Module - 5 Quantum Mechanics

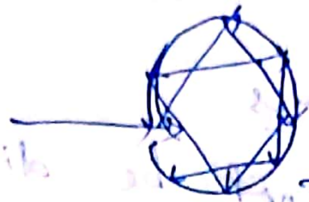
Advantages of Quantum Mechanics

1. The Particles which can not be directly observe with the help of eye can be explained with quantum mechanics.
2. According Quantum mechanics atom is stable which can not be explained in classical mechanics.
3. Spectra of hydrogen atom can be explained by this mechanics.
4. Specific heat radiation of Solids canbe explained in this mechanics.
5. Black body radiation can be explained in this mechanics.
6. Heisenberg's uncertainty principle, Bohr's correspondence principle can be explained in this mechanics.
7. photoelectric effect, Schroedinger's eqn can be explained in this mechanics.

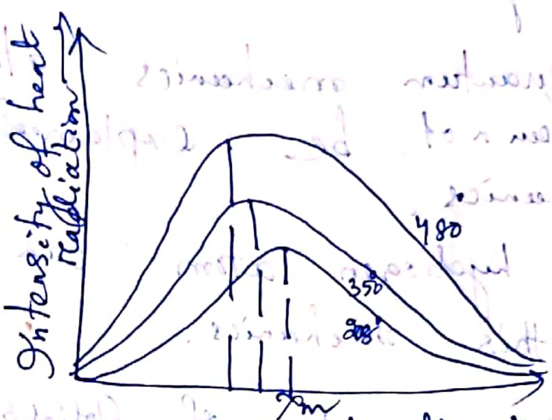
Black Body radiation

Black body is that type of body which absorbs all types of radiation incident on it.

But in this universe there is no perfectly black body.

for ex:- A hollow sphere with Small hole
the light passes through
Now the sphere by the hole
the light is internally
the light and no light is

reflected
transmitted out.

Characteristics of Black body radiation



1) The area under each curve represents the total heat radiation for a particular temp.

2) For a particular temp. the intensity of heat radiation first increases, attains maximum, then decreases.

3) With increase in temperature, the maximum value of wavelength decreases.

4) With increase in temperature, the area of radiation increases.

Photo electric effect:-

The ejection of electron from the metallic Surface when light of certain frequency incident on it. The minimum frequency is known as threshold frequency.

~~Let ν_0 is the~~

Let's consider a photon of frequency ν falls on a metal Surface, then energy of photon $E = h\nu$

Let ν_0 is the threshold frequency then energy required for rejection of electron $E = h\nu_0$

If m be the mass of electron and v its velocity, then the rest amount of energy helps gaining K.E of electron.

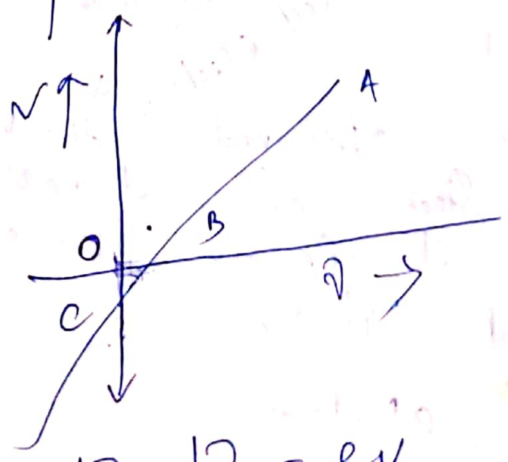
i.e $\boxed{h\nu - h\nu_0 = \frac{1}{2}mv^2}$

$h\nu_0 \rightarrow$ work function.

Again we know that $\boxed{h\nu - h\nu_0 = eV}$

$V \rightarrow$ potential

If we draw a graph betn potential V and frequency 'ν', then the nature of the graph is as follows.



