Multidegree of freedom (MDOF) system

The system having more than one DOF are known as multidegree of freedom. for multidegree of freedom, the equations of motion can be expressed interms of influence coefficients.

stiffness influence coefficients

For a vibrating system with MDOF, the stiffness coefficients, denoted by k_{ij}. i.e defined as force at point i due to unit displacement given at point J, when all the points other than J are held fixed i.e zero displacements of all the points other than j

A matrix of stiffness influences coefficients as shown in below is called stiffness matrix.

$$k_{ij} = \begin{bmatrix} k_{11}, & k_{12}, & \cdots, & k_{1n} \\ \vdots & & & \\ k_{n1}, & k_{n2}, & \cdots, & k_{nn} \end{bmatrix}$$

Flexibility influence coefficients

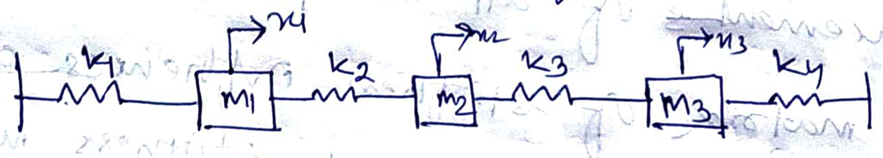
In multi degree of freedom system, the deflection produced at one point due to load application at that point is usually not restricted to that point alone. when one part of the system moves proportional deflections occurs in other parts also.

Thus the flexibility influence coefficients, denoted as a_{ij} i.e. deflection at point i due to unit load acting at point j

A matrix of flexibility influence coefficients as shown in below is referred as flexibility matrix.

$$a_{ij} = \begin{bmatrix} a_{11}, & a_{12}, & \cdots & a_{1n} \\ \vdots & & & \\ a_{n1}, & a_{n2}, & \cdots & a_{nn} \end{bmatrix} = \begin{bmatrix} \frac{1}{k_1} & \frac{1}{k_1} & \cdots \\ \frac{1}{k_1} & \frac{1}{k_1}+\frac{1}{k_2} & \end{bmatrix}$$

Q. Determine the stiffness coefficients of 3DOF system as shown in figure.



Soln

The equation of motions

$$m_1 \ddot{x}_1 + k_1 x_1 + k_2 (x_1 - x_2) = 0$$

$$m_2 \ddot{x}_2 + k_3 (x_2 - x_3) - k_2 (x_1 - x_2) = 0$$

$$m_3 \ddot{x}_3 - k_3 (x_2 - x_3) + k_4 k_3 = 0$$

Rearranging:-

$$m_1 \ddot{x}_1 + (k_1 + k_2) x_1 - k_2 x_2 = 0$$

$$m_2 \ddot{x}_2 - k_2 x_1 + (k_2 + k_3) x_2 - k_3 x_3 = 0$$

$$m_3 \ddot{x}_3 - k_3 x_2 + (k_3 + k_4) x_3 = 0$$

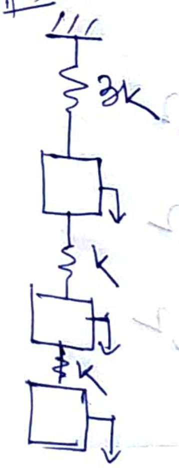
$$\begin{bmatrix} M_1 & 0 & 0 \\ 0 & M_2 & 0 \\ 0 & 0 & M_3 \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \end{Bmatrix} + \begin{bmatrix} k_1+k_2 & -k_2 & 0 \\ -k_2 & k_2+k_3 & -k_3 \\ 0 & -k_3 & k_3+k_4 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = 0$$

Thus the stiffness coefficient matrix

$$k_{ij} = \begin{bmatrix} k_{11} & k_{12} & k_{13} \\ k_{21} & k_{22} & k_{23} \\ k_{31} & k_{32} & k_{33} \end{bmatrix} = \begin{bmatrix} k_1+k_2 & k_2 & 0 \\ -k_2 & k_2+k_3 & -k_3 \\ 0 & -k_3 & k_3+k_4 \end{bmatrix}$$

Q Find the influence coefficients of the spring mass system as shown in fig.

