Fourier Integrals

(1) Let $f(x)$ be a function then the fourier Integral of the function $f(x)$ is

$$f(x) = \int_0^{\infty} [A(\omega)\cos\omega x + B(\omega)\sin\omega x]\, d\omega. \quad -(1)$$

where $A(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\cos\omega v\, dv \quad -(2)$

$B(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\sin\omega v\, dv \quad -(3)$

(2) **Fourier cosine Integral.**

If $f(x)$ is an even function then $B(\omega)=0$ and the fourier Integral ~~eq~~ eq^n (1) reduces to the fourier cosine Integral.

$$f(x) = \int_0^{\infty} A(\omega)\cos\omega x\, d\omega$$

where $A(\omega) = \frac{2}{\pi}\int_0^{\infty} f(v)\cos\omega v\, dv$

(3) **Fourier sine Integral.**

If $f(x)$ is an odd function then $A(\omega)=0$ eq^n (1) reduces to The fourier integral fourier sine Integral

$f(x) = \int_0^{\infty} B(\omega)\sin\omega x\, d\omega$ where

$$B(\omega) = \frac{2}{\pi}\int_0^{\infty} f(v)\sin\omega v\, dv.$$

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Exercise Ch. 10.8

① proof that $\int_0^{\infty} \frac{\cos xw + w \sin xw}{1+w^2} dw = \begin{cases} 0 & x<0 \\ \frac{\pi}{2} & x=0 \\ \pi e^{-x} & x>0 \end{cases}$

Ans We have to use Right hand side for ~~[illegible]~~ ~~[illegible]~~ prove this questions.

$\cos xw$ & $\sin xw$ are the functions so

~~[illegible]~~ First we have to compare left hand side to the fourier integral.

so we have to find fourier integral by using the R.H.S. as $f(x)$

so let $f(x) = \begin{cases} 0 & x<0 \\ \frac{\pi}{2} & x=0 \\ \pi e^{-x} & x>0 \end{cases}$

fourier integral we know

$$f(x) = \int_0^{\infty} [A(w) \cos wx + B(w) \sin wx] dw$$

where $A(w) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(v) \cos wv \, dv$

& $B(w) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(v) \sin wv \, dv$

$$A(\omega) = \frac{1}{\pi}\left[\int_{-\infty}^{0} f(v)\cos\omega v\,dv + \int_{0}^{\infty} f(v)\cos\omega v\,dv\right]$$

$$= \frac{1}{\pi}\left[\underbrace{\int_{-\infty}^{0} 0\times\cos\omega v\,dv}_{0} + \int_{0}^{\infty}\pi e^{-v}\cos\omega v\,dv\right]$$

$$= \frac{1}{\cancel{\pi}}\left[\cancel{\pi}\int_{0}^{\infty} e^{-v}\cos\omega v\,dv\right]$$

$$= \left[\frac{e^{-v}}{(-1)^2+\omega^2}\left[-\cos\omega v + \omega\sin\omega v\right]\right]_{0}^{\infty}$$

$$\Rightarrow \underbrace{\frac{e^{-\infty}}{1+\omega^2}}_{\to 0}\left(-\cos\omega\infty + \omega\sin\omega\infty\right) - \frac{e^{0}}{1+\omega^2}\left(-\cos 0 + \omega\underbrace{\sin 0}_{0}\right)$$

$$\Rightarrow -\frac{1}{1+\omega^2}\left(-\underbrace{\cos 0}_{-1}\right) = \frac{1}{1+\omega^2} \rightarrow A(\omega)$$

$$B(\omega) = \frac{1}{\pi}\left[\int_{\infty}^{0} f(v)\sin\omega v\,dv + \int_{0}^{\infty} f(v)\sin\omega v\,dv\right]$$

$$= \frac{1}{\pi}\left[\underbrace{\int_{-\infty}^{0} 0\times\sin\omega v\,dv}_{0} + \int_{0}^{\infty}\pi e^{-v}\sin\omega v\,dv\right]$$

$$= \frac{1}{\cancel{\pi}}\left[\cancel{\pi}\int_{0}^{\infty} e^{-v}\sin\omega v\,dv\right] = \left[\frac{e^{-v}}{(-1)^2+\omega^2}\left(-\sin\omega v - \omega\cos\omega v\right)\right]_{0}^{\infty}$$

$$= -\frac{e^{0}}{1+\omega^2}\left(-\underbrace{\sin 0}_{0} - \omega\cos 0\right) = \frac{\omega}{1+\omega^2} \rightarrow B(\omega)$$

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By putting the value of $A(w)$ & $B(w)$ in the fourier integral we get

$$f(x) = \int_0^\infty \left(A(w)\cos wx + B(w)\sin wx\right) dw$$

$$= \int_0^\infty \left(\frac{1}{1+w^2}\cos wx + \frac{w}{1+w^2}\sin wx\right) dw$$

$$= \int_0^\infty \frac{\cos xw + w\sin xw}{1+w^2} = f(x) = \begin{cases} 0 & x<0 \\ \frac{\pi}{2} & x=0 \\ \pi e^{-x} & x>0 \end{cases}$$

proof

(2) $\int_0^\infty \frac{\sin w \cos xw}{w} dw = \begin{cases} \frac{\pi}{2} & 0<x<1 \\ \frac{\pi}{4} & x=1 \\ 0 & x>1 \end{cases}$

Ans let $f(x) = \begin{cases} \frac{\pi}{2} & 0<x<1 \\ \frac{\pi}{4} & x=1 \\ 0 & x>1 \end{cases}$

In left hand side there is only the function $\cos xw$ so we have to use fourier cosine integral.