(1) $h\nu - h\nu_0 = eV$

$\Rightarrow V = \frac{h}{e}\nu - \frac{h}{e}\nu_0$

Comparing this eqn with $y = mx + c$

We get $m = \frac{h}{e}$

$\Rightarrow \tan\theta = \frac{h}{e}$

$\Rightarrow h = e\tan\theta$

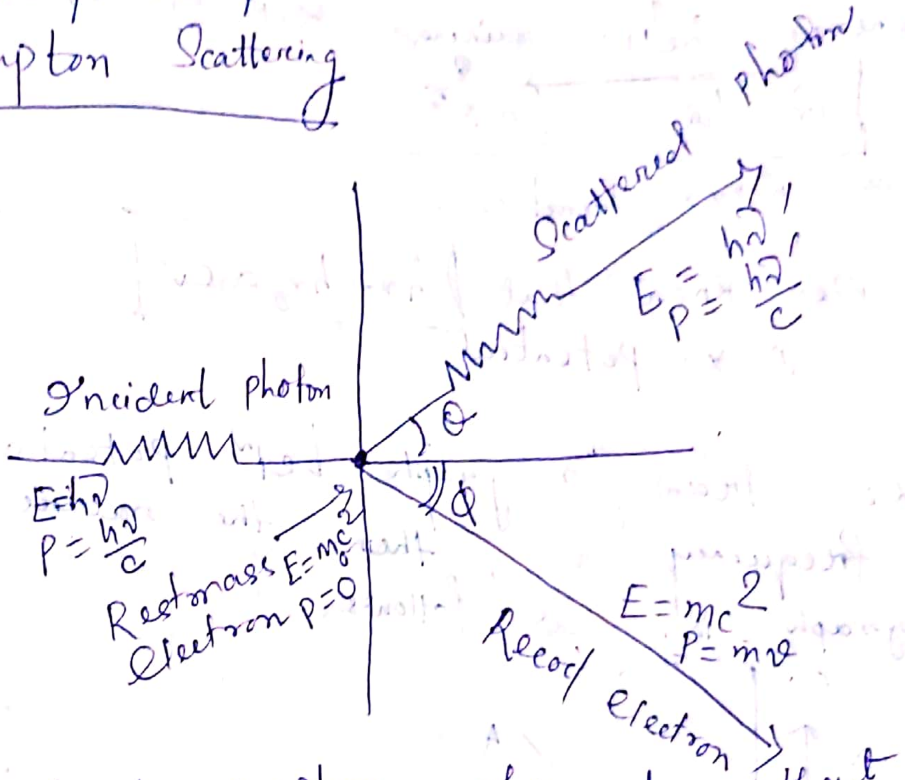
(ii) When $V = 0$, then $\nu = \nu_0$ is represented by OB part of the curve.

(iii) When $\nu = 0$

$eV = -h\nu_0$

$\Rightarrow V = -\frac{h}{e}\nu_0$ which is represented by OC part of the curve.

Compton Scattering



In 1921 Compton observed that when a photon strikes on rest mass electron after collision it forms two component

(i) Scattered photon

(2) Recoil electron.

Now The change in wavelength depends on angle of Scattering.

Here
Before Collision

Energy of incident photon. $E = h\nu$
momentum of " " $p = \frac{h\nu}{c}$

Energy of rest mass electron $E = m_0 c^2$
momentum of " " $p = 0$

After collision

Energy of scattered photon $E = h\nu'$
Momentum of scattered " $p = \frac{h\nu'}{c}$
Energy of recoil electron $E = mc^2$
Momentum " " " $p = mv$
Angle of scattering $= \theta$
" " recoil $= \phi$

Applying conservation of energy i.e

$$h\nu + m_0 c^2 = h\nu' + mc^2 \quad \text{—— (1)}$$

Applying conservation of momentum in x-axis

$$\frac{h\nu}{c} + 0 = \frac{h\nu'}{c} \cos\theta + mv \cos\phi \quad \text{—— (2)}$$

in y-axis

$$0 = \frac{h\nu'}{c} \sin\theta - mv \sin\phi \quad \text{—— (3)}$$

$$\Rightarrow h\nu' \sin\theta = mvc \sin\phi \quad \text{—— (4)}$$

from eqn (2) $h\nu - h\nu' \cos\theta = mvc \cos\phi$ —— (5)

Squaring and adding eqn (4) and (5)

$$m^2v^2c^2\sin^2\phi + m^2v^2c^2\cos^2\phi = h^2\nu'^2\sin^2\theta + (h\nu - h\nu'\cos\theta)^2$$

$$\Rightarrow \quad '' = h^2\nu'^2\sin^2\theta + h^2\nu^2 + h^2\nu'^2\cos^2\theta - 2h^2\nu\nu'\cos\theta$$

$$\Rightarrow m^2v^2c^2 = h^2\nu^2 + h^2\nu'^2 - 2h^2\nu\nu'\cos\theta \quad \text{——} (6)$$

from eq^n (1)

$$h\nu + m_0c^2 = h\nu' + mc^2$$

$$\Rightarrow mc^2 = h(\nu - \nu') + m_0c^2$$

Squaring it we get

$$m^2c^4 = h^2(\nu^2 + \nu'^2 - 2\nu\nu') + 2h(\nu-\nu')m_0c^2 + m_0^2c^4 \quad \text{——} (7)$$

Substracting eq^n (6) from eq^n (7)

$$m^2c^4 - m^2v^2c^2 = \cancel{h^2\nu^2} + \cancel{h^2\nu'^2} - 2h^2\nu\nu' + 2h(\nu-\nu')m_0c^2 + m_0^2c^4$$

$$-(\cancel{h^2\nu^2} + \cancel{h^2\nu'^2} - 2h^2\nu\nu'\cos\theta)$$

$$= -2h^2\nu\nu'(1-\cos\theta) + 2h(\nu-\nu')m_0c^2 + m_0^2c^4$$

$$\Rightarrow m^2c^2(c^2 - v^2) = -2h^2\nu\nu'(1-\cos\theta) + 2h(\nu-\nu')m_0c^2 + m_0^2c^4$$

