MODULE-I

Kinematic fundamental: Basic Kinematic concepts and definitions, Degrees of freedom, Elementary Mechanism : Link, joint, Kinematic Pair, Classification of kinematic pairs, Kinematic chain and mechanism, Gru ebler's criterion, Inversion of mechanism, Grashof criteria, Four bar linkage and their inversions, Single slider crank mechanism, Double slider crank mechanism and their inversion. Transmission angle and toggle position, Mechanical advantage.

Kinematic Analysis : Graphical analysis of position, velocity and acceleration of four bar and Slider crank mechanisms. Instantaneous centre method, Aronhold-Kennedy Theorem, Rubbing velocity at a Pin-joint.Coriolis component of acceleration.

Simple Mechanisms

Features

5.1. Introduction

We have already discussed that a machine is a device which receives energy and transforms it into some useful work. A machine consists of a number of parts or bodies. In this chapter, we shall study the mechanisms of the various parts or bodies from which the machine is assembled. This is done by making one of the parts as fixed, and the relative motion of other parts is determined with respect to the fixed part.

5.2. Kinematic Link or Element

Each part of a machine, which moves relative to some other part, is known as a ***kinematic link*** (or simply link) or ***element***. A link may consist of several parts, which are rigidly fastened together, so that they do not move relative to one another. For example, in a reciprocating steam engine, as shown in Fig. 5.1, piston, piston rod and crosshead constitute one link ; connecting rod with big and small end bearings constitute a second link ; crank, crank shaft and flywheel a third link and the cylinder, engine frame and main bearings a fourth link.

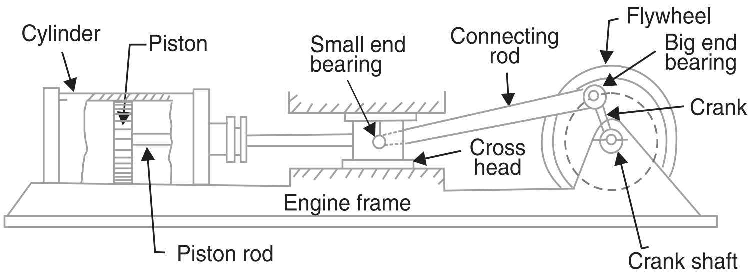
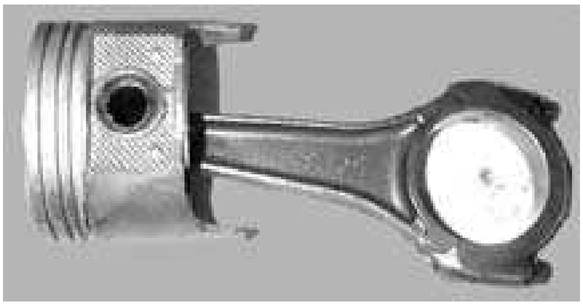


Fig. 5.1. Reciprocating steam engine.

A link or element need not to be a rigid body, but it must be a ***resistant body***. A body is said to be a resistant body if it is capable of transmitting the required forces with negligible deformation. Thus a link should have the following two characteristics:

1. It should have relative motion, and

2. It must be a resistant body.



Piston and piston rod of an IC engine.

5.3. Types of Links

In order to transmit motion, the driver and the follower may be connected by the following three types of links :

1. *Rigid link*. A rigid link is one which does not undergo any deformation while transmitting motion. Strictly speaking, rigid links do not exist. However, as the deformation of a connecting rod, crank etc. of a reciprocating steam engine is not appreciable, they can be considered as rigid links.

2. *Flexible link*. A flexible link is one which is partly deformed in a manner not to affect the transmission of motion. For example, belts, ropes, chains and wires are flexible links and transmit tensile forces only.

3. *Fluid link*. A fluid link is one which is formed by having a fluid in a receptacle and the motion is transmitted through the fluid by pressure or compression only, as in the case of hydraulic presses, jacks and brakes.

5.4. Structure

It is an assemblage of a number of resistant bodies (known as members) having no relative motion between them and meant for carrying loads having straining action. A railway bridge, a roof truss, machine frames etc., are the examples of a structure.

5.5. Difference Between a Machine and a Structure

The following differences between a machine and a structure are important from the subject point of view :

1. The parts of a machine move relative to one another, whereas the members of a structure do not move relative to one another.

