Government College of Engineering Keonjhar

LECTURE NOTES

MATHS-II

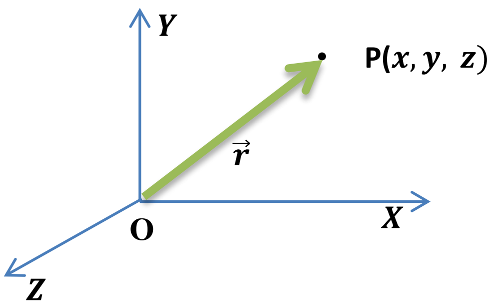
VECTOR CALCULUS

Module - III (10 Hours)

Syllabus: Vector differential calculus: vector and scalar functions and fields, Derivatives, Curves, tangents and arc Length, gradient, divergence, curl.

SCALAR- Quantities having only magnitude

VECTOR- ***Magnitude as well as direction***



POSITION VECTOR

Position vector OP at point P(x, y, z)

$$\vec{r} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$$

MAGNITUDE OF VECTOR

$$r = |\vec{r}| = \sqrt{x^2 + y^2 + z^2}$$

UNIT VECTOR

$$\hat{r} = \frac{\vec{r}}{|\vec{r}|} = \frac{x\hat{\imath} + y\hat{\jmath} + z\hat{k}}{\sqrt{x^2 + y^2 + z^2}}$$

VECTOR PRODUCT

DOT Product- $\vec{a}.\vec{b} = |\vec{a}||\vec{b}|cos\theta$

$$\vec{a} = a_1\hat{\imath} + a_2\hat{\jmath} + a_3\hat{k}$$

$$\vec{b} = b_1\hat{\imath} + b_2\hat{\jmath} + b_3\hat{k}$$

$\hat{\imath}.\hat{\jmath} = \hat{\jmath}.\hat{k} = \hat{k}.\hat{\imath} = 0,\ \hat{\imath}.\hat{\imath} = \hat{\jmath}.\hat{\jmath} = \hat{k}.\hat{k} = 1$

$$\vec{a}.\vec{b} = a_1b_1 + a_2b_2 + a_3b_3$$

CROSS Product- $\vec{a} \times \vec{b} = |\vec{a}||\vec{b}|sin\theta$

$$\vec{a} = a_1\hat{\imath} + a_2\hat{\jmath} + a_3\hat{k}$$

$$\vec{b} = b_1\hat{\imath} + b_2\hat{\jmath} + b_3\hat{k}$$

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

POINT FUNCTION

SCALAR POINT FUNCTION: A scalar point function is a function that assigns a real number (i.e. a scalar) to each point of some region of space.

$$\phi(x,\ y,\ z) = x^2y + 4z^2 + yz$$

VECTOR POINT FUNCTION: A vector function is a function that assigns a vector to a set of real variables. Its general form is

$$\vec{F} = F_1\hat{\imath} + F_2\hat{\jmath} + F_3\hat{k}$$

e.g.

$$\vec{F} = x^2yz\hat{\imath} + z^3\hat{\jmath} + xy\hat{k}$$

GRADIENT, DIVERGENCE AND CURL

DEL OPERATOR

Vector differential operator

$$\vec{\nabla} = \hat{\imath}\frac{\partial}{\partial x} + \hat{\jmath}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}$$

GRADIENT OF SCALAR FUNCTION ϕ

The gradient of scalar function is defined as

$$grad\ \phi = \vec{\nabla}\phi = \hat{\imath}\frac{\partial \phi}{\partial x} + \hat{\jmath}\frac{\partial \phi}{\partial y} + \hat{k}\frac{\partial \phi}{\partial z}$$

- $\vec{\nabla}\phi$ ***is normal vector to the surface*** ϕ

DIVERGENCE OF VECTOR FUNCTION $\vec{F}$

If $\vec{F} = F_1\hat{\imath} + F_2\hat{\jmath} + F_3\hat{k}$ **, then divergence of vector function is defined as**

$$Div\,\vec{F} = \vec{\nabla}.\vec{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