where $f(x) = \int_0^\infty A(w)\cos wx\, dw$

& $A(w) = \frac{2}{\pi}\int_0^\infty f(v)\cos wv\, dv$

$$A(w) = \frac{2}{\pi}\left[\int_0^1 f(v)\cos wv\,dv + \int_1^{\infty} f(v)\cos wv\,dv\right]$$

$$= \frac{2}{\pi}\left[\int_0^1 \frac{\pi}{2}\cos wv\,dv + \int_1^{\infty} 0\cos wv\,dv\right]$$

$$\Rightarrow \frac{2}{\pi}\left[\frac{\pi}{2}\int_0^1 \cos wv\,dv\right] : \left.\frac{\sin wv}{w}\right]_0^1 : \frac{\sin w}{w} - 0$$

$$= \frac{\sin w}{w} \rightarrow A(w)$$

By putting the value of $A(w)$ in fourier cosine Integral,

$$f(x) = \int_0^{\infty} A(w)\cos wx\,dw$$

$$= \int_0^{\infty} \frac{\sin w}{w}\cos xw\,dw \quad : \quad f(x) = \begin{cases} \frac{\pi}{2} & 0 \le x < 1 \\ \frac{\pi}{4} & x = 1 \\ 0 & x > 1 \end{cases}$$

(6) $$\int_0^{\infty} \frac{w^3 \sin xw}{w^4 + 4}\,dw = \frac{\pi}{2} e^{-x}\cos x \text{ if } x > 0.$$

There the function only $\sin xw$ so we have to find out Fourier Sine Integral.

where $A(w) = 0$, $B(w) = \frac{2}{\pi}\int_0^{\infty} f(v)\sin wv\,dv$

$$B(w) = \frac{2}{\pi}\left[\int_0^{\infty} \frac{\pi}{2} e^{-v}\cos v \sin wv\,dv\right]$$

$$= \int_0^{\infty} e^{-v}\cos v \sin wv\,dv$$

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$2\sin A\cos B = \sin(A+B) + \sin(A-B)$

$2\cos A\sin B = \sin(A+B) - \sin(A-B)$

$$\cos v \sin \omega v = \frac{1}{2}\left(\sin(v+\omega v) - \sin(v-\omega v)\right)$$

$$= \frac{1}{2}\left(\sin v(1+\omega) - \sin v(1-\omega)\right)$$

$$\therefore B(\omega) = \int_0^\infty e^{-v}\cos v \sin \omega v \, dv.$$

$$= \int_0^\infty e^{-v}\left(\frac{1}{2}\sin(1+\omega)v - \sin(1-\omega)v\right) dv$$

$$= \frac{1}{2}\left[\int_0^\infty e^{-v}\sin(1+\omega)v \, dv - \int_0^\infty e^{-v}\sin(1-\omega)v \, dv\right]$$

$$= \frac{1}{2}\left[\frac{e^{-v}}{1+(1+\omega)^2}\left(-\sin(1+\omega)v - (1+\omega)\cos(1+\omega)v\right)\right]_0^\infty$$

$$-\frac{1}{2}\left[\frac{e^{-v}}{1+(1-\omega)^2}\left(-\sin(1-\omega)v - (1-\omega)\cos(1-\omega)v\right)\right]_0^\infty$$

$$-\frac{1}{2}\left(\frac{e^{0}}{1+(1+\omega)^2}\left(-\underbrace{\sin(1+\omega)0}_{0} - (1+\omega)\cos 0\right)\right)$$

$$+\frac{1}{2}\left(\frac{e^{0}}{1+(1-\omega)^2}\left(-\underbrace{\sin 0}_{0} - (1-\omega)\cos 0\right)\right)$$

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$$= \frac{-1}{2}\left(\frac{1}{1+1+w^2+2w}\left(-(1+w)\right)\right) + \frac{1}{2}\left(\frac{1}{1+1+w^2-2w}(w-1)\right)$$

$$= \frac{1}{2}\left(\frac{1+w}{w^2+2w+2} + \frac{w-1}{w^2-2w+2}\right)$$

$$= \frac{1}{2}\left(\frac{1+w}{(w^2+2)+2w} + \frac{w-1}{(w^2+2)-2w}\right)$$

$$= \frac{1}{2} \times \frac{(1+w)\{w^2+2-2w\} + (w-1)\{w^2+2+2w\}}{\{(w^2+2)+2w\}\{w^2+2-2w\}}$$

$$= \frac{1}{2} \times \frac{\cancel{w^2}+\cancel{2}-2\cancel{w}+w^3+\cancel{2w}-2\cancel{w^2}+w^3+\cancel{2w}+2\cancel{w^2}-\cancel{w^2}-\cancel{2}-\cancel{2w}}{(w^2+2)^2-(2w)^2}$$

$$= \frac{1}{2} \times \frac{2w^3}{w^4+4+\cancel{4w^2}-\cancel{4w^2}} = \frac{1}{2} \times \frac{2w^3}{w^4+4} = \frac{w^3}{w^4+4} \rightarrow B(w)$$

By putting the value of $B(w)$ in Fourier sine integral $f(x) = \int_0^\infty B(w)\sin wx\,dw$ we get.

$$f(x) = \int_0^\infty \frac{w^3}{w^4+4}\sin wx\,dw$$ where $f(x) = \frac{\pi}{2}e^{-x}\cos x$ if $x > 0$

$$\therefore \int_0^\infty \frac{w^3 \sin xw}{w^4+4}\,dw = \frac{\pi}{2}e^{-x}\cos x \quad \text{if } x > 0.$$

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