2. A machine transforms the available energy into some useful work, whereas in a structure no energy is transformed into useful work.

3. The links of a machine may transmit both power and motion, while the members of a structure transmit forces only.

5.6. Kinematic Pair

The two links or elements of a machine, when in contact with each other, are said to form a pair. If the relative motion between them is completely or successfully constrained (*i.e.* in a definite direction), the pair is known as ***kinematic pair***.

First of all, let us discuss the various types of constrained motions.

5.7. Types of Constrained Motions

Following are the three types of constrained motions :

1. ***Completely constrained motion***. When the motion between a pair is limited to a definite direction irrespective of the direction of force applied, then the motion is said to be a completely constrained motion. For example, the piston and cylinder (in a steam engine) form a pair and the motion of the piston is limited to a definite direction (*i.e.* it will only reciprocate) relative to the cylinder irrespective of the direction of motion of the crank, as shown in Fig. 5.1.

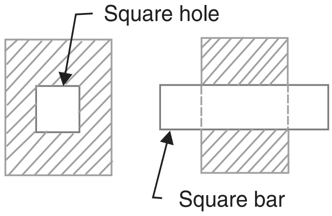


Fig. 5.2. Square bar in a square hole.

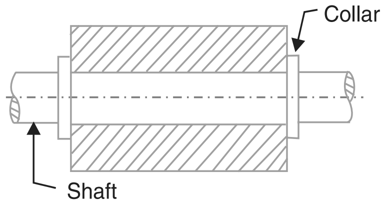


Fig. 5.3. Shaft with collars in a circular hole.

The motion of a square bar in a square hole, as shown in Fig. 5.2, and the motion of a shaft with collars at each end in a circular hole, as shown in Fig. 5.3, are also examples of completely constrained motion.

2. ***Incompletely constrained motion***. When the motion between a pair can take place in more than one direction, then the motion is called an incompletely constrained motion. The change in the direction of impressed force may alter the direction of relative motion between the pair. A circular bar or shaft in a circular hole, as shown in Fig. 5.4, is an example of an incompletely constrained motion as it may either rotate or slide in a hole. These both motions have no relationship with the other.

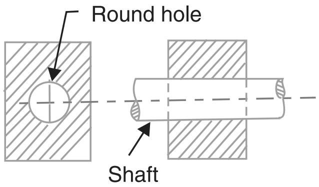


Fig. 5.4. Shaft in a circular hole.

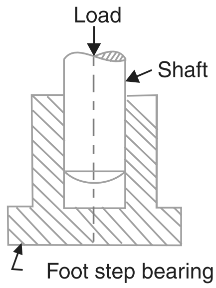


Fig. 5.5. Shaft in a foot step bearing.

3. ***Successfully constrained motion.*** When the motion between the elements, forming a pair,is such that the constrained motion is not completed by itself, but by some other means, then the motion is said to be successfully constrained motion. Consider a shaft in a foot-step bearing as shown in Fig. 5.5. The shaft may rotate in a bearing or it may move upwards. This is a case of incompletely constrained motion. But if the load is placed on the shaft to prevent axial upward movement of the shaft, then the motion of the pair is said to be successfully constrained motion. The motion of an I.C. engine valve (these are kept on their seat by a spring) and the piston reciprocating inside an engine cylinder are also the examples of successfully constrained motion.

5.8. Classification of Kinematic Pairs

The kinematic pairs may be classified according to the following considerations :

1. ***According to the type of relative motion between the elements.*** The kinematic pairs according to type of relative motion between the elements may be classified as discussed below:

(*a*) ***Sliding pair.*** When the two elements of a pair are connected in such a way that one can only slide relative to the other, the pair is known as a sliding pair. The piston and cylinder, cross-head and guides of a reciprocating steam engine, ram and its guides in shaper, tail stock on the lathe bed etc. are the examples of a sliding pair. A little consideration will show, that a sliding pair has a completely constrained motion.

(*b*) ***Turning pair***. When the two elements of a pair are connected in such a way that one can only turn or revolve about a fixed axis of another link, the pair is known as turning pair. A shaft with collars at both ends fitted into a circular hole, the crankshaft in a journal bearing in an engine, lathe spindle supported in head stock, cycle wheels turning over their axles etc. are the examples of a turning pair. A turning pair also has a completely constrained motion.