Ans



The influence coefficients can be found as

$$a_{11} = a_{12} = a_{13} = \frac{1}{3k}$$

By maxwell reciprocal thm, $a_{12} = a_{21}$, $a_{13} = a_{31}$

Hence $a_{11} = a_{12} = a_{13} = a_{21} = a_{31} = \frac{1}{3k}$

$$a_{22} = a_{23} = a_{32} = \frac{1}{3k} + \frac{1}{k} = \frac{4}{3k}$$

$$a_{33} = \frac{1}{3k} + \frac{1}{k} + \frac{1}{k} = \frac{7}{3k}$$

Thus the influence coefficient matrix

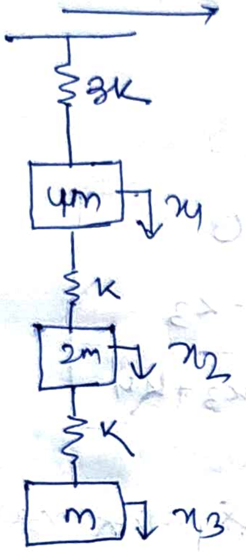
$$a_{ij} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} \frac{1}{3k} & \frac{1}{3k} & \frac{1}{3k} \\ \frac{1}{3k} & \frac{4}{3k} & \frac{4}{3k} \\ \frac{1}{3k} & \frac{4}{3k} & \frac{7}{3k} \end{bmatrix}$$

Matrix Iteration method

With this help of this method the natural frequencies and corresponding mode shapes are determined.

→ Use of influence coefficients is made in the analysis.

Q. → Find the fundamental frequency and corresponding mode shapes for the 3 DOF system shown in figure by matrix iteration method.



Solution The equation for the system in terms of influence coefficients can be written as

$$x_1 = a_{11}(4m)x_1\omega^2 + a_{12}(2m)x_2\omega^2 + a_{13}\,m\,x_3\omega^2$$

$$x_2 = a_{21}(4m)x_1\omega^2 + a_{22}(2m)x_2\omega^2 + a_{23}\,m\,x_3\omega^2$$

$$x_3 = a_{31}(4m)x_1\omega^2 + a_{32}(2m)x_2\omega^2 + a_{33}\,m\,x_3\omega^2$$

Influence coefficients are

$$a_{11} = a_{12} = a_{13} = a_{31} = a_{21} = 1/3k$$

$$a_{22} = 1/3k + \frac{1}{k} = \frac{4}{3k} = a_{23} + a_{32}$$

$$a_{33} = \frac{1}{3k} + \frac{1}{k} + \frac{1}{k} = 7/3k$$

Substituting these values in above equations

$$x_1 = \frac{4m}{3k}\omega^2 x_1 + \frac{2m}{3k}\omega^2 x_2 + \frac{m}{3k}\omega^2 x_3$$

$$x_2 = \frac{4m}{3k}\omega^2 x_1 + \frac{8m}{3k}\omega^2 x_2 + \frac{4m}{3k}\omega^2 x_3$$

$$x_3 = \frac{4m}{3k}\omega^2 x_1 + \frac{8m}{3k} x_2 + \frac{7m}{3k}\omega^2 x_3$$

The above equation in matrix form.

$$\begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \frac{m\omega^2}{3k}\begin{bmatrix} 4 & 2 & 1 \\ 4 & 8 & 4 \\ 4 & 8 & 7 \end{bmatrix}\begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix}$$

The iteration process may be started by assuming any suitable value for x_1, x_2, x_3. Let's assume

$$\begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 1 \\ 2 \\ 3 \end{Bmatrix}$$

