}{8} \times (-0.0075) + (-1) \times (0.0050) + \frac{3}{8}(-0.0025) + \delta^*_{HE} = 0$$

$$\boxed{\delta^*_{HE} = -0.006875\ m}$$

–ve sign means from right to left.

Principle of superposition :-

Assumptions

1) Material is isotropic, homogenous and linearly elastic in which Hooke's Law is valid.

2) Temperature is constant

3) Supports are unyielding.

on a beam, truss or frame which may be determinate or indeterminate.

→ The resultant value of any stress function due to multiple loading is equal to the sum of effects of individual loading. The stress function maybe SF, B.M, reaction slope, deflection, stress or strain.

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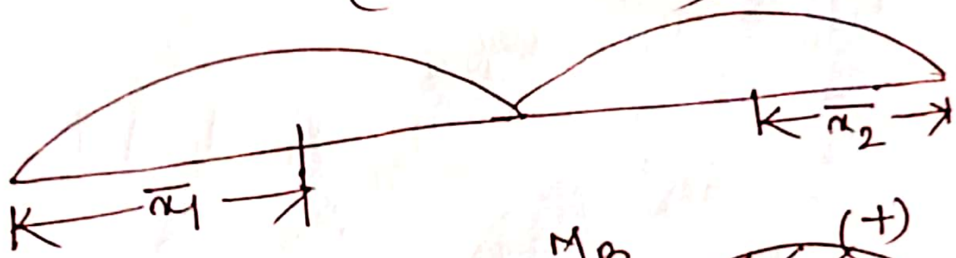
Three Moment equation :-

The three moment equation express the relationship between the moment at the 3 successive supports.
→ The support moments can be determined by the application of 3 moment equations.
This method is most suitable for the analysis of continuous beam.

(Continuous beam)



[Free BMD]

[Final BMD]

According to 3 moment equation, support moment M_A, M_B and M_C at the support A, B and C are given by the relation

$$M_A\left(\frac{L_1}{I_1}\right) + 2M_B\left(\frac{L_1}{I_1}+\frac{L_2}{I_2}\right) + M_C\left(\frac{L_2}{I_2}\right)$$

$$= \frac{6a_1\overline{x}_1}{I_1 L_1} + \frac{6a_2\overline{x}_2}{I_2 L_2}.$$

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where a_1 = area of free BMD for span AB

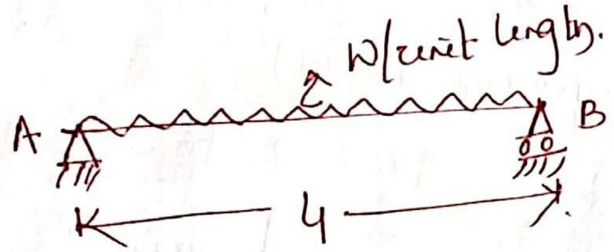
a_2 = area of free BMD for span BC

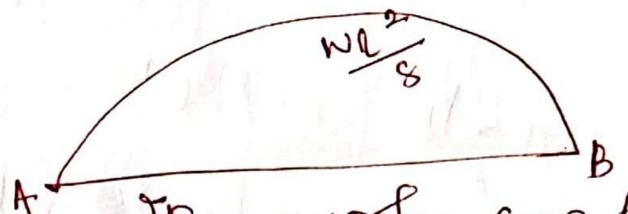
$\bar{x}_1$ = centroidal distance of free BMD on AB from A.

$\bar{x}_2$ = centroidal distance of free BMD on BC from C.