(*c*) ***Rolling pair***. When the two elements of a pair are connected in such a way that one rolls over another fixed link, the pair is known as rolling pair. Ball and roller bearings are examples of rolling pair.

(*d*) ***Screw pair.*** When the two elements of a pair are connected in such a way that one element can turn about the other by screw threads, the pair is known as screw pair. The lead screw of a lathe with nut, and bolt with a nut are examples of a screw pair.

(*e*) ***Spherical pair***. When the two elements of a pair are connected in such a way that one element (with spherical shape) turns or swivels about the other fixed element, the pair formed is called a spherical pair. The ball and socket joint, attachment of a car mirror, pen stand etc., are the examples of a spherical pair.

2. ***According to the type of contact between the elements***. The kinematic pairs according to the type of contact between the elements may be classified as discussed below :

(*a*) ***Lower pair***. When the two elements of a pair have a surface contact when relative motion takes place and the surface of one element slides over the surface of the other, the pair formed is known as lower pair. It will be seen that sliding pairs, turning pairs and screw pairs form lower pairs.

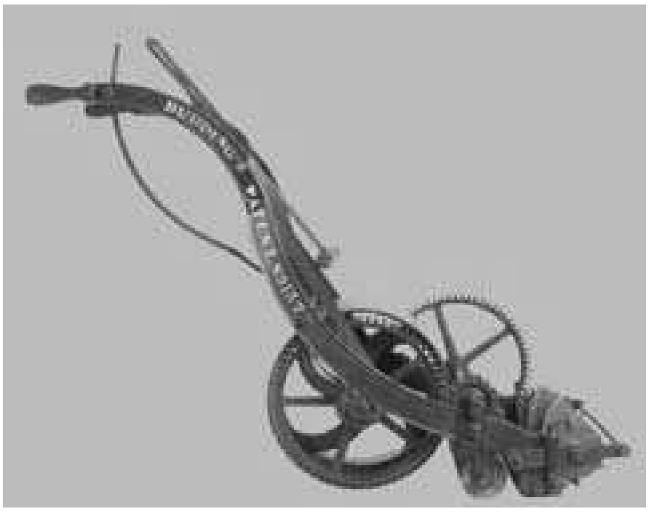
(*b*) ***Higher pair***. When the two elements of a pair have a line or point contact when relative motion takes place and the motion between the two elements is partly turning and partly sliding,then the pair is known as higher pair. A pair of friction discs, toothed gearing, belt and rope drives, ball and roller bearings and cam and follower are the examples of higher pairs.

3. ***According to the type of closure***. The kinematic pairs according to the type of closure between the elements may be classified as discussed below :

(*a*) ***Self closed pair***. When the two elements of a pair are connected together mechanically in such a way that only required kind of relative motion occurs, it is then known as self closed pair. The lower pairs are self closed pair.

(*b*) ***Force - closed pair***. When the two elements of a pair are not connected mechanically but are kept in contact by the action of external forces, the pair is said to be a force-closed pair. The cam and follower is an example of force closed pair, as it is kept in contact by the forces exerted by spring and gravity.

5.9. Kinematic Chain



Lawn-mover is a combination of kinematic links.

When the kinematic pairs are coupled in such a way that the last link is joined to the first link to transmit definite motion (*i.e.* completely or successfully constrained motion), it is called a ***kinematic chain.*** In other words, a kinematic chain may be defined as a combination of kinematic pairs, joined in such a way that each link forms a part of two pairs and the relative motion between the links or elements is completely or successfully constrained. For example, the crankshaft of an engine forms a kinematic pair with the bearings which are fixed in a pair, the connecting rod with the crank forms a second kinematic pair, the piston with the connecting rod forms a third pair and the piston with the cylinder forms a fourth pair. The total combination of these links is a kinematic chain.

If each link is assumed to form two pairs with two adjacent links, then the relation between the number of pairs (p) forming a kinematic chain and the number of links (l) may be expressed in the form of an equation :

$$l = 2p - 4 \qquad \ldots (i)$$

Since in a kinematic chain each link forms a part of two pairs, therefore there will be as many links as the number of pairs.