Applying relativistic mass formula

$$m = \frac{m_0}{\sqrt{1 - v^2/c^2}}$$

$$\Rightarrow \frac{m_0^2c^2}{\cancel{c^2-v^2}} \times c^2 \times \cancel{(c^2-v^2)} = -2h^2\nu\nu'(1-\cos\theta) + 2h(\nu-\nu')m_0c^2 + m_0^2c^4$$

$$\Rightarrow \cancel{m_0^2c^4} = -2h^2\nu\nu'(1-\cos\theta) + 2h(\nu-\nu')m_0c^2 + \cancel{m_0^2c^4}$$

$$\Rightarrow \cancel{2}h^{\cancel{2}}\nu\nu'(1-\cos\theta) = \cancel{2}\cancel{h}(\nu-\nu')m_0c^2$$

$$\Rightarrow \frac{\nu-\nu'}{\nu\nu'} = \frac{h(1-\cos\theta)}{m_0c^2}$$

$$\Rightarrow \frac{1}{\lambda'} - \frac{1}{\lambda} = \frac{h}{m_0 c}(1-\cos\theta)$$

$$\Rightarrow \frac{\lambda'}{c} - \frac{\lambda}{c} = \frac{h}{m_0 c^2}(1-\cos\theta)$$

$$\Rightarrow \boxed{\lambda' - \lambda = \frac{h}{m_0 c}(1-\cos\theta)}$$

The change in wavelength depends upon angle of scattering.

i) If $\theta = 0$

then $\lambda' - \lambda = 0$

No change in wavelength.

(ii) $\theta = \frac{\pi}{2}$

$$\lambda' - \lambda = \frac{h}{m_0 c}\left\{1 - \cos\left(\frac{\pi}{2}\right)\right\} = \frac{h}{m_0 c}$$

(iii) $\theta = \pi$

$$\lambda' - \lambda = \frac{h}{m_0 c}[1-(-1)] = \frac{2h}{m_0 c}$$

De-broglie hypothesis of matter waves

De-broglie postulated that a free particle with mass m moving with speed v should have a wavelength λ related to its momentum exactly the same way as for photon.

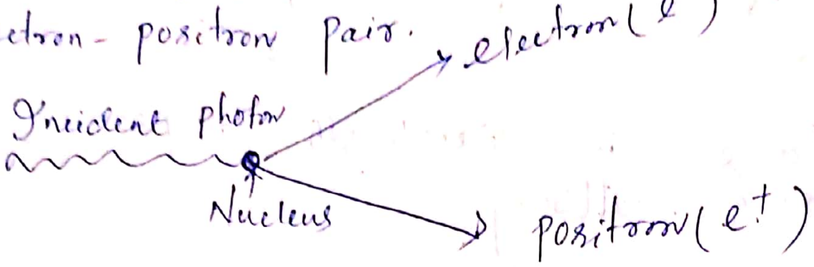
A photon of light of frequency ν has the momentum $p = \frac{h\nu}{c}$

Since $\lambda\nu = c \Rightarrow p = \frac{h}{\lambda}$

$$\Rightarrow \boxed{\lambda = \frac{h}{p} = \frac{h}{mv}}$$

Pair production :-

Pair production is the process that results in the conversion of a photon into an electron - positron pair.



i.e when a photon incident on a nucleus then after collision it is devide into electron and positron pair.

Heisenberg's uncertainty principle -

We can not determine both momentum and position of a particle simoltaneously. Hence ~~then~~ according to Heisenberg the product of uncertainty in position and uncertainty in momentum is greater than equal to $\frac{h}{4\pi}$

Hence $\Delta x \cdot \Delta p_x \geq \frac{\hbar}{2}$

Application :-

① Electron cannot present inside the nucleus where as neutron and proton can reside inside the nucleus.

Generally the radius of the ~~nucleus~~ nucleus is 10^{-14} m.

If any particle reside inside the necleus its uncertainty in position must be equal to ~~less~~ less than or equal to diameter i.e $(2\times10^{-14} m)$.

According to Heisenberg's uncertainty formula

$$\Delta x . \Delta p_x \geq \frac{h}{4\pi} \quad \text{Let } \Delta x . \Delta p_x \simeq \frac{h}{2\pi}$$

$$\Rightarrow \Delta p_x \cancel{\geq} \frac{h}{2\pi \times \Delta x}$$

$$= 0.527 \times 10^{-20} \ N.s$$

According to Einstein's relativistic formula

$E = mc^2 \rightarrow$ energy.