Special case-1

When span carries UDL over entire span





[Free BMD for span AB]

a_1 = Area of free BMD on AB

$$= \frac{2}{3} \times l_1 \times \frac{w_1 l_1^2}{8} = \frac{w_1 l_1^3}{12}$$

$\bar{x}_1$ = centroidal distance of free BMD on AB from A

$$= l_1/2.$$

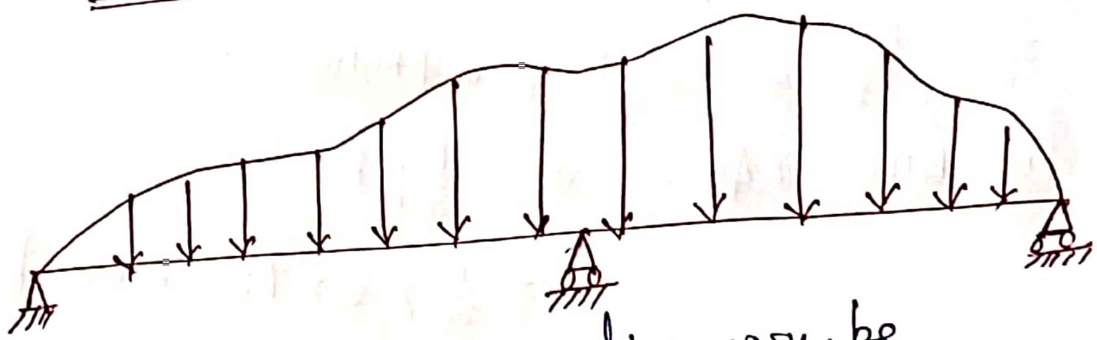
$$\frac{6 a_1 \bar{x}_1}{l_1} = \frac{6 \times \frac{w_1 l_1^3}{12} \times \frac{l_1}{2}}{l_1} = \frac{w_1 l_1^3}{4}.$$

(9)

Hence 3 moment equation can be written as when span AB and AC carries udL over entire span.

$$\boxed{M_A l_1 + 2M_B(l_1+l_2) + M_C l_2 = \frac{w_1 l_1^3}{4} + \frac{w_2 l_2^3}{4}}$$