Another relation between the number of links (l) and the number of joints (j) which constitute a kinematic chain is given by the expression :

$$j = \frac{3}{2}l - 2 \qquad \ldots (ii)$$

The equations (*i*) and (*ii*) are applicable only to kinematic chains, in which lower pairs are used. These equations may also be applied to kinematic chains, in which higher pairs are used. In that case each higher pair may be taken as equivalent to two lower pairs with an additional element or link.

Let us apply the above equations to the following cases to determine whether each of them is a kinematic chain or not.

1. Consider the arrangement of three links AB, BC and CA with pin joints at A, B and C as shown in Fig. 5.6. In this case,

Number of links, $l = 3$

Number of pairs, $p = 3$

and number of joints, $j = 3$

From equation (*i*), $l = 2p - 4$

or $3 = 2 \times 3 - 4 = 2$

i.e. L.H.S. > R.H.S.

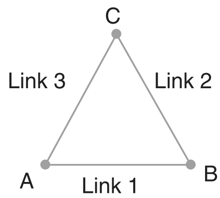


Fig. 5.6. Arrangement of three links.

Now from equation (*ii*),

$$j = \frac{3}{2}l - 2 \quad \text{or} \quad 3 = \frac{3}{2} \times 3 - 2 = 2.5$$

i.e. L.H.S. > R.H.S.

Since the arrangement of three links, as shown in Fig. 5.6, does not satisfy the equations (*i*) and (*ii*) and the left hand side is greater than the right hand side, therefore it is not a kinematic chain and hence no relative motion is possible. Such type of chain is called ***locked chain*** and forms a rigid frame or structure which is used in bridges and trusses.

2. Consider the arrangement of four links AB, BC, CD and DA as shown in Fig. 5.7. In this case

$$l = 4,\ p = 4, \text{ and } j = 4$$

From equation (*i*),

$$l = 2p - 4$$

$$4 = 2 \times 4 - 4 = 4$$

i.e. L.H.S. = R.H.S.

From equation (*ii*),

$$j = \frac{3}{2}l - 2$$

$$4 = \frac{3}{2} \times 4 - 2 = 4$$

i.e. L.H.S. = R.H.S.

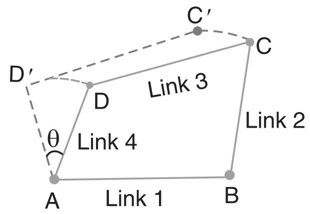


Fig. 5.7. Arrangement of four links.

Since the arrangement of four links, as shown in Fig. 5.7, satisfy the equations (*i*) and (*ii*), therefore it is a ***kinematic chain of one degree of freedom***.

A chain in which a single link such as AD in Fig. 5.7 is sufficient to define the position of all other links, it is then called a kinematic chain of one degree of freedom.

A little consideration will show that in Fig. 5.7, if a definite displacement (say θ) is given to the link AD, keeping the link AB fixed, then the resulting displacements of the remaining two links BC and CD are also perfectly definite. Thus we see that in a four bar chain, the relative motion is completely constrained. Hence it may be called as a ***constrained kinematic chain***, and it is the basis of all machines.

3. Consider an arrangement of five links, as shown in Fig. 5.8. In this case,

$$l = 5,\ p = 5, \text{ and } j = 5$$

From equation (*i*),

$$l = 2p - 4 \quad \text{or} \quad 5 = 2 \times 5 - 4 = 6$$

i.e. L.H.S. < R.H.S.

From equation (*ii*),

$$j = \frac{3}{2}l - 2 \quad \text{or} \quad 5 = \frac{3}{2} \times 5 - 2 = 5.5$$

i.e. L.H.S. < R.H.S.

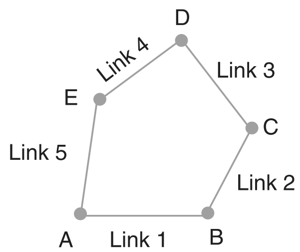


Fig. 5.8. Arrangement of five links.

Since the arrangement of five links, as shown in Fig. 5.8 does not satisfy the equations and left hand side is less than right hand side, therefore it is not a kinematic chain. Such a type of chain is called ***unconstrained chain*** *i.e.* the relative motion is not completely constrained. This type of chain is of little practical importance.