$E_0 = m_0c^2 \rightarrow$ rest mass energy

Total energy $E^2 = m^2c^4 + m_0^2c^4$

$$= p^2c^2 + m_0^2c^4$$

$$\Rightarrow E = \sqrt{p^2c^2 + m_0^2c^4}.$$

Since the rest mass energy value is very less, hence it can be neglected as comparision to 1st term.

Hence $E = \sqrt{p^2c^2} = pc$

$$= \frac{0.527 \times 10^{-20} \times 3\times10^8}{1.6\times10^{-19}}$$

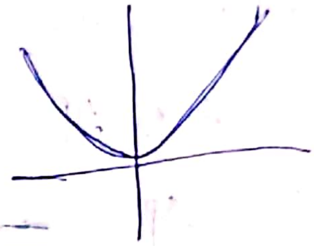
$$=$$

$$= 9.8 \text{ MeV} \simeq 10 \text{ MeV}.$$

But during radioactive decay the particle which are emitted have energy 3 to 4 MeV. Since high amount of energy required for electron to present inside the nuclei. So electron cannot reside inside the nucleu

(iii) Ground state energy of a harmonic Oscillator.

Total energy of the S.H.O



$$E = K.E + P.E$$

$$= \frac{p^2}{2m} + \frac{1}{2} m\omega^2 x^2$$

Uncertainty in momentum Δp and uncertainty in position Δx.

$$E = \frac{(\Delta p)^2}{2m} + \frac{1}{2} m\omega^2 (\Delta x)^2$$

Applying Heisenberg's uncertainty principle

$$\Delta x \cdot \Delta p_x = \frac{\hbar}{2}$$

$$\Delta p = \frac{\hbar}{2\Delta x}$$

$$E = \frac{1}{2m}\left(\frac{\hbar}{2\Delta x}\right)^2 + \frac{1}{2} m\omega^2 (\Delta x)^2$$

$$= \frac{\hbar^2}{8m(\Delta x)^2} + \frac{1}{2} m\omega^2 (\Delta x)^2$$

$$= \frac{\hbar^2}{8mx^2} + \frac{1}{2} m\omega^2 x^2$$

Now minimum energy can be find out by

$$\frac{dE}{dx} = 0$$

$$\Rightarrow \frac{\hbar^2}{8m}(-2)x^{-3} + \frac{1}{2} m\omega^2 \times 2x = 0$$

$$\Rightarrow \frac{+\hbar^2}{8mx^3}(2) + \frac{1}{2} m\omega^2 \times 2x = 0$$

$$\Rightarrow \frac{-\hbar^2}{4mx^3} + m\omega^2 x = 0$$

$$\Rightarrow \frac{\hbar^2}{4mx^3} = m\omega^2 x$$

$$\Rightarrow \frac{\hbar^2}{4m^2\omega^2} = x^4$$

$$\Rightarrow x^2 = \frac{\hbar}{2m\omega}$$

Hence Energy $E = \frac{\hbar^2}{8m}\left(\frac{2m\omega}{\hbar}\right) + \frac{1}{2}m\omega^2\left(\frac{\hbar}{2m\omega}\right)$

$$= \frac{\hbar\omega}{4} + \frac{\hbar\omega}{4}$$

$$= \frac{\hbar\omega}{2}$$

Hence the ground state energy of a harmonic oscillator is $\boxed{E_0 = \frac{\hbar\omega}{2}}$

Probability density

Probability density is defined as the probability of finding the particle in space.

If $\psi(x,t)$ represent the wave function then $\psi^*(x,t)$ represent the complex conjugate of the wave function, then probability $P(x)\,dx$ of a particle between x and $x+dx$ is given by

$$\boxed{P=\int_x^{x+dx} P(x)\,dx = \int_x^{x+dx} \psi^*(x,t)\,\psi(x,t)\,dx}$$

Normalization of wave function:-

If the probability of finding the particle is unity then the wave function is normalized.

i.e $$P=\int_x^{x+dx} P(x)\,dx$$

$$= \int_x^{x+dx} \psi^*(x,t)\,\psi(x,t)\,dx = 1$$

If wave function varies from ($x\to-\infty$ to $x\to+\infty$) then $\int_{-\infty}^{+\infty} \psi^*(x,t)\,\psi(x,t)\,dx = 1$ in one dimension.

Observables:-

It is the linear operator representing the state space of quantum states. It corresponds to physical quantity i.e energy, spin etc.

Eigen value & Eigen function :-

In an wave equation when the operator operating on a function, produces a constant times the function is called an eigen value equation. The function is called eigen function and the numerical value is called eigen value.

for ex:- $\frac{d}{dx}(e^{2x}) = 2e^{2x}$

Here $\frac{d}{dx} \rightarrow$ operator

$2 \rightarrow$ eigen value

$e^{2x} \rightarrow$ eigen function

Operators :-

operator is a function over a space of physical state to another space of physical state.

Few operators used in Quantum:-

$E \rightarrow i\hbar \frac{\partial}{\partial t}$ (energy operator)

$p \rightarrow -i\hbar \frac{\partial}{\partial x}$ (momentum operator.