Special case - 2 :- When EI not constant

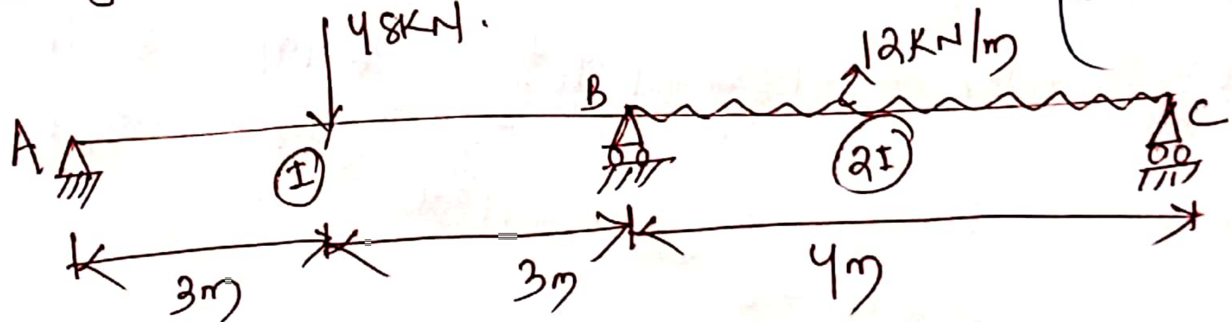


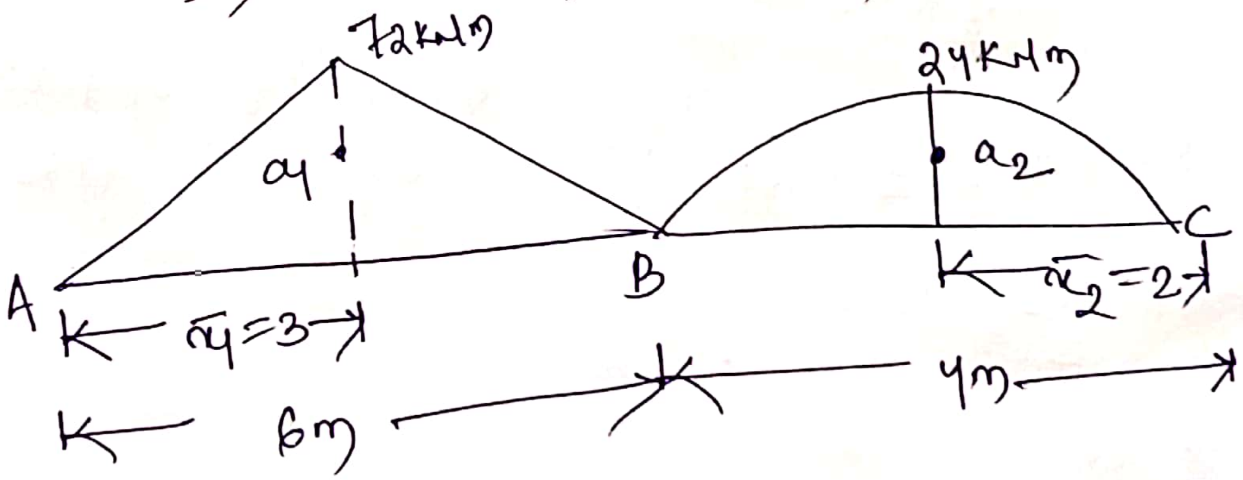
The three moment equation will be

$$M_A \frac{l_1}{I_1} + 2M_B\left(\frac{l_1}{I_1} + \frac{l_2}{I_2}\right) + M_C\left(\frac{l_2}{I_2}\right) = \frac{6a_1\bar{x}_1}{I_1 l_1} + \frac{6a_2\bar{x}_2}{I_2 l_2}$$

Example

Analyse the continuous beam as shown in figure.





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Maximum ordinate of free BMD on AB $= \frac{WL}{4}$

$W = 48KN$

$L = 6$

$(B.M)$ on AB $= \frac{WL}{4} = \frac{48 \times 6}{4} = 72 KNm.$

Maximum ordinate of free BMD on BC

$= \frac{WL^2}{8} = \frac{12 \times 4^2}{8} = 24 KNm.$

Area of free BMD on AB $= a_1 = \frac{1}{2} \times b \times h$

$= \frac{1}{2} \times 6 \times 72 = 216$ unit

Area of free BMD on BC $= a_2 = \frac{1}{2} \times 4 \times 24 = 48$ unit

centroidal distance of free BMD on AB $= \bar{x}_1 = 3m$

on BC $= \bar{x}_2 = 2m.$

Applying 3 moment equation for span AB and BC

$$M_A \frac{L_1}{I_1} + 2M_B \left(\frac{L_1}{I_1} + \frac{L_2}{I_2}\right) + M_C \frac{L_2}{I_2} = \frac{6a_1\bar{x}_1}{I_1 L_1} + \frac{6a_2\bar{x}_2}{I_2 L_2}$$

Since A and C are simple supports

$$\boxed{M_A = M_C = 0}$$

$$\Rightarrow 0 + 2M_B \left[\frac{6}{I} + \frac{4}{2I}\right] + 0 = \frac{6 \times 216 \times 3}{6 \times I} + \frac{6 \times 48 \times 2}{4 \times 2I}$$

$$\Rightarrow 16\frac{M_B}{I} = \frac{648}{I} + \frac{72}{I}$$

$$\Rightarrow 16M_B = 720$$

$$\Rightarrow M_B = 45 KNm.$$

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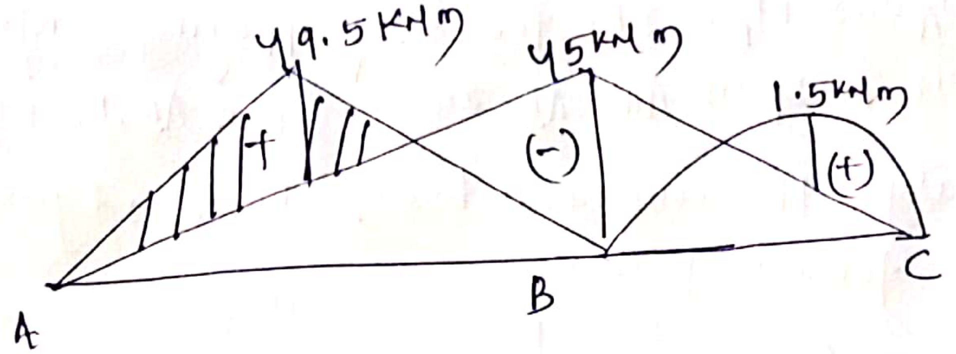