CURL OF VECTOR FUNCTION $\vec{F}$

If $\vec{F} = F_1\hat{\imath} + F_2\hat{\jmath} + F_3\hat{k}$**, then Curl of vector function is defined as**

$$Curl\,\vec{F} = \vec{\nabla} \times \vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$$

Question 1

Prove that

(i) $\vec{\nabla} r^n = n r^{n-2}\vec{r}$

(ii) $\vec{\nabla}.\vec{r} = 3$

(iii) Curl grad $\phi = 0$

Soln. :

$$grad\ \phi = \vec{\nabla}\phi = \hat{\imath}\frac{\partial\phi}{\partial x} + \hat{\jmath}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}$$

(i) $\vec{r} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$

$$r = |\vec{r}| = \sqrt{x^2 + y^2 + z^2}$$

$$r^n = (x^2 + y^2 + z^2)^{n/2}$$

$$\vec{\nabla} r^n = \left(\hat{\imath}\frac{\partial}{\partial x} + \hat{\jmath}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(x^2 + y^2 + z^2)^{n/2}$$

$$= \hat{\imath}\frac{\partial(x^2 + y^2 + z^2)^{n/2}}{\partial x} + \hat{\jmath}\frac{\partial(x^2 + y^2 + z^2)^{n/2}}{\partial y} + \hat{k}\frac{\partial(x^2 + y^2 + z^2)^{n/2}}{\partial z}$$

$$\vec{\nabla} r^n = \left(\hat{\imath}\frac{\partial}{\partial x} + \hat{\jmath}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(x^2 + y^2 + z^2)^{n/2}$$

$$= \hat{\imath}\frac{\partial(x^2 + y^2 + z^2)^{n/2}}{\partial x} + \hat{\jmath}\frac{\partial(x^2 + y^2 + z^2)^{n/2}}{\partial y} + \hat{k}\frac{\partial(x^2 + y^2 + z^2)^{n/2}}{\partial z}$$

$$= \frac{n}{2}.2x(x^2 + y^2 + z^2)^{(n-2)/2}\hat{\imath} + \frac{n}{2}.2y(x^2 + y^2 + z^2)^{(n-2)/2}\hat{\jmath} + \frac{n}{2}.2z(x^2 + y^2 + z^2)^{(n-2)/2}\hat{k}$$

$$= n(x^2 + y^2 + z^2)^{(n-2)/2}\{x\hat{\imath} + y\hat{\jmath} + z\hat{k}\}$$

$= n r^{n-2}\vec{r}$ **Proved.**

(ii) $\vec{r} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$

$$Div\,\vec{F} = \vec{\nabla}.\vec{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

$$\vec{\nabla}.\vec{r} = \left(\hat{\imath}\frac{\partial}{\partial x} + \hat{\jmath}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right).x\hat{\imath} + y\hat{\jmath} + z\hat{k}$$

$$= \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z}$$

= 1 + 1 + 1 =3

(iii)

$$Curl\,\vec{F} = \vec{\nabla} \times \vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$$

Here, $\vec{F} = grad\,\phi = \vec{\nabla}\phi = \hat{\imath}\frac{\partial\phi}{\partial x} + \hat{\jmath}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}$

Curl grad $\phi = \vec{\nabla} \times (\vec{\nabla}\phi)$

$$= \left(\hat{\imath}\frac{\partial}{\partial x} + \hat{\jmath}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) \times \left(\hat{\imath}\frac{\partial\phi}{\partial x} + \hat{\jmath}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}\right)$$

$$= \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial\phi}{\partial x} & \frac{\partial\phi}{\partial y} & \frac{\partial\phi}{\partial z} \end{vmatrix}$$