4. Consider an arrangement of six links, as shown in Fig. 5.9. This chain is formed by adding two more links in such a way that these two links form a pair with the existing links as well as form themselves a pair. In this case

$$l = 6,\ p = 5, \text{ and } j = 7$$

From equation (*i*),

$$l = 2p - 4 \quad \text{or} \quad 6 = 2 \times 5 - 4 = 6$$

i.e. L.H.S. = R.H.S.

From equation (*ii*),

$$j = \frac{3}{2}l - 2 \quad \text{or} \quad 7 = \frac{3}{2} \times 6 - 2 = 7$$

i.e. L.H.S. = R.H.S.

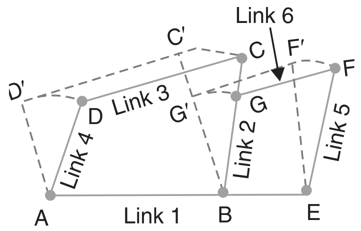


Fig. 5.9. Arrangement of six links.

Since the arrangement of six links, as shown in Fig. 5.9, satisfies the equations (*i.e.* left hand side is equal to right hand side), therefore it is a kinematic chain.

Note : A chain having more than four links is known as **compound kinematic chain.**

5.10. Types of Joints in a Chain

The following types of joints are usually found in a chain :

1. *Binary joint*. When two links are joined at the same connection, the joint is known as binary joint. For example, a chain as shown in Fig. 5.10, has four links and four binary joins at A, B, C and D.

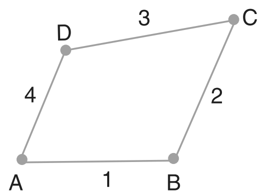


Fig. 5.10. Kinematic chain with all binary joints.

In order to determine the nature of chain, *i.e.* whether the chain is a locked chain (or structure) or kinematic chain or unconstrained chain, the following relation between the number of links and the number of binary joints, as given by A.W. Klein, may be used :

$$j + \frac{h}{2} = \frac{3}{2}l - 2 \qquad \ldots (i)$$

where j = Number of binary joints,

h = Number of higher pairs, and

l = Number of links.

When $h = 0$, the equation (*i*), may be written as

$$j = \frac{3}{2}l - 2 \qquad \ldots (ii)$$

Applying this equation to a chain, as shown in Fig. 5.10, where $l = 4$ and $j = 4$, we have

$$4 = \frac{3}{2} \times 4 - 2 = 4$$

Since the left hand side is equal to the right hand side, therefore the chain is a kinematic chain or constrained chain.

2. *Ternary joint*. When three links are joined at the same connection, the joint is known as ternary joint. It is equivalent to two binary joints as one of the three links joined carry the pin for the other two links. For example, a chain, as shown in Fig. 5.11, has six links. It has three binary joints at A, B and D and two ternary joints at C and E. Since one ternary joint is equivalent to two binary joints, therefore equivalent binary joints in a chain, as shown in Fig. 5.11, are $3 + 2 \times 2 = 7$

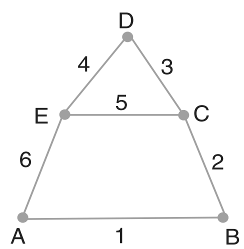


Fig. 5.11. Kinematic chain having binary and ternary joints.

Let us now determine whether this chain is a kinematic chain or not. We know that $l = 6$ and $j = 7$, therefore from equation (*ii*),

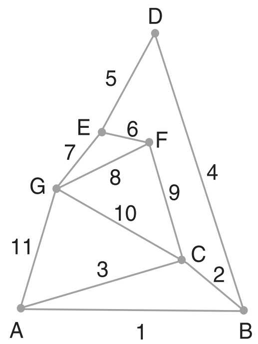
$$j = \frac{3}{2} l - 2$$

or $$7 = \frac{3}{2} \times 6 - 2 = 7$$

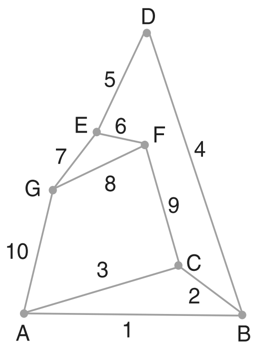
Since left hand side is equal to right hand side, therefore the chain, as shown in Fig. 5.11, is a kinematic chain or constrained chain.