1st iteration

$$\begin{Bmatrix} 1 \\ 2 \\ 3 \end{Bmatrix} = \frac{m\omega^2}{3k}\begin{bmatrix} 4 & 2 & 1 \\ 4 & 8 & 4 \\ 4 & 8 & 7 \end{bmatrix}\begin{Bmatrix} 1 \\ 2 \\ 3 \end{Bmatrix}$$

$$= \frac{m\omega^2}{3k}\begin{Bmatrix} 11 \\ 32 \\ 41 \end{Bmatrix} = \frac{m\omega^2}{3k}\underbrace{(11)}_{\text{Taken common.}}\begin{bmatrix} 1 \\ 2.9 \\ 3.7 \end{bmatrix}$$

For second Iteration

$x_1 = 1, \quad x_2 = 2.9, \quad x_3 = 3.7$

$$\begin{Bmatrix} 1 \\ 2.9 \\ 3.7 \end{Bmatrix} = \frac{m\omega^2}{3k}\begin{bmatrix} 4 & 2 & 1 \\ 4 & 8 & 4 \\ 4 & 8 & 7 \end{bmatrix}\begin{Bmatrix} 1 \\ 2.9 \\ 3.7 \end{Bmatrix} = \frac{m\omega^2}{3k}\begin{bmatrix} 13.5 \\ 42 \\ 53.1 \end{bmatrix} = \frac{m\omega^2}{3k}\underbrace{(13.5)}_{\text{common}}\begin{bmatrix} 1 \\ 3.1 \\ 3.9 \end{bmatrix}$$

3rd Iteration

$x_1 = 1, \quad x_2 = 3.1 ; x_3 = 3.9$

$$\begin{Bmatrix} 1 \\ 3.1 \\ 3.9 \end{Bmatrix} = \frac{m\omega^2}{3k} \begin{bmatrix} 4 & 2 & 1 \\ 4 & 8 & 4 \\ 4 & 8 & 7 \end{bmatrix} \begin{Bmatrix} 1 \\ 3.1 \\ 3.9 \end{Bmatrix}$$

$$= \frac{m\omega^2}{3k} \begin{bmatrix} 14.1 \\ 44.4 \\ 56.1 \end{bmatrix} = \frac{m\omega^2}{3k} (14.1) \begin{bmatrix} 1 \\ 3.15 \\ 3.98 \end{bmatrix}$$

4th Iteration

$x_1 = 1, \quad x_2 = 3.15 \quad x_3 = 3.98$

$$\begin{Bmatrix} 1 \\ 3.15 \\ 3.98 \end{Bmatrix} = \frac{m\omega^2}{3k} \begin{bmatrix} 4 & 2 & 1 \\ 4 & 8 & 4 \\ 4 & 8 & 7 \end{bmatrix} \begin{Bmatrix} 1 \\ 3.15 \\ 3.98 \end{Bmatrix}$$

$$= \frac{m\omega^2}{3k} \begin{bmatrix} 14.28 \\ 45.12 \\ 57.06 \end{bmatrix} = \frac{m\omega^2}{3k} (14.28) \begin{bmatrix} 1 \\ 3.16 \\ 4 \end{bmatrix}$$

The obtained values varry close to assumed value

Therefor mode shapes are

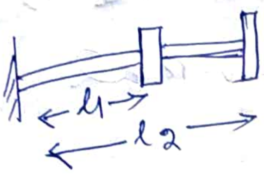
$$\begin{Bmatrix} 1 \\ 3.16 \\ 4 \end{Bmatrix}$$

and the natural frequeny obtained as

$$\frac{m\omega^2}{3k} (14.28) = 1$$

$$\omega = \sqrt{\frac{3k}{14.28\, m}} = 0.458 \sqrt{k/m} \text{ rad/sec.}$$

Q. Determine the flexibility / Influence coefficients given figure.

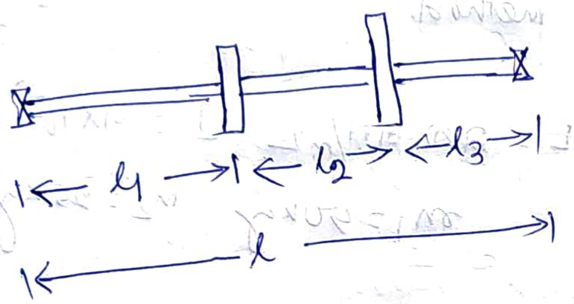


d = dia of shaft.