Note

In this case linear parabolic profile is used so Area of linear parabola $= \frac{1}{2} \times b \times h$ is taken.

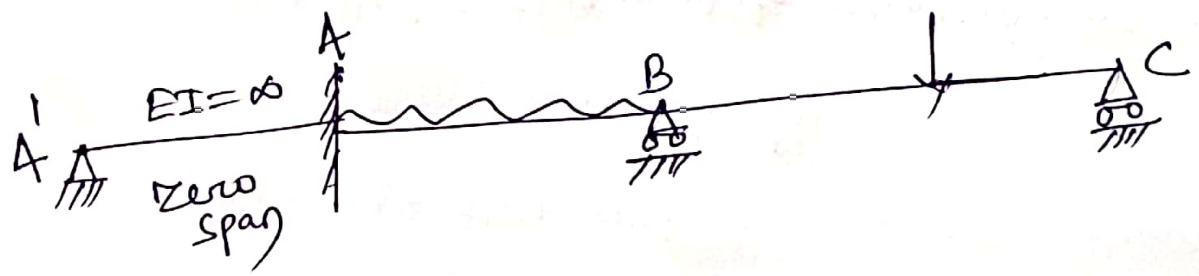
→ In case of cubic parabola Area of cubic parabola $= \frac{3}{4} bh$.

→ In parabola Area $= \frac{2}{3} \times b \times h$.

Application of 3 Moment equation to continuous beams with fixed ends :-

Above beam can be analyzed by 3 Moment equation If an imaginary zero span A'A of length 0 with flexural rigidity ($EI=\infty$) is added at fixed end A

Add a member AA' of zero span



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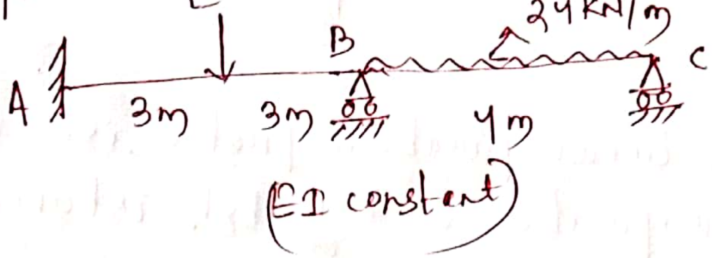
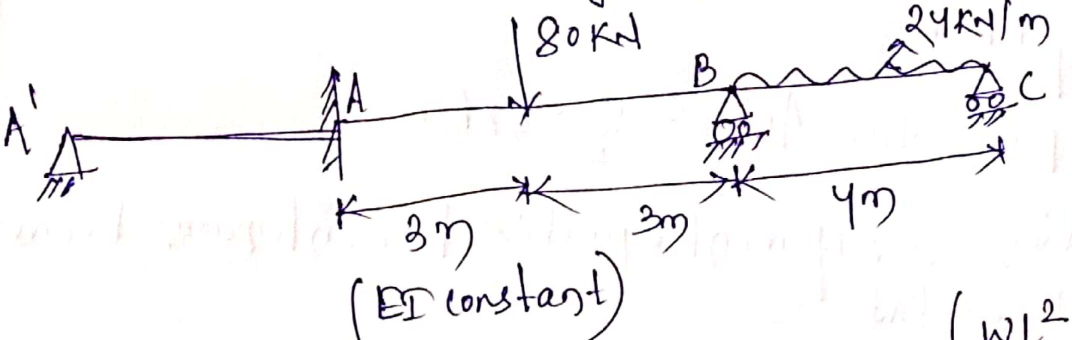
Now M_A and M_B can be found by applying 3 Moment equation for span AA', AB and Span AB, BC separately.