$K.E = -\frac{\hbar^2 \nabla^2}{2m}$ (Kinetic energy operator)

Hamiltonian operator

$$\hat{H} \rightarrow -\frac{\hbar^2 \nabla^2}{2m} + V$$

Expectation value

Expectation value is the probabilistic expected value of the result.

In Quantum mechanics the expectation value of an operator can be written as

$$\langle \hat{A} \rangle = \frac{\int_{-\infty}^{+\infty} \psi^*(x) \langle \hat{A} \rangle \psi(x)\, dx}{\int_{-\infty}^{+\infty} |\psi(x)|^2\, dx} \quad \text{in one dimension}$$

Q:- Find out the expectation value of Kinetic energy of a particle in one dimension box $\psi(x) = \sqrt{\frac{2}{a}} \sin\left(\frac{n\pi x}{a}\right)$. The width of the box is a.

$$\langle K.E \rangle = \frac{\int_0^a \psi^*(x) \langle K.E \rangle \psi(x)\, dx}{\int_0^a \psi^*(x)\psi(x)\, dx}$$

$$= \frac{\int_0^a \sqrt{\frac{\cancel{2}}{a}} \sin\left(\frac{n\pi x}{a}\right)\left(\frac{p^2}{2m}\right)\sqrt{\frac{\cancel{2}}{a}} \sin\left(\frac{n\pi x}{a}\right) dx}{\int_0^a \frac{\cancel{2}}{a} \sin^2\left(\frac{n\pi x}{a}\right) dx}$$

$$= \frac{1}{2m} \frac{\int_0^a \sin\left(\frac{n\pi x}{a}\right)\left(-\hbar^2 \frac{\partial^2}{\partial x^2}\right) \sin\left(\frac{n\pi x}{a}\right) dx}{\int_0^a \sin^2\left(\frac{n\pi x}{a}\right) dx}$$

$$= \frac{-\frac{\hbar^2}{2m}\int_0^a \sin\left(\frac{n\pi x}{a}\right)\left\{-\left(\frac{n\pi}{a}\right)^2\right\}\sin\left(\frac{n\pi x}{a}\right)dx}{\int_0^a \sin^2\left(\frac{n\pi x}{a}\right)dx}$$

$$= -\frac{\hbar^2}{2m}\times -\frac{n^2\pi^2}{a^2}\frac{\cancel{\int_0^a \sin^2\left(\frac{n\pi x}{a}\right)dx}}{\cancel{\int_0^a \sin^2\left(\frac{n\pi x}{a}\right)dx}}$$

$$= \frac{n^2\pi^2\hbar^2}{2ma^2}$$

(ii) Expectation value of position $\langle x\rangle$

$$\psi(x) = \sqrt{\frac{2}{a}}\int \sin\left(\frac{n\pi x}{a}\right) \cancel{dx}$$

$$\langle x\rangle = \frac{\int_0^a \sqrt{\frac{2}{a}}\sin\left(\frac{n\pi x}{a}\right) x \sqrt{\frac{2}{a}}\sin\left(\frac{n\pi x}{a}\right)dx}{\frac{2}{a}\int_0^a \sin^2\left(\frac{n\pi x}{a}\right)dx}$$

$$= \frac{\int_0^a x \sin^2\left(\frac{n\pi x}{a}\right)dx}{\int_0^a \sin^2\left(\frac{n\pi x}{a}\right)dx}$$

$$= \frac{x\int_0^a \frac{1-\cos 2\left(\frac{n\pi x}{a}\right)}{2}dx}{\int_0^a \sin^2\left(\frac{n\pi x}{a}\right)dx}$$

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$$= \frac{\frac{1}{2}\int_0^a \left\{x - x\cos\left(\frac{2n\pi x}{a}\right)\right\} dx}{\int_0^a \sin^2\left(\frac{n\pi x}{a}\right) dx}$$

$$\left[\int x \cos nx \, dx = \frac{1}{n} x \sin nx + \frac{1}{n^2} \cos nx + c \right]$$

$$= \frac{\frac{1}{2} \times \left[\int_0^a x \, dx - \int_0^a x \cos\left(\frac{2n\pi x}{a}\right) dx \right]}{\int_0^a \sin^2\left(\frac{n\pi x}{a}\right) dx}$$

$$= \frac{\frac{1}{2}\left(\frac{x^2}{2}\right)\Big|_0^a - 0}{\frac{a}{2}}$$

$$\left[\int_0^a \sin^2\left(\frac{n\pi x}{a}\right) dx \right.$$

$$= \frac{1}{2}\int_0^a \left\{1 - \cos\left(\frac{2n\pi x}{a}\right)\right\} dx$$