$I = \frac{\pi d^4}{64}$, EI = flexural rigidity.

$$a_{11} = \frac{l_1^3}{3EI}, \quad a_{22} = \frac{l_2^3}{3EI}, \quad a_{12} = a_{21} = \frac{l_1^3}{6EI}(3l_2 - l_1)$$

Q. Determine the Influence coefficients of the system



$$a_{11} = \frac{l_1^3 (l_2 + l_3)}{3EI(l_1 + l_2 + l_3)} \qquad a_{22} = \frac{(l_1 + l_2) l_3^3}{3EI(l_1 + l_2 + l_3)}$$

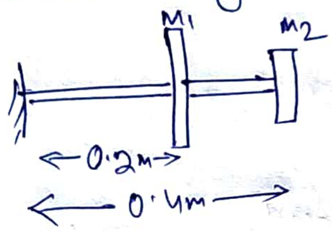
$$a_{12} = a_{21} = \frac{l_1 l_2 (l^2 - l_1^2 - l_2^2)}{6EIl}$$

Stodola Method

It is an iterative process to obtain the fundamental mode of vibration of several DOF system.

Q. Determine the fundamental frequency and principal mode of vibration for the system shown in figure using Stodola method.

So



$E = 210\ GN/m^2$ $\quad I = 4 \times 10^{-7} m^2$

$m_1 = 50kg$ $\quad m_2 = 25kg$

Solⁿ

$$a_{11} = \frac{(0.2)^3}{3EI} = 3.33 \times 10^{-8} \qquad a_{12} = a_{21} = 8.31 \times 10^{-8}$$

$$a_{22} = \frac{(0.4)^3}{3EI} = 2.66 \times 10^{-7}$$

For 1st trial Assume $x_1 = 1$ $\quad x_2 = 1$

The Inertia forces, $f_1 = m_1 w^2 x_1 = 50w^2$

$f_2 = m_2 w^2 x_2 = 25w^2$

Corresponding deflection.

$$x_1' = f_1 a_{11} + f_2 a_{12}$$
$$= 374.75 \times 10^{-8} w^2$$
$$x_2' = f_2 a_{21} + f_2 a_{22}$$
$$= 1081.5 \times 10^{-8} w^2$$