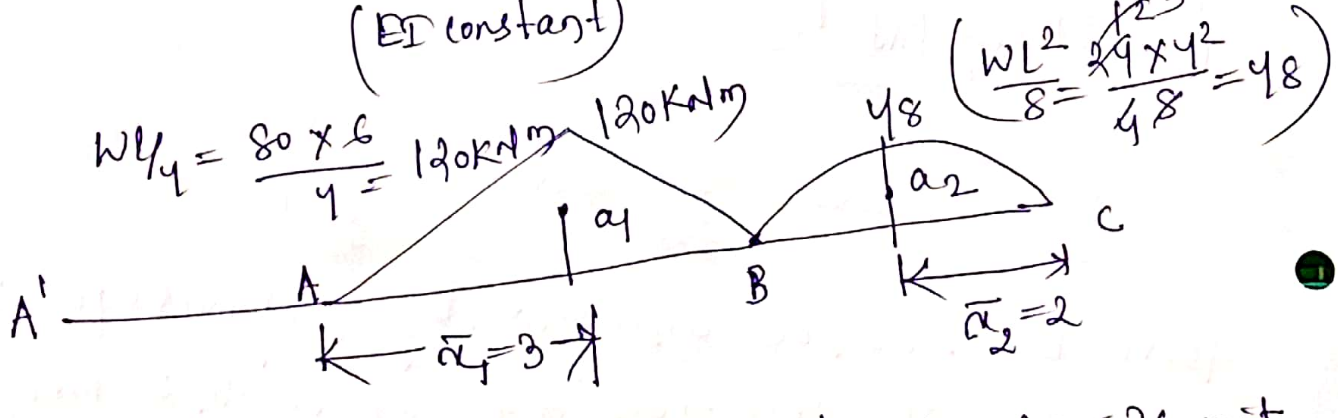
Q. Analyze the continuous beam shown in figure by using 3 Moment equation?



Solution

Let us introduce an imaginary zero span $A'A$.





Area of free BMD on $AB = (a_1) = \frac{1}{2} \times 6 \times 120 = 360$ unit

Area of free BM on $BC = (a_2) = \frac{2}{3} \times 4 \times 48 = 128$ unit

centroidal distance of free BMD on AB from A

$$\bar{x}_1 = 3m.$$

centroidal distance of free BMD on BC from B

$$\bar{x}_2 = 2m.$$

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(11)

Applying 3 moment equation for span AA' and AB

$$M_{A'} \times 0 + (2M_A(0+6)) + M_B \times 6 = \frac{6 a_1 \bar{x}_1}{l_1}$$

$$\Rightarrow 0 + 12M_A + 6M_B = \frac{6 \times 360 \times 3}{6}$$

$$\Rightarrow 12M_A + 6M_B = 1080 \quad \text{--- (1)}$$

Applying 3 moment equation for span AB and BC

$$\Rightarrow M_A \times 6 + 2M_B(6+4) + M_C \times 4 = \frac{6 a_1 \bar{x}_1}{l_1} + \frac{6 a_2 \bar{x}_2}{l_2}$$

$$\Rightarrow 6M_A + 20M_B + 4M_C = \frac{6 \times 360 \times 3}{6} + \frac{6 \times 128 \times 2}{4}$$