3. *Quaternary joint.* When four links are joined at the same connection, the joint is called a quaternary joint. It is equivalent to three binary joints. In general, when l number of links are joined at the same connection, the joint is equivalent to $(l - 1)$ binary joints.

For example consider a chain having eleven links, as shown in Fig. 5.12 (*a*). It has one binary joint at *D*, four ternary joints at *A, B, E* and *F*, and two quaternary joints at *C* and *G*. Since one quaternary joint is equivalent to three binary joints and one ternary joint is equal to two binary joints, therefore total number of binary joints in a chain, as shown in Fig. 5.12 (*a*), are



(*a*) Looked chain having binary, ternary and quaternary joints.



(*b*) Kinematic chain having binary and ternary joints.

Fig. 5.12

$$1 + 4 \times 2 + 2 \times 3 = 15$$

Let us now determine whether the chain, as shown in Fig. 5.12 (*a*), is a kinematic chain or not. We know that $l = 11$ and $j = 15$. We know that,

$$j = \frac{3}{2} l - 2, \quad \text{or} \quad 15 = \frac{3}{2} \times 11 - 2 = 14.5, \text{ i.e., L.H.S.} > \text{R.H.S.}$$

Since the left hand side is greater than right hand side, therefore the chain, as shown in Fig. 5.12 (*a*) , is not a kinematic chain. We have discussed in Art 5.9 , that such a type of chain is called locked chain and forms a rigid frame or structure.

If the link *CG* is removed, as shown in Fig. 5.12 (*b*), it has ten links and has one binary joint at *D* and six ternary joints at *A, B, C, E, F* and *G*.

Therefore total number of binary joints are $1 + 2 \times 6 = 13$. We know that

$$j = \frac{3}{2} l - 2, \quad \text{or} \quad 13 = \frac{3}{2} \times 10 - 2 = 13, \text{ i.e. L.H.S.} = \text{R.H.S.}$$

Since left hand side is equal to right hand side, therefore the chain, as shown in Fig. 5.12 (*b*), is a kinematic chain or constrained chain.

5.11. Mechanism

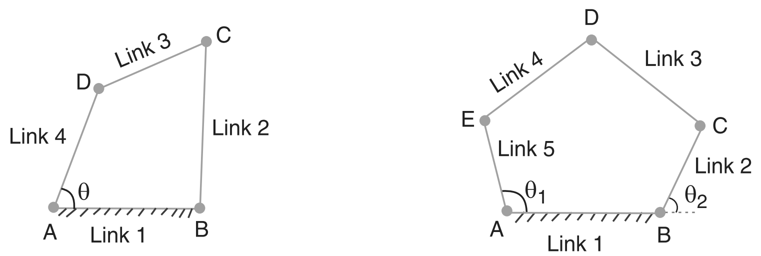
When one of the links of a kinematic chain is fixed, the chain is known as ***mechanism***. It may be used for transmitting or transforming motion *e.g.* engine indicators, typewriter etc.

A mechanism with four links is known as ***simple mechanism***, and the mechanism with more than four links is known as ***compound mechanism***. When a mechanism is required to transmit power or to do some particular type of work, it then becomes a ***machine***. In such cases, the various links or elements have to be designed to withstand the forces (both static and kinetic) safely.

A little consideration will show that a mechanism may be regarded as a machine in which each part is reduced to the simplest form to transmit the required motion.

5.12. Number of Degrees of Freedom for Plane Mechanisms

In the design or analysis of a mechanism, one of the most important concern is the number of degrees of freedom (also called movability) of the mechanism. It is defined as the number of input parameters (usually pair variables) which must be independently controlled in order to bring the mechanism into a useful engineering purpose. It is possible to determine the number of degrees of freedom of a mechanism directly from the number of links and the number and types of joints which it includes.

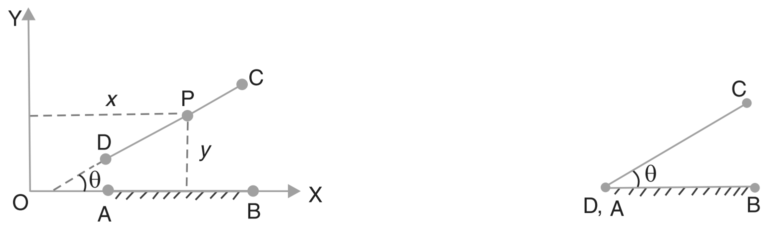


(*a*) Four bar chain. (*b*) Five bar chain.