$$= \frac{1}{2}\left\{ x\Big|_0^a \cancel{+ \frac{\sin\left(\frac{2n\pi x}{a}\right)}{\frac{2n\pi}{a}}\Big|_0^a} \right\}$$

$$\left. = \frac{a}{2} \right.$$

$$= \frac{\frac{a^2}{4}}{\frac{a}{2}} = \frac{a}{2}$$

Schroedingers equation :-

(Time independent wave equation)-

We know that $\psi(x,t) = A \sin(kx - \omega t)$

$$\Rightarrow \frac{\partial^2 \psi}{\partial x^2} = -k^2 \psi(x,t)$$

$$\& \ \frac{\partial^2 \psi}{\partial t^2} = -\omega^2 \psi(x,t)$$

$$\Rightarrow \frac{\frac{\partial^2 \psi}{\partial x^2}}{\frac{\partial^2 \psi}{\partial t^2}} = \frac{k^2}{\omega^2} = \frac{1}{v^2}$$

$$\Rightarrow \boxed{\frac{\partial^2 \psi}{\partial x^2} = \frac{1}{v^2} \frac{\partial^2 \psi}{\partial t^2}} \longrightarrow (*)$$

Total energy $E = T + V$

$$= \frac{p^2}{2m} + V$$

$$\Rightarrow \frac{p^2}{2m} = E - V$$

$$\Rightarrow p = \sqrt{2m(E-V)} \longrightarrow (1)$$

According to de-broglie hypothesis $\lambda = \frac{h}{p}$

Apply eqn (1) $\Rightarrow \lambda = \frac{h}{\sqrt{2m(E-V)}} \longrightarrow (2)$

Let's consider a wave function $\psi(x,t) = e^{i\omega t}$

$$\Rightarrow \frac{\partial^2 \psi}{\partial t^2} = (i\omega)^2 e^{i\omega t}$$

$$\Rightarrow \frac{\partial^2 \psi}{\partial t^2} = -\omega^2 \psi \longrightarrow (3)$$

Putting the value of eq^n(3) in eq^n(*)

$$\frac{\partial^2\psi}{\partial x^2} = \frac{1}{v^2}(-\omega^2\psi)$$

$$= -\frac{(2\pi\cancel{\nu})^2}{(\cancel{\nu}\lambda)^2}\psi$$

$$= -\frac{4\pi^2}{\lambda^2}\psi$$

$$\Rightarrow \frac{\partial^2\psi}{\partial x^2} + \frac{4\pi^2}{\lambda^2}\psi = 0 \quad \text{------} (4)$$

Putting the value of eq^n (2) in eq^n (4)

$$\frac{\partial^2\psi}{\partial x^2} + \frac{4\pi^2}{\frac{h^2}{2m(E-V)}}\psi = 0$$

$$\Rightarrow \frac{\partial^2\psi}{\partial x^2} + \frac{8\pi^2 m}{h^2}(E-V)\psi = 0$$

$$\Rightarrow \boxed{\frac{\partial^2\psi}{\partial x^2} + \frac{2m}{\hbar^2}(E-V)\psi = 0}$$ which is the time independent Schroedinger's eq^n.

In three dimension it can be written as

$$\boxed{\nabla^2\psi + \frac{2m}{\hbar^2}(E-V)\psi = 0}$$

For free particle $V=0$

Hence the eq^n is $\nabla^2\psi + \frac{2mE}{\hbar^2}\psi = 0$

Time dependent Schroedinger's eq^n

Let's consider the wave function $\psi = e^{-i\omega t}$

$$\Rightarrow \frac{\partial\psi}{\partial t} = -i\omega \times e^{-i\omega t}$$

$$= -i\omega\psi$$

$$= -i(2\pi\nu)\psi$$

$$= -i\, 2\pi\left(\frac{E}{h}\right)\psi$$

$$\Rightarrow E\psi = -\frac{h}{i}\frac{1}{2\pi}\frac{\partial\psi}{\partial t}$$

$$\Rightarrow E\psi = -\frac{1}{i}\hbar\frac{\partial\psi}{\partial t}$$

$$\Rightarrow E\psi = i\hbar\frac{\partial\psi}{\partial t} \quad \text{—— (1)}$$

Putting the eq^n in time independent Schroedinger's eq^n

$$\frac{\partial^2\psi}{\partial x^2} + \frac{2m}{\hbar^2}(E-V)\psi = 0$$

$$\Rightarrow \frac{\hbar^2}{2m}\nabla^2\psi + (E-V)\psi = 0$$

$$\Rightarrow E\psi = \left(-\frac{\hbar^2\nabla^2}{2m} + V\right)\psi$$

$$\Rightarrow i\hbar\frac{\partial\psi}{\partial t} = \left(-\frac{\hbar^2\nabla^2}{2m} + V\right)\psi$$

$$\Rightarrow \boxed{E\psi = H\psi}$$

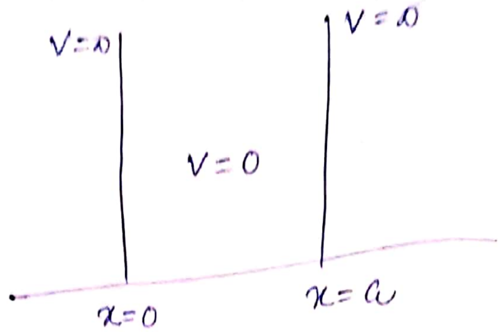
where $H \rightarrow$ Hamiltonian operator

$$H \rightarrow -\frac{\hbar^2\nabla^2}{2m} + V$$

Which is the time dependent Schroedinger's eq^n.