Curl grad $\phi = \vec{\nabla} \times (\vec{\nabla}\phi)$

$$= \left(\hat{\imath}\frac{\partial}{\partial x} + \hat{\jmath}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) \times \left(\hat{\imath}\frac{\partial\phi}{\partial x} + \hat{\jmath}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}\right)$$

$$= \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial\phi}{\partial x} & \frac{\partial\phi}{\partial y} & \frac{\partial\phi}{\partial z} \end{vmatrix}$$

$$= \hat{\imath}\left(\frac{\partial^2\phi}{\partial y\partial z} - \frac{\partial^2\phi}{\partial z\partial y}\right) + \hat{\jmath}\left(\frac{\partial^2\phi}{\partial z\partial x} - \frac{\partial^2\phi}{\partial x\partial z}\right) + \hat{k}\left(\frac{\partial^2\phi}{\partial x\partial y} - \frac{\partial^2\phi}{\partial y\partial x}\right)$$

$$= 0\hat{\imath} + 0\hat{\jmath} + 0\hat{k} \quad = 0$$

IMPORTANT NOTES

1. Solenoidal Vector

If div $\vec{F} = 0$

$$\vec{\nabla}.\vec{F} = 0$$

Then $\vec{F}$ is called Solenoidal vector

2. Irrotational Vector

If Curl $\vec{F} = 0$

$$\vec{\nabla} \times \vec{F} = 0$$

Then $\vec{F}$ is called irrotational vector

3. Directional Derivative

Directional derivative of scalar function ϕ at point $P(x, y, z)$ in the direction of unit vector $\hat{a}$ is

$$D.D.\ of\ \phi = \frac{d\phi}{ds} = grad\ \phi\ .\hat{a}$$

Question 1

Show that

$$\vec{F} = (x + 3y)\hat{\imath} + (y - 2z)\hat{\jmath} + (x - 2z)\hat{k}$$

is solenoidal.

Soln. : If div $\vec{F} = 0$

$$\vec{\nabla}.\vec{F} = 0$$

Then $\vec{F}$ is called Solenoidal vector

Given $\vec{F} = (x + 3y)\hat{\imath} + (y - 2z)\hat{\jmath} + (x - 2z)\hat{k}$

$$\vec{\nabla}.\vec{F} = \left(\hat{\imath}\frac{\partial}{\partial x} + \hat{\jmath}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right).\{(x + 3y)\hat{\imath} + (y - 2z)\hat{\jmath} + (x - 2z)\hat{k}\}$$

$$= \frac{\partial(x + 3y)}{\partial x} + \frac{\partial(y - 2z)}{\partial y} + \frac{\partial(x - 2z)}{\partial z}$$

$$= 1 + 1 - 2$$

$$= 0$$

So, $\vec{\nabla}.\vec{F} = 0$

Hence $\vec{F}$ is solenoidal.

Question 2

Determine the value of $a, \ b \ and \ c$ so that

$\vec{F} = (x + 2y + az)\hat{\imath} + (bx - 3y - z)\hat{\jmath} + (4x + cy + 2z)\hat{k}$

is irrotational.

Soln. : If Curl $\vec{F} = 0$

$$\vec{\nabla} \times \vec{F} = 0$$

Then $\vec{F}$ is called irrotational vector

$\vec{F} = (x + 2y + az)\hat{\imath} + (bx - 3y - z)\hat{\jmath} + (4x + cy + 2z)\hat{k}$

$$\vec{\nabla} \times \vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x + 2y + az & bx - 3y - z & 4x + cy + 2z \end{vmatrix}$$

$$= \hat{\imath}\left(\frac{\partial(4x + cy + 2z)}{\partial y} - \frac{\partial(bx - 3y - z)}{\partial z}\right) - \hat{\jmath}\left(\frac{\partial(4x + cy + 2z)}{\partial x} - \frac{\partial(x + 2y + az)}{\partial z}\right)$$
$$+\hat{k}\left(\frac{\partial(bx - 3y - z)}{\partial x} - \frac{\partial(x + 2y + az)}{\partial y}\right)$$

$$= \hat{\imath}\left(\frac{\partial(4x + cy + 2z)}{\partial y} - \frac{\partial(bx - 3y - z)}{\partial z}\right) - \hat{\jmath}\left(\frac{\partial(4x + cy + 2z)}{\partial x} - \frac{\partial(x + 2y + az)}{\partial z}\right)$$
$$+\hat{k}\left(\frac{\partial(bx - 3y - z)}{\partial x} - \frac{\partial(x + 2y + az)}{\partial y}\right)$$