Fig. 5.13

Consider a four bar chain, as shown in Fig. 5.13 (*a*). A little consideration will show that only one variable such as θ is needed to define the relative positions of all the links. In other words, we say that the number of degrees of freedom of a four bar chain is one. Now, let us consider a five bar chain, as shown in Fig. 5.13 (*b*). In this case two variables such as θ_1 and θ_2 are needed to define completely the relative positions of all the links. Thus, we say that the number of degrees of freedom is * two.

In order to develop the relationship in general, consider two links *AB* and *CD* in a plane motion as shown in Fig. 5.14 (*a*).



(*a*) (*b*)

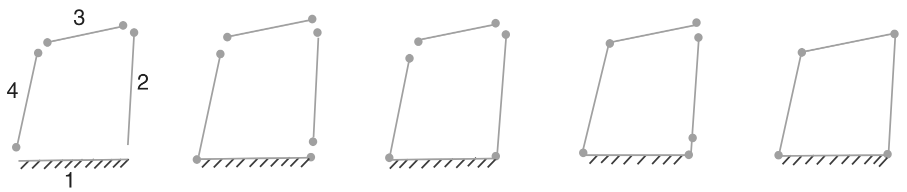
Fig. 5.14. Links in a plane motion.

The link *AB* with co-ordinate system *OXY* is taken as the reference link (or fixed link). The position of point *P* on the moving link *CD* can be completely specified by the three variables, *i.e.* the

* The differential of an automobile requires that the angular velocity of two elements be fixed in order to know the velocity of the remaining elements. The differential mechanism is thus said to have two degrees of freedom. Many computing mechanisms have two or more degrees of freedom.

co-ordinates of the point P denoted by x and y and the inclination θ of the link CD with X-axis or link AB. In other words, we can say that each link of a mechanism has three degrees of freedom before it is connected to any other link. But when the link CD is connected to the link AB by a turning pair at A, as shown in Fig. 5.14 (*b*), the position of link CD is now determined by a single variable θ and thus has one degree of freedom.

From above, we see that when a link is connected to a fixed link by a turning pair (*i.e.* lower pair), two degrees of freedom are destroyed. This may be clearly understood from Fig. 5.15, in which the resulting four bar mechanism has one degree of freedom (*i.e.* $n = 1$).



(*a*) $n = 9$ (*b*) $n = 7$ (*c*) $n = 5$ (*d*) $n = 3$ (*e*) $n = 1$

Fig. 5.15. Four bar mechanism.

Now let us consider a plane mechanism with l number of links. Since in a mechanism, one of the links is to be fixed, therefore the number of movable links will be $(l - 1)$ and thus the total number of degrees of freedom will be $3 (l - 1)$ before they are connected to any other link. In general, a mechanism with l number of links connected by j number of binary joints or lower pairs (*i.e.* single degree of freedom pairs) and h number of higher pairs (*i.e.* two degree of freedom pairs), then the number of degrees of freedom of a mechanism is given by

$$n = 3 (l - 1) - 2j - h \qquad \text{... } (i)$$

This equation is called Kutzbach criterion for the movability of a mechanism having plane motion.

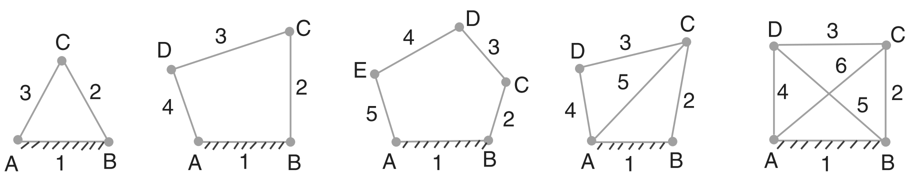
If there are no two degree of freedom pairs (*i.e.* higher pairs), then $h = 0$. Substituting $h = 0$ in equation (*i*), we have

$$n = 3 (l - 1) - 2j \qquad \text{... } (ii)$$

5.13. Application of Kutzbach Criterion to Plane Mechanisms

We have discussed in the previous article that Kutzbach criterion for determining the number of degrees of freedom or movability (n) of a plane mechanism is

$$n = 3 (l - 1) - 2j - h$$



(*a*) Three-bar mechanism. (*b*) Four bar mechanism. (*c*) Five bar mechanism. (*d*) Five bar mechanism. (*e*) Six bar mechanism.