$$\Rightarrow 6M_A + 20M_B + 4M_C = 1464$$

Since support C is simply supported

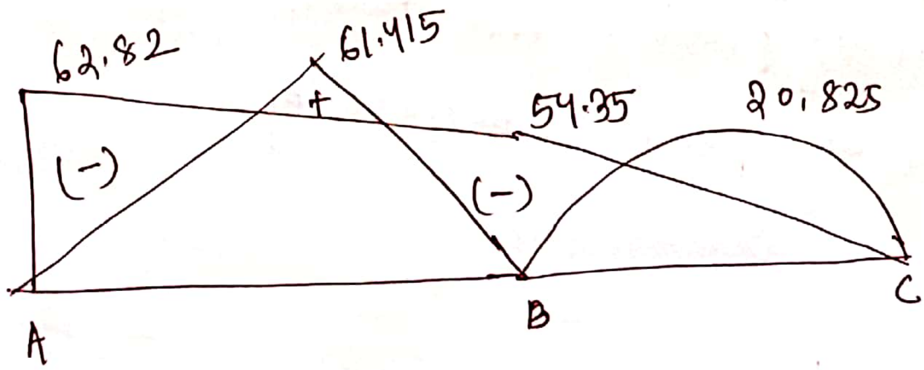
$$M_C = 0.$$

$$\Rightarrow 6M_A + 20M_B = 1464. \quad \text{--- (2)}$$

On solving equation (1) and (2) we get

$$M_A = 62.82 \text{ KNm}$$

$$M_B = 54.35 \text{ KNm}$$



Unit Load Method

Unit Load Method also referred to as method of virtual work was developed by John Bernoulli in 1717.
In this method a virtual unit load is applied at point of deflection (Δ) and in the direction of deflection.

By applying principal of work and energy
External virtual work = Internal virtual work

$$\Delta = \int_0^L \frac{Mm_1}{EI} dx$$

Where structure consist more than one members then

$$\boxed{\Delta = \sum \frac{Mm_1}{EI} dx}$$

M = Bending Moment at any section due to given loading.

m_1 = Bending Moment at any section is applied, when external loading is removed and a unit load is applied at a point where deflection is required.

→ The unit load method can also be used to find rotation at any point in a structure.

Rotation at any point can be found by

$$\boxed{\theta = \sum \frac{Mm_2}{EI} dx}$$

M = Bending Moment at any section due to given loading.

m_2 = Bending moment at any section, when external loading is removed and a unit concentrated moment is applied at a point where rotation is required.

Steps :-

(1) Find external support reactions

(2) find expression of M for all members due to given Loading.

(3) Remove external loading and apply unit load on unit Moment at that point where slope or deflection is required.

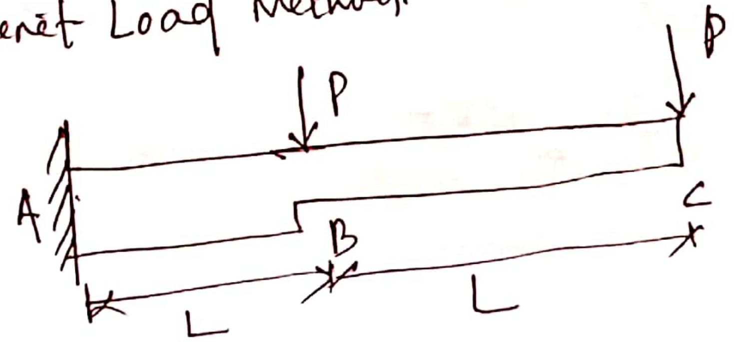
(4) Find expression of m for all members due to unit load.

(5) Apply formula for slope and deflection.

$$\Delta = \Sigma \frac{M m_1}{EI} dx$$

$$\theta = \Sigma \frac{M m_2}{EI} dx.$$

Q. Determine the deflection and rotation at the free end of the cantilever as shown in figure. use unit Load method.



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Solution:-

Deflection at free end is given by

$$\Delta = \sum \frac{M m_1}{EI} dx$$

$$\Rightarrow \int_0^L \frac{-Px(-x)}{EI} dx$$

$$+ \int_0^L \frac{[P(L+x)+Px][x+L]}{E(2I)} dx$$

Portion	CB	~~BA~~
Origin	C	B
Limit	0–L	0–L
M	$-Px$	$-P(L+x)+Px$
m_1	$-x$	$-(x+L)$
m_2	-1	-1
MOI	I	2I

$$\Rightarrow \int_0^L \frac{Px^2 dx}{EI} + \int_0^L \frac{P(x+L)^2 + Px(x+L)\,dx}{2EI}$$

$$\Rightarrow \int_0^L \frac{Px^2 dx}{EI} + \int_0^L \frac{P}{2EI}\left[x^2+L^2+2Lx+x^2+Lx\right]dx$$

$$\Rightarrow \int_0^L \frac{Px^2 dx}{EI} + \int_0^L \frac{P}{2EI}\left[2x^2+3Lx+L^2\right]dx$$

$$\Rightarrow \frac{P}{2EI}\left[\int_0^L 2x^2 dx + \int_0^L (2x^2+3Lx+L^2)dx\right]$$

$$\Rightarrow \frac{P}{2EI}\left[2\left[\frac{x^3}{3}\right]_0^L + \left[2\cdot\frac{x^3}{3} + 3\cdot\frac{Lx^2}{2} + L^2\cdot x\right]_0^L\right]$$

$$\Rightarrow \frac{P}{2EI}\left[2\cdot\frac{L^3}{3} + \frac{2}{3}L^3 + \frac{3}{2}L^3 + L^3\right]$$

$$\Rightarrow \boxed{\Delta = \frac{23PL^3}{12EI}}$$

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Rotation at free end is given by

$$\theta = \sum \frac{M m_2}{EI} dx$$

$$\Rightarrow \int_0^L \frac{(-Px)(-1)dx}{EI} + \int_0^L \frac{[-P(L+x)+Px](-1)dx}{2EI}$$

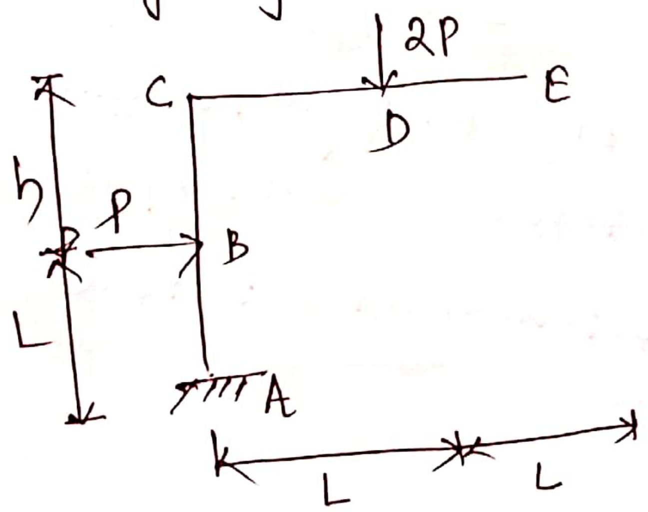
$$\Rightarrow \int_0^L \frac{Px\,dx}{EI} + \int_0^L \frac{[P(L+x)+Px]}{2EI}dx$$

$$\Rightarrow \frac{P}{2EI}\left[\int_0^L 2x\,dx + \int_0^L (2x+L)dx\right]$$

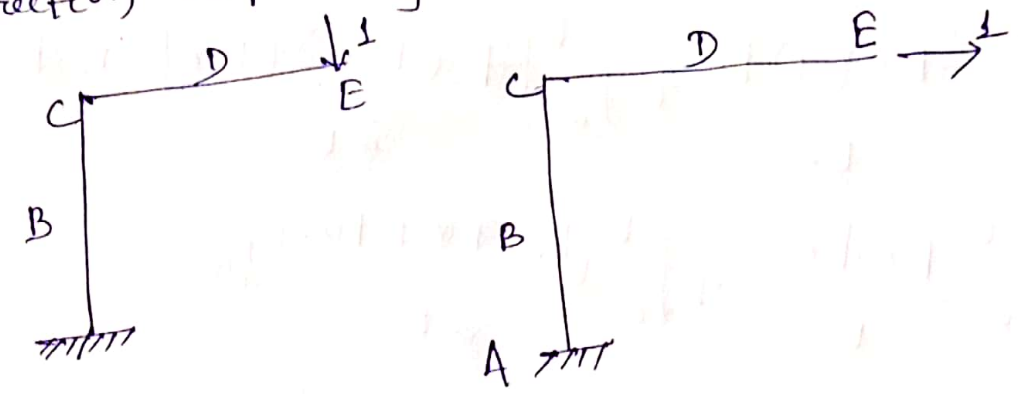
$$\Rightarrow \frac{P}{2EI}\left[2\left[\frac{x^2}{2}\right]_0^L + 2\left[\frac{x^2}{2}\right]_0^L + L\left[x\right]_0^L\right]$$

