Fourier cosine and sine Transforms

We know If $f(x)$ is an even function then the fourier integral is fourier cosine Integral.

$$f(x) = \int_0^\infty A(w)\cos wx \, dw \quad \text{where } A(w) = \frac{2}{\pi}\int_0^\infty f(v)\cos w(v)\,dv$$

Fourier cosine ~~series~~ Transforms

To find out the fourier cosine Transform we have to use fourier cosine Integral.

where we have to choose

$$A(w) = \sqrt{\frac{2}{\pi}}\,\hat{f}_c(w)$$

where $\hat{f}_c(w) \rightarrow$ fourier cosine Transforms

$c \rightarrow$ assume for cosine

$$A(w) = \sqrt{\frac{2}{\pi}}\,\hat{f}_c(w)$$

$$\Rightarrow \frac{2}{\pi}\int_0^\infty f(v)\cos wv\,dv = \sqrt{\frac{2}{\pi}}\,\hat{f}_c(w)$$

for writing $v = x$ we get

$$\sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos wx\,dx = \hat{f}_c(w)$$ which is

fourier cosine Transform

Also we know fourier integral

$$f(x) = \int_0^\infty A(w)\cos wx\,dw = \sqrt{\frac{2}{\pi}}\int_0^\infty \hat{f}_c(w)\cos wx\,dw$$

by using $A(w) = \sqrt{\frac{2}{\pi}}\,\hat{f}_c(w)$

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty \hat{f}_c(\omega) \cos\omega x \, d\omega$$ is called

Inverse Fourier cosine Transform of $\hat{f}_c(\omega)$.

Ex-1 Find $\hat{f}_c(\omega)$ for $f(x) = x^2$ if $0<x<1$, $f(x) = 0$ if $x>1$.

Ans
$$f(x) = \begin{cases} x^2 & 0<x<1 \\ 0 & x>1 \end{cases}$$

We know Fourier cosine Transform

$$\hat{f}_c(\omega) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos\omega x \, dx$$

$$= \sqrt{\frac{2}{\pi}} \left[\int_0^1 x^2 \cos\omega x \, dx + \int_1^\infty 0 \cos\omega x \, dx \right]$$

$$= \sqrt{\frac{2}{\pi}} \left[x^2 \int \cos\omega x \, dx - \int \frac{dx^2}{dx} \int \cos\omega x \, dx \, dx \right]_0^1$$

$$= \sqrt{\frac{2}{\pi}} \left[x^2 \frac{\sin\omega x}{\omega} - \int 2x \frac{\sin\omega x}{\omega} dx \right]_0^1$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{\sin\omega}{\omega} - \frac{2}{\omega} \int x \sin\omega x \, dx \right]_0^1$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{\sin\omega}{\omega} - \frac{2}{\omega} \left(x \int \sin\omega x \, dx - \int \frac{dx}{dx} \int \sin\omega x \, dx \, dx \right) \right]_0^1$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{\sin\omega}{\omega} - \frac{2}{\omega} \left\{ -\frac{x\cos\omega x}{\omega} - \int -\frac{\cos\omega x}{\omega} dx \right. \right]_0^1$$

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$$= \sqrt{\frac{2}{\pi}} \left[\frac{\sin w}{w} - \frac{2}{w} \left\{ -\frac{x\cos wx}{w} + \frac{\sin wx}{w^2} \right]_0^1 \right.$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{\sin w}{w} + \frac{2}{w} \left\{ \frac{x\cos wx}{w} - \frac{\sin wx}{w^2} \right\} \right]_0^1$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{\sin w}{w} + \frac{2}{w} \left\{ \frac{\cos w}{w} - \frac{\sin w}{w^2} \right\} \right]$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{\sin w}{w} + \frac{2\cos w}{w^2} - \frac{2\sin w}{w^3} \right]$$

(2) Find $f(x)$ inverse fourier cosine Transform if $f(x) = 1$ if $0 \leq x \leq 1$, $f(x) = -1$ if $1 < x < 2$, $f(x) = 0$ if $x > 2$.