Fig. 5.16. Plane mechanisms.

The number of degrees of freedom or movability (n) for some simple mechanisms having no higher pair (*i.e.* $h = 0$), as shown in Fig. 5.16, are determined as follows :

1. The mechanism, as shown in Fig. 5.16 (*a*), has three links and three binary joints, *i.e.* $l = 3$ and $j = 3$.

$$\therefore \quad n = 3\,(3 - 1) - 2 \times 3 = 0$$

2. The mechanism, as shown in Fig. 5.16 (*b*), has four links and four binary joints, *i.e.* $l = 4$ and $j = 4$.

$$\therefore \quad n = 3\,(4 - 1) - 2 \times 4 = 1$$

3. The mechanism, as shown in Fig. 5.16 (*c*), has five links and five binary joints, *i.e.* $l = 5$, and $j = 5$.

$$\therefore \quad n = 3\,(5 - 1) - 2 \times 5 = 2$$

4. The mechanism, as shown in Fig. 5.16 (*d*), has five links and six equivalent binary joints (because there are two binary joints at *B* and *D*, and two ternary joints at *A* and *C*), *i.e.* $l = 5$ and $j = 6$.

$$\therefore \quad n = 3\,(5 - 1) - 2 \times 6 = 0$$

5. The mechanism, as shown in Fig. 5.16 (*e*), has six links and eight equivalent binary joints (because there are four ternary joints at *A, B, C* and *D*), *i.e.* $l = 6$ and $j = 8$.

$$\therefore \quad n = 3\,(6 - 1) - 2 \times 8 = -1$$

It may be noted that

(*a*) When $n = 0$, then the mechanism forms a structure and no relative motion between the links is possible, as shown in Fig. 5.16 (*a*) and (*d*).

(*b*) When $n = 1$, then the mechanism can be driven by a single input motion, as shown in Fig. 5.16 (*b*).

(*c*) When $n = 2$, then two separate input motions are necessary to produce constrained motion for the mechanism, as shown in Fig. 5.16 (*c*).

(*d*) When $n = -1$ or less, then there are redundant constraints in the chain and it forms a statically indeterminate structure, as shown in Fig. 5.16 (*e*).

The application of Kutzbach's criterion applied to mechanisms with a higher pair or two degree of freedom joints is shown in Fig. 5.17.

Fig. 5.17. Mechanism with a higher pair.

In Fig. 5.17 (*a*), there are three links, two binary joints and one higher pair, *i.e.* $l = 3$, $j = 2$ and $h = 1$.

$$\therefore \quad n = 3\,(3 - 1) - 2 \times 2 - 1 = 1$$

In Fig. 5.17 (*b*), there are four links, three binary joints and one higher pair, *i.e.* $l = 4$, $j = 3$ and $h = 1$

$$\therefore \quad n = 3\,(4 - 1) - 2 \times 3 - 1 = 2$$

Here it has been assumed that the slipping is possible between the links (*i.e.* between the wheel and the fixed link). However if the friction at the contact is high enough to prevent slipping, the joint will be counted as one degree of freedom pair, because only one relative motion will be possible between the links.

5.14. Grubler's Criterion for Plane Mechanisms

The Grubler's criterion applies to mechanisms with only single degree of freedom joints where the overall movability of the mechanism is unity. Substituting $n = 1$ and $h = 0$ in Kutzbach equation, we have

$$1 = 3\,(l - 1) - 2\,j \qquad \text{or} \qquad 3l - 2j - 4 = 0$$

This equation is known as the Grubler's criterion for plane mechanisms with constrained motion.

A little consideration will show that a plane mechanism with a movability of 1 and only single degree of freedom joints can not have odd number of links. The simplest possible machanisms of this type are a four bar mechanism and a slider-crank mechanism in which $l = 4$ and $j = 4$.

5.15. Inversion of Mechanism

We have already discussed that when one of links is fixed in a kinematic chain, it is called a mechanism. So we can obtain as many mechanisms as the number of links in a kinematic chain by fixing, in turn, different links in a kinematic chain. This method of obtaining different mechanisms by fixing different