Application of Schroedinger's equation

Free particle and particle in a box



Let us consider a particle of mass m confined in one dimensional potential box of length a. Since the particle is moving between $x=0$ to $x=a$ along x-axis, so its potential energy is zero within the box. The particle does not exist outside the potential box. Hence the potential energy is infinity outside the box.

$$V(x) = \begin{cases} 0 & \text{for } 0 < x < a \\ \infty & \text{for } x < 0 \text{ and } x > a \end{cases}$$

According to Schroedinger's eq^n

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{2m}{\hbar^2}(E-V)\psi = 0 \quad \text{—(1)}$$

Since potential $v(x) = 0$ for $0 < x < a$.

Hence $$\frac{\partial^2 \psi}{\partial x^2} + \frac{2mE}{\hbar^2}\psi = 0 \quad \text{—(2)}$$

$$\Rightarrow \frac{\partial^2 \psi}{\partial x^2} + k^2\psi = 0 \quad \text{—(3)} \quad \text{where } k = \sqrt{\frac{2mE}{\hbar^2}}$$

It is a second order homogeneous linear differential eq^n.

whose solution is

$$\psi(x) = A\sin kx + B\cos kx \quad (4)$$

Wave function must be continuous at the boundary.

Hence $\psi(x) = 0$ at $x = 0$ & at $x = a$

Putting $\psi(x) = 0$ at $x = 0$

$$0 = 0 + B\cos 0$$

$$\Rightarrow B = 0$$

Hence $\psi(x) = A\sin kx \quad (5)$

Putting $\psi(x) = 0$ at $x = a$

$$0 = A\sin ka$$

$$\Rightarrow \sin ka = 0$$

$$\Rightarrow ka = n\pi$$

$$\Rightarrow \boxed{k = \frac{n\pi}{a}}$$

Hence the $\psi(x) = A\sin\left(\frac{n\pi x}{a}\right) \quad (6)$

To find the value of A, apply normalization condition i.e

$$\int_{-\infty}^{+\infty} \psi^*(x)\,\psi(x)\,dx = 1$$

$$\Rightarrow \int_0^a \psi^*(x)\,\psi(x)\,dx = 1$$

$$\Rightarrow \int_0^a A^2\sin^2\left(\frac{n\pi x}{a}\right)dx = 1$$

$$\Rightarrow \frac{A^2}{2}\int_0^a \left[1 - \cos\frac{2n\pi x}{a}\right] dx = 1$$

$$\Rightarrow \frac{A^2}{2}\int_0^a dx - \frac{A^2}{2}\int_0^a \cancel{\cos\left(\frac{2n\pi x}{a}\right) dx}^{\,0} = 1$$

$$\Rightarrow \frac{A^2}{2} a = 1$$

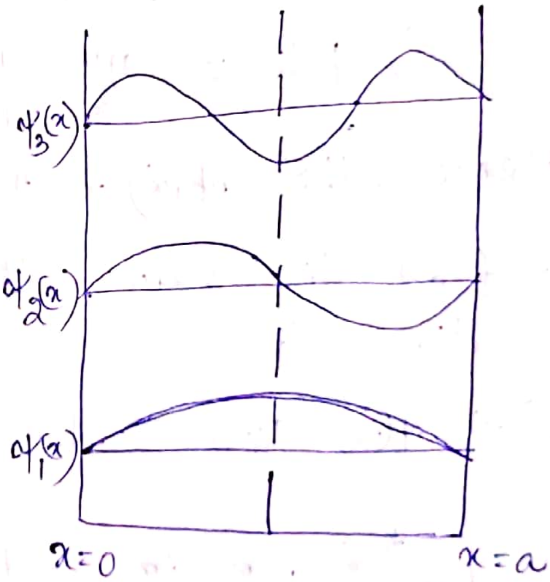
$$\Rightarrow A^2 = \frac{2}{a}$$

$$\Rightarrow A = \sqrt{2/a}$$

Hence $\psi_n(x) = \sqrt{\frac{2}{a}} \sin\left(\frac{n\pi x}{a}\right)$ —— (7)

The function $\psi_n(x)$ is given by eq^n (7) is known as eigen function.

The first three eigen functions are given in this figure.