Ans

$$f(x) = \begin{cases} 1 & 0 \leq x \leq 1 \\ -1 & 1 < x < 2 \\ 0 & x > 2 \end{cases}$$

To find out the $f(x) \rightarrow$ inverse fourier cosine Transform we have to find first fourier cosine Transform.

$$\hat{f}_c(w) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos wx \, dx$$

$$= \sqrt{\frac{2}{\pi}} \left[\int_0^1 1 \times \cos wx \, dx + \int_1^2 -1 \times \cos wx \, dx + \int_2^\infty 0 \times \cos wx \, dx \right]$$

$$= \sqrt{\frac{2}{\pi}} \left[\left(\frac{\sin wx}{w} \right)_0^1 - \left(\frac{\sin wx}{w} \right)_1^2 \right]$$

$$= \sqrt{\frac{2}{\pi}} \left(\frac{\sin w}{w} - 0 - \frac{\sin 2w}{w} + \frac{\sin w}{w} \right)$$

$$= \sqrt{\frac{2}{\pi}} \left(\frac{2\sin w}{w} - \frac{\sin 2w}{w} \right) \rightarrow \hat{f}_c(w)$$

The inverse fourier cosine Transform is

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty \hat{f}_c(w) \cos wx \, dw$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty \sqrt{\frac{2}{\pi}} \left(2\frac{\sin w}{w} - \frac{\sin 2w}{w} \right) \cos wx \, dw$$

$$= \frac{2}{\pi} \int_0^\infty \left(\frac{2\sin w}{w} - \frac{\sin 2w}{w} \right) \cos wx \, dw$$

By using fourier cosine Integral

$$\left[\text{Hint:-} \int_0^\infty \frac{\sin w \cos xw}{w} \, dw = \begin{cases} \frac{\pi}{2} & 0 < x < 1 \\ 0 & x > 1 \end{cases} \right]$$

$$f(x) = \frac{2}{\pi} \begin{cases} \frac{\pi}{2} & 0 < x < 1 \\ -\frac{\pi}{2} & 1 < x < 2 \end{cases} = \begin{cases} 1 & 0 < x < 1 \\ -1 & 1 < x < 2 \end{cases}$$

~~where~~ we found by using fourier cosine integral of $\left(f(x) = \begin{cases} \frac{\pi}{2} & 0 < x < 1 \\ -\frac{\pi}{2} & 1 < x < 2 \end{cases} \right)$

Fourier sine Transforms

We also find fourier sine Transform by using fourier sine integral

$$f(x) = \int_0^\infty B(w) \sin wx \, dw$$

where $B(w) = \frac{2}{\pi} \int_0^\infty f(v) \sin wv \, dv$

and using $B(w) = \sqrt{\frac{2}{\pi}} \hat{f}_s(w)$

Where $\hat{f}_s(w)$ is the fourier sine Transforms.

$s \to$ use for sine.

$$B(w) = \frac{2}{\pi} \int_0^\infty f(v) \sin wv \, dv$$

$$\Rightarrow \sqrt{\frac{2}{\pi}} \hat{f}_s(w) = \frac{2}{\pi} \int_0^\infty f(v) \sin wv \, dv$$

Setting $v = x$

$$\hat{f}_s(w) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin wx \, dx$$

is the fourier sine Transform

And $f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty \hat{f}_s(w) \sin wx \, dw$

is the inverse fourier sine Transform

Ex-3 find $F_s(e^{-ax})$, $a>0$

Ans $F_s(e^{-ax})$, $a>0$ → fourier sine Transform

$$F_s(f(x)) = F_s(e^{-ax})$$

$$f(x) = e^{-ax}, \quad a>0$$

we know fourier sine Transform

$$\hat{f}_s(w) = F_s(f(x)) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin wx \, dx$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-ax} \sin wx \, dx, \quad a>0.$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{e^{-ax}}{(-a)^2 + w^2} \left((-a) \sin wx - w \cos wx \right) \right]_0^\infty$$

$$= \sqrt{\frac{2}{\pi}} \left[-\frac{e^{-ax}}{a^2 + w^2} \left(a \sin wx + w \cos wx \right) \right]_0^\infty$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{e^0}{a^2+w^2} \left(a \sin 0 + w \cos 0 \right) \right]$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{1}{a^2+w^2} \times w \right] = \sqrt{\frac{2}{\pi}} \frac{w}{a^2+w^2}$$

Linearity

If $f(x)$ be a piecewise continuous function and absolutely integrable on every finite integral then the fourier cosine & sine Transforms exist and also ~~a~~ Linear.

$$F_c(af+bg) = aF_c(f) + bF_c(g)$$

$$F_s(af+bg) = aF_s(f) + bF_s(g)$$

If ~~f~~ f and g are two ~~piece wise~~ piecewise continuous functions

Derivatives

If $f(x)$ be continuous and absolutely integrable on the x-axis and $f'(x)$ exist which is piecewise continuous on every finite ~~a~~ integral.