obtained ratio is are different than assumed

$$x_1' : x_2' = 1 : 2.89$$

2nd trial

$x_1' = 1 \qquad x_2' = 2.89$

The inertia forces $F_1' = m_1 w^2 x_1' = 50 w^2$

$F_2 = m_2 w^2 x_2' = 72.25 w^2$

$x_1'' = a_{11} f_1' + a_{12} f_2' = 768.34 \times 10^{-8} w^2$

$x_2'' = a_{21} f_1' + a_{22} f_2' = 2338.35 \times 10^{-8} w^2$

So ratios are $= x_1'' : x_2'' = 1 : 3$

3rd trial

$x_1'' = 1 \qquad x_2'' = 3$

$F_1'' = m_1 w^2 x_1'' = 50 w^2 \qquad f_2'' = m w^2 x_2'' = 75 w^2$

$x_1''' = a_{11} f_1'' + a_{12} F_2'' = 791.25 \times 10^{-8} w^2$

$x_2''' = a_{21} f_1'' + a_{22} f_2'' = 2619.75 \times 10^{-8} w^2$

$x_1''' : x_2''' = 1 : 3.31$

4th trial $\quad x_1''' = 1 \qquad x_2''' = 3.31$

$F_1''' = w^2 x_1''' = 50 w^2 \qquad f_2''' = m w^2 x_2''' = 82.75 w^2$

$x_1^{iv} = a_{11} F_1''' + a_{12} F_2''' = 855.8 \times 10^{-8} w^2$

$x_2^{iv} = a_{21} f_1''' + a_{22} f_2''' = 2617.65 \times 10^{-8} w^2$

$x_1^{iv} : x_2^{iv} = 855.5 : 2617.65 = 1 : 1.31$

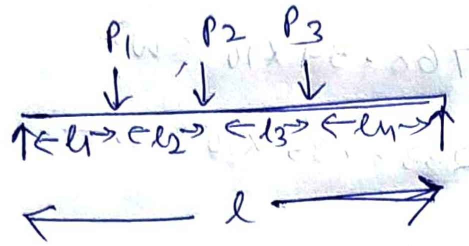
This method lies very close to trial ratio, so the Principal modes are $\begin{Bmatrix} 1 \\ 3.1 \end{Bmatrix}$ and the corresponding fundamental frequency is obtained as

$x_1^{iv} = 1$

$855.8 \times 10^{-8} w^2 = 1 \quad , \quad \boxed{w = 341.83 \text{ rad/sec}}$

* Dunkerley method

This method is useful for estimating the fundamental natural frequency of structure. (Transverse vibration)



Here P_1, P_2 & P_3 are the concentration loads and y_1, y_2 & y_3 are the corresponding deflections.

According to Dunkerley's method

$$\frac{1}{w_n^2} = \frac{1}{w_1^2} + \frac{1}{w_2^2} + \dots + \frac{1}{w_s^2}$$

Where w_n = natural frequency of whole system

$w_1, w_2, \dots$ etc = natural frequency of individual point load.

w_s = Natural frequency of transverse vibration because of weight of shaft or (UDL)

* The general form of expression for deflection under concentrated load is given by

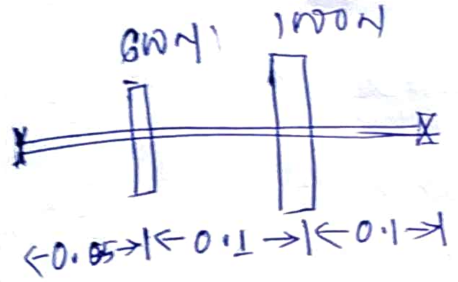
$$y = \frac{w\, l_1^2 l_2^2}{3EIl}$$

ex If 2 loads are there

$$y_1 = \frac{P_1 l_1^2 (l - l_1)^2}{3EIl}, \quad y_2 = \frac{P_2 (l_1+l_2)^2 [l - (l_1+l_2)]^2}{3EIl}$$

1. Determine the fundamental natural frequency of the figure below using dunkerley method.

Take $E = 196 \cdot 10^9 \, N/m^2$ $\quad I = 4 \times 10^{-7} m^4$



Consider 1st the beam with single point load of 600N alone.

The deflection due to this load is given by

$$y_1 = \frac{600 (0.05)^2 (0.2)^2}{3 \times (196 \times 10^9) \times 4 \times 10^{-7} \times 0.25} = 1.02 \times 10^{-6} M$$

Now deflection due to load of 1000N alone calculated

$$y_2 = \frac{1000 (0.15)^2 (0.1)^2}{3 \times (196 \times 10^9) \times (4 \times 10^{-7}) \times 0.25} = 3.83 \times 10^{-6} M$$

Natural frequency under individual point loads

$$\omega_1 = \sqrt{g/y_1} = \sqrt{\frac{9.81}{1.02 \times 10^{-6}}} = 3101.23 \text{ rad/sec.}$$

& $$\omega_2 = \sqrt{g/y_2} = \sqrt{\frac{9.81}{3.83 \times 10^{-6}}} = 1600.42 \text{ rad/sec}$$

Using dunkerley method

$$\frac{1}{\omega_n^2} = \frac{1}{\omega_1^2} + \frac{1}{\omega_2^2} = , \quad \omega_n = 1422.2 \text{ rad/sec}$$