The corresponding energy eigen values are

$$E = \frac{\hbar^2 k^2}{2m}$$

$$= \frac{\hbar^2}{2m}\frac{n^2\pi^2}{a^2}$$

$$E_n = \frac{n^2 h^2}{8ma^2}$$

for $n = 1$

$$E_1 = \frac{h^2}{8ma^2}$$

This is the lowest possible energy which is zero point energy.

Problem :-

Find the maximum kinetic energy of photo electrons ejected from a potassium surface for which $\phi_0 = 2.1ev$, by photons of wavelengths 2000 and 5000 A^0. What is the threshold frequency and corresponding wavelength?

Solⁿ :- $\phi_0 = 2.1 ev$

We know that $h\nu - \phi_0 = \frac{1}{2}mv^2$

For $\lambda = 2000A^0$. $\Rightarrow h\nu = \frac{hc}{\lambda} = \frac{6.62\times10^{-34}\times3\times10^8}{2000\times10^{-10}\times1.6\times10^{-19}}$

$= 6.204 ev$

$\Rightarrow \frac{1}{2}mv^2 = h\nu - \phi_0$

$= 6.204 ev - 2.1 ev = 4.104 ev$

Threshold frequency, $\nu_0 = \frac{\phi_0}{h} = \frac{2.1 ev}{6.62\times10^{-34}}$

$= 5.08\times10^{14}$ Hz

Corresponding wavelength $\lambda_0 = \frac{c}{\nu_0} = \frac{3\times10^8 m/s}{5.08\times10^{14} Hz} = 5900 \mathring{A}$

Problem :- An X-ray photon is found to have its wavelength doubled on being scattered through 90°. Calculate the wavelength and energy of incident photon.

Solⁿ :- According to Compton's Scattering

$$\lambda' - \lambda = \frac{h}{m_0c}(1 - \cos\theta)$$

Here $\theta = 90^\circ$, $\lambda' = 2\lambda$

Hence $2\lambda - \lambda = \frac{h}{m_0 c} \cancel{\cos 90})(1-\cos\theta)$

$$= \frac{h}{m_0 c}(1-\cos 90^\circ)$$

$$= \frac{h}{m_0 c}$$

$$= \frac{6.62 \times 10^{-34}}{9.1 \times 10^{-31} \times 3 \times 10^{8}} = 0.0245 \, A^\circ$$

Energy of incident photon $E = h\nu$

$$= \frac{hc}{\lambda}$$

$$= \frac{6.62 \times 10^{-34} \times 3 \times 10^{8}}{0.0245 \times 10^{-10}} \, J$$

Problem:-

Green light has a wavelength of about $5461 A^\circ$. Through what potential difference must an electron be accelerated to have this wavelength?

Solⁿ:- We know that K.E gained by electron is equal to electric potential energy.

Hence $\frac{1}{2}mv^2 = eV$

$$V = \frac{mv^2}{2e} = \frac{1}{2e} m\left(\frac{h}{m\lambda}\right)^2$$

$$= \frac{1}{2e}\frac{h^2}{m\lambda^2}$$

$$= \frac{(6.62 \times 10^{-34})^2}{2 \times 1.6 \times 10^{-19} \times 9.1 \times 10^{-31} \times (5461 \times 10^{-10})^2}$$

$$= 5.06 \times 10^{-6} \, V = 5 \mu V$$

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Problem :- An excited atom has an average life time of order 10^{-8} s. What is the minimum uncertainty in this frequency of photon?

Solⁿ:- We know that

$$\Delta E \cdot \Delta t = \frac{h}{4\pi}$$

$$\Rightarrow h\Delta\nu \cdot \Delta t = \frac{h}{4\pi}$$

$$\Rightarrow \Delta\nu = \frac{h}{4\pi \times h\Delta t} = \frac{1}{4\pi \times \Delta t} = \frac{1}{4 \times 3.14 \times 10^{-8}}$$

$$= 7.9 \times 10^{8}\,s$$

Problem:- A particle limited to the x-axis has the wave function $\phi(z) = bz$ betⁿ $z=0$ and $z=2$ the wave function $\phi(z) = 0$ else where

(a) Find the Probability that the particle can be found between $Z=0$ and $Z=0.5$

(b) Find the expectation value $\langle z \rangle$ of the particle position.

Solⁿ:- probability density function is given by

(a) $$\int_{z_1}^{z_2} |\phi|^2 \, dz = b^2 \int_0^{0.5} z^2 \, dz$$

$$= b^2 \left[\frac{z^3}{3}\right]_0^{0.5} = b^2 \left(\frac{0.125}{3}\right)$$

$$= b^2(0.042) = 0.042\, b^2$$

(b) Expectation value is given by

$$\langle x \rangle = \frac{b^2 \int_0^{0.5} z \cdot z^2 \, dz}{b^2 \int_0^{0.5} z^2 \, dz} = 3.72$$

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