$$= (c + 1)\hat{\imath} + (-a + 4)\hat{\jmath} + (b - 2)\hat{k}$$

For Irrotational Vector

$$\vec{\nabla} \times \vec{F} = 0$$

$\Rightarrow (c + 1)\hat{\imath} + (-a + 4)\hat{\jmath} + (b - 2)\hat{k} = 0\hat{\imath} + 0\hat{\jmath} + 0\hat{k}$

$$c + 1 = 0, \ -a + 4 = 0, \ b - 2 = 0$$

So, for $a = 4, \ b = 2, \ c = -1$, *$\vec{F}$ is called irrotational vector*

Question 3

Find the directional derivative of

$\boldsymbol{\phi = x^2 - y^2 + 2z^2}$ **at Point P (1, 2, 3) in the direction of line PQ, where Q is the point (5, 0, 4).**

Soln.: Directional derivative of scalar function $\boldsymbol{\phi}$ at point $\boldsymbol{P(x, y, z)}$ in the direction of unit vector $\boldsymbol{\hat{a}}$ is

$$D.D.\ of\ \phi = \frac{d\phi}{ds} = grad\,\phi\,.\hat{a}$$

Step I- Find scalar function (surface)

$$\phi = x^2 - y^2 + 2z^2$$

Step II- Find Gradient of $\boldsymbol{\phi}$

$$\vec{\nabla}\phi = \left(\hat{\imath}\frac{\partial}{\partial x} + \hat{\jmath}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(x^2 - y^2 + 2z^2)$$

$$= \hat{\imath}\frac{\partial(x^2 - y^2 + 2z^2)}{\partial x} + \hat{\jmath}\frac{\partial(x^2 - y^2 + 2z^2)}{\partial y} + \hat{k}\frac{\partial(x^2 - y^2 + 2z^2)}{\partial z}$$

Step I- Find scalar function (surface)

$$\phi = x^2 - y^2 + 2z^2$$

Step II- Find Gradient of $\boldsymbol{\phi}$

$$\vec{\nabla}\phi = \hat{\imath}\frac{\partial(x^2 - y^2 + 2z^2)}{\partial x} + \hat{\jmath}\frac{\partial(x^2 - y^2 + 2z^2)}{\partial y} + \hat{k}\frac{\partial(x^2 - y^2 + 2z^2)}{\partial z}$$

$$grad\phi = \vec{\nabla}\phi = 2x\hat{\imath} - 2y\hat{\jmath} + 4z\hat{k}$$

Step III- Substitttute Point P (1, 2, 3) in $\boldsymbol{grad\phi}$

$$\vec{\nabla}\phi = 2(1)\hat{\imath} - 2(2)\hat{\jmath} + 4(3)\hat{k}$$

$$\Rightarrow \vec{\nabla}\phi = 2\hat{\imath} - 4\hat{\jmath} + 12\hat{k}$$

Step IV- Find Unit Vector $\boldsymbol{\hat{a}}$

Given $\vec{a} = \overrightarrow{PQ} = \vec{Q} - \vec{P}$

$$= \left(5\hat{\imath} + 0\hat{\jmath} + 4\hat{k}\right) - (\hat{\imath} + 2\hat{\jmath} + 3\hat{k})$$

$$= \left(4\hat{\imath} - 2\hat{\jmath} + \hat{k}\right)$$

$$\hat{a} = \frac{\vec{a}}{|\vec{a}|} = \frac{4\hat{\imath} - 2\hat{\jmath} + \hat{k}}{\sqrt{4^2 + 2^2 + 1}} = \frac{4\hat{\imath} - 2\hat{\jmath} + \hat{k}}{\sqrt{21}}$$

Step V – Apply definition of Directional derivative

$$D.D.\ of\ \phi = grad\ \phi\,.\hat{a}$$

$\vec{\nabla}\phi = 2\hat{\imath} - 4\hat{\jmath} + 12\hat{k}$ and $\hat{a} = \frac{4\hat{\imath}-2\hat{\jmath}+\hat{k}}{\sqrt{21}}$

$$grad\ \phi\,.\hat{a} = \left(2\hat{\imath} - 4\hat{\jmath} + 12\hat{k}\right).\left(\frac{4\hat{\imath} - 2\hat{\jmath} + \hat{k}}{\sqrt{21}}\right)$$

$$= \frac{2.4 + 4.2 + 12.1}{\sqrt{21}} = \frac{28}{\sqrt{21}}\quad Ans.$$

<u>Question 4</u>

Find the directional derivative of

$\phi = xy^2 + yz^3$ **at**

Point (2, -1, 1) in the direction of normal to the surface $x\log z - y^2 = -4$ at $(-1,\ 2, 1)$.

Soln.: Directional derivative of scalar function ϕ at point $P(x, y, z)$ in the direction of unit vector $\hat{a}$ is

$$D.D.\ of\ \phi = \frac{d\phi}{ds} = grad\ \phi\,.\hat{a}$$

Step I- Find scalar function (surface)

$$\phi = xy^2 + yz^3$$

Step II- Find Gradient of ϕ

$\vec{\nabla}\phi = \left(\hat{\imath}\frac{\partial}{\partial x} + \hat{\jmath}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(xy^2 + yz^3)$

$$= \hat{\imath}\frac{\partial(xy^2 + yz^3)}{\partial x} + \hat{\jmath}\frac{\partial(xy^2 + yz^3)}{\partial y} + \hat{k}\frac{\partial(xy^2 + yz^3)}{\partial z}$$

$= y^2\hat{\imath} + (2xy + z^3)\hat{\jmath} + 3yz^2\hat{k}$

$$D.D.\ of\ \phi = grad\ \phi\,.\hat{a}$$

Step III- Substittute Point (2, -1, 1) in $grad\phi$

$$\vec{\nabla}\phi = y^2\hat{\imath} + (2xy + z^3)\hat{\jmath} + 3yz^2\hat{k}$$

$$\Rightarrow \vec{\nabla}\phi = \hat{\imath} - 3\hat{\jmath} - 3\hat{k}$$

Step IV- Find Unit Vector $\hat{a}$

Given $\vec{a}$ = normal to the surface $x\log z - y^2 = -4$ at $(-1,\ 2, 1)$.

$$Let \qquad \psi = x\log z - y^2 + 4$$

$$\vec{a} = \vec{\nabla}\psi$$

$$= \hat{\imath}\frac{\partial(x\log z - y^2 + 4)}{\partial x} + \hat{\jmath}\frac{\partial(x\log z - y^2 + 4)}{\partial y} + \hat{k}\frac{\partial(x\log z - y^2 + 4)}{\partial z}$$

$$\vec{a} = logz\hat{\imath} - 2y\hat{\jmath} + \frac{x}{z}\hat{k}$$

$$\vec{a} = logz\hat{\imath} - 2y\hat{\jmath} + \frac{x}{z}\hat{k}$$

At (-1, 2, 1), $\vec{a} = -4\hat{\jmath} - \hat{k}$

$$\hat{a} = \frac{\vec{a}}{|\vec{a}|} = \frac{-4\hat{\jmath} - \hat{k}}{\sqrt{4^2 + 1}} = \frac{-4\hat{\jmath} - \hat{k}}{\sqrt{17}}$$

Step V – Apply definition of Directional derivative

$$D.D.\ of\ \phi = \ grad\ \phi\,.\hat{a}$$

$\vec{\nabla}\phi = \hat{\imath} - 3\hat{\jmath} - 3\hat{k}$ **and** $\hat{a} = \frac{-4\hat{\jmath} - \hat{k}}{\sqrt{17}}$

$$grad\ \phi\,.\hat{a} = (\hat{\imath} - 3\hat{\jmath} - 3\hat{k}).\left(\frac{-4\hat{\jmath} - \hat{k}}{\sqrt{17}}\right)$$

$$= \frac{(-3).(-4) + (-3).(-1)}{\sqrt{17}} = \frac{15}{\sqrt{21}} \quad Ans.$$

CURVES AND ARC LENGTH

1) Arc Length: The arc length of a curve y = f(x) over the interval [a,b] is L $= \int ds$

where, $ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}$; $y = f(x)$ **, a ≤ x ≤ b**

$$ds = \sqrt{1 + \left(\frac{dx}{dy}\right)^2} \qquad \text{if} \quad x = g(y),\ c \le y \le d$$

***2) Plane curve*: Given a smooth curve *C* defined by the function** $\vec{r}(t) = f(t)\hat{\imath} + g(t)\hat{\jmath}$. **where *t* lies within the interval the arc length of *C* over the interval is L** $= \int \sqrt{\left(\frac{df}{dt}\right)^2 + \left(\frac{dg}{dt}\right)^2}\, dt.$ **i.e. d**$s = \sqrt{\left(\frac{df}{dt}\right)^2 + \left(\frac{dg}{dt}\right)^2}$

3) ***Space curve*****: Given a smooth curve *C* defined by the function** $\vec{r}(t) = f(t)\hat{\imath} + g(t)\hat{\jmath} + h(t)\hat{k}$**. where *t* lies within the interval the arc length of *C* over the interval is** L = $\int \sqrt{\left(\frac{df}{dt}\right)^2 + \left(\frac{dg}{dt}\right)^2 + \left(\frac{dh}{dt}\right)^2}\, dt$.

EXAMPLE 1

Suppose $\vec{r}(t) = cost\hat{\imath} + sint\hat{\jmath} + t\hat{k}$**. What is the distance along the helix from (1,0,0) to (**$cost, \sin t$**, t).**

Sol. We know that this curve is a helix.

$$\vec{r}(t) = cost\hat{\imath} + sint\hat{\jmath} + t\hat{k}$$

$$\vec{r}'^{(t)} = -sint\hat{\imath} + cost\hat{\jmath} + \hat{k}$$

The distance along the helix from (1,0,0) to ($cost, \sin t$**, t)**

$$s = \int |r'(t)|dt = \int_0^t \sqrt{(-Sint)^2 + (Cost)^2 + 1}dt = \sqrt{2}[t]_0^t = \sqrt{2}t$$

Note: The value of t that gets us distance s along the helix is t = $\frac{s}{\sqrt{2}}$**, and so the same curve is given by** $\vec{r}(s) = cos\frac{s}{\sqrt{2}}\hat{\imath} + sin\frac{s}{\sqrt{2}}\hat{\jmath} + \frac{s}{\sqrt{2}}\hat{k}$

EXAMPLE 2

Find the length of the cycloid $\vec{r}(t) = \langle t - \sin t,\ 1 - \cos t\rangle$ **generated by the unit circle .**

Sol. We know that given cycloid is

$$\vec{r}(t) = (t - sint)\hat{\imath} + (1 - cost)\hat{\jmath}$$

$f(t) = t - sint, \qquad g(t) = 1 - cost$

$$\text{L} = \int \sqrt{\left(\frac{df}{dt}\right)^2 + \left(\frac{dg}{dt}\right)^2}$$

$$\frac{df}{dt} = 1 - cost, \qquad \frac{dg}{dt} = sint$$

$$\text{L} = \int \sqrt{\left(\frac{df}{dt}\right)^2 + \left(\frac{dg}{dt}\right)^2}\, dt = \int \sqrt{(1 - cost)^2 + (sint)^2}\, dt$$

$$\mathbf{L} = \int \sqrt{(1-cost)^2 + (sint)^2}\, dt = \int \sqrt{1 - 2cost + (-cost)^2 + (sint)^2}\, dt$$

$$= \int \sqrt{2(1-cost)}dt = \sqrt{2}\int \sqrt{2sin^2\left(\frac{t}{2}\right)}\, dt = 2\int sin\left(\frac{t}{2}\right) dt$$

The cycloid is generated by the unit circle, Thus t will vary from 0 to 2π

$$\mathbf{L} = 2\int_0^{2\pi} sin\left(\frac{t}{2}\right) dt = 2\left[-2cos\left(\frac{t}{2}\right)\right]_0^{2\pi} = -4[cos\pi - cos0] = -4(-2) = 8$$

<u>Assignment</u>

Evaluate following integral for curve C: $x = t^2,\ y = t,\quad 0 \le t \le 1$

$$I = \int xy\, ds$$

- There will be separate question set for Assignments and Practice.