$$\theta = \frac{P}{2EI}\left[2\cdot\frac{L^2}{2} + 2\cdot\frac{L^2}{2} + L\cdot L\right] = \frac{3PL^2}{2EI}$$

Q Determine the vertical and horizontal deflection at the free end of the bent shown in figure. Assume the flexural rigidity EI throughout is bent is uniform.



Solution :- For vertical and horizontal deflection at E apply a unit load at E in vertical and horizontal direction respectively.



for given bent

Taking outer face as reference face.

Portion	ED	DC	CB	BA
origin	E	D	C	B
Limit	0–L	0–L	0–L	0–L
M	0	$-2Px$	$-2PL$	$-2PL-Px$
m_1	$-x$	$-(L+x)$	$-2L$	$-2L$
m_2	0	0	$-x$	$-(x+L)$
MOI	I	I	I	I.

M = Expression of BM due to given Loading.

m_1 = Moments due to vertical unit load at free end.

m_2 = Moments due to horizontal unit load at the free end.

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(14)

The vertical deflection of free end is given by

$$\Delta_E = \Sigma \frac{M m_1}{EI} dx$$

$$= 0 + \int_0^L \frac{(-2Px)(-L+x)dx}{EI} + \int_0^L \frac{(-2PL)(-2PL)dx}{EI}$$

$$+ \int_0^L \frac{[(-2PL)+Px]\cdot(-2L)}{EI} dx$$

$$= \frac{1}{EI}\left[\int_0^L [2Px - 2Px^2 + 4PL^2 + 4PL^2 + 2PLx] dx\right.$$

$$= \frac{1}{EI}\left[\int_0^L [-2Px^2 + 4PLx + 8PL^2] dx = \frac{2P}{EI}\int_0^L [-x^2 + 2Lx + 4L^2] dx\right.$$

$$= \frac{2P}{EI}\left\{\left[-\frac{x^3}{3}\right]_0^L + 2L\left[\frac{x^2}{2}\right]_0^L + 4L^2[x]_0^L\right\}$$

$$= \frac{2P}{EI}\left[-\frac{L^3}{3} + L^3 + 4L^3\right] = \frac{28PL^3}{3EI} (\downarrow)$$

The horizontal deflection of free end is given by

Δ_E in horizontal direction

$$= 0 + 0 + \int_0^L \frac{(-2PL)(-x)dx}{EI} + \int_0^L \frac{(2PL+Px).(-x+L)dx}{EI}$$

$$= \frac{2P}{EI}\int_0^L Lx\,dx + \int_0^L \frac{(2PLx + Px^2 + 2PL^2 + PLx)}{EI} dx$$

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$$= \frac{2P}{EI}\int_0^L Lx\,dx + \frac{P}{EI}\int_0^L \left(x^2 + 3Lx + 2L^2\right)dx$$

$$= \frac{P}{EI}\left[\int_0^L \left(5Lx + x^2 + 2L^2\right)dx\right]$$

$$= \frac{P}{EI} \times \left[\frac{5L^3}{2} + \frac{L^3}{3} + 2L^3\right] = \frac{29PL^3}{6EI} (\longrightarrow)$$

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