$f(x) \to 0$ as $x \to \infty$ then

1st
$$F_c\{f'(x)\} = \omega F_s\{f(x)\} - \sqrt{\frac{2}{\pi}} f(0)$$

$$F_s\{f'(x)\} = -\omega F_c\{f(x)\}$$

2nd
$$F_c\{f''(x)\} = -\omega^2 F_c\{f(x)\} - \sqrt{\frac{2}{\pi}} f'(0)$$

$$F_s\{f''(x)\} = -\omega^2 F_s\{f(x)\} + \sqrt{\frac{2}{\pi}}\, \omega f(0)$$

$$\mathcal{F}_c\{f'(x)\} = w\,\mathcal{F}_s\{f(x)\} - \sqrt{\tfrac{2}{\pi}}\,f(0)$$

Ans $\mathcal{F}_c\{f'(x)\} = \sqrt{\frac{2}{\pi}} \int_0^\infty f'(x)\cos wx\,dx$

$$= \sqrt{\frac{2}{\pi}}\left[\cos wx \int f'(x)\,dx - \int \frac{d\cos wx}{dx}\int f'(x)\,dx\,dx\right]_0^\infty$$

$$= \sqrt{\frac{2}{\pi}}\left(\cos wx\, f(x) + w\int_0^\infty f(x)\sin wx\,dx\right]_0^\infty$$

$$= \cancel{\sqrt{\frac{2}{\pi}}\left[-f(0) + w\,\hat{f}_s(w)\right]}$$

$$= -\sqrt{\frac{2}{\pi}}\,f(0) + w\sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\sin wx\,dx$$

$$= -\sqrt{\frac{2}{\pi}}\,f(0) + w\,\hat{f}_s(w)$$

$$\mathcal{F}_s\{f'(x)\} = -w\,\mathcal{F}_c\, f(x)$$

Ans $\mathcal{F}_s\{f'(x)\} = \sqrt{\frac{2}{\pi}}\int_0^\infty f'(x)\sin wx\,dx$

$$= \sqrt{\frac{2}{\pi}}\left[\sin wx\int f'(x)\,dx - \int \frac{d\sin wx}{dx}\int f'(x)\,dx\,dx\right]_0^\infty$$

$$= \sqrt{\frac{2}{\pi}}\left[f(x)\sin wx - w\int f(x)\cos wx\,dx\right]_0^\infty$$

$$= -\sqrt{\frac{2}{\pi}}\,w\int_0^\infty f(x)\cos wx\,dx$$

$$= -w\sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos wx\,dx = -w\,\hat{f}_c\{f(x)\}$$

$$\hat{f}_c\{f''(x)\} = -w^2\hat{f}_c\{f(x)\} - \sqrt{\frac{2}{\pi}}f'(0)$$

<u>Ans</u> $$\hat{f}_c\{f''(x)\} = \sqrt{\frac{2}{\pi}}\int_0^\infty f''(x)\cos wx\,dx$$

$$= \sqrt{\frac{2}{\pi}}\left[\cos wx\int f''(x)\,dx - \int\frac{d\cos wx}{dx}\int f''(x)\,dx\,dx\right]_0^\infty$$

$$= \sqrt{\frac{2}{\pi}}\left[f'(x)\cos wx + w\int f'(x)\sin wx\,dx\right]_0^\infty$$

$$= \sqrt{\frac{2}{\pi}}\left(f'(x)\cos wx\right]_0^\infty + w\sqrt{\frac{2}{\pi}}\int_0^\infty f'(x)\sin wx\,dx.$$

$$= -\sqrt{\frac{2}{\pi}}f'(0) + w\,\hat{f}_s\{f'(x)\}$$

$$= -\sqrt{\frac{2}{\pi}}f'(0) + w \times -w\,\hat{f}_c\{f(x)\}$$

$$= -w^2\hat{f}_c\{f(x)\} - \sqrt{\frac{2}{\pi}}f'(0)$$

$$\hat{f}_s\{f''(x)\} = -w^2\hat{f}_s\{f(x)\} + \sqrt{\frac{2}{\pi}}\,w\,f(0)$$

$$\hat{f}_s\{f''(x)\} = \sqrt{\frac{2}{\pi}}\int_0^\infty f''(x)\sin wx\,dx$$

$$= \sqrt{\frac{2}{\pi}}\left[\int f''(x)\,dx\,\sin wx - \int\frac{d\sin wx}{dx}\int f''(x)\,dx\,dx\right]_0^\infty$$

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$$= \sqrt{\frac{2}{\pi}} \left[\sin wx \, f'(x) - \int w \cos wx \, f'(x) \, dx \right]_0^\infty$$

$$= \sqrt{\frac{2}{\pi}} \left[\sin wx \, f'(x) \right]_0^\infty - w \sqrt{\frac{2}{\pi}} \int_0^\infty f'(x) \cos wx \, dx$$

$$= 0 - w \, F_c\{ f'(x) \}$$

$$= -w \left[w \, F_s\{ f(x) \} - \sqrt{\frac{2}{\pi}} f(0) \right]$$

$$= -w^2 F_s\{ f(x) \} + w \sqrt{\frac{2}{\pi}} f(0)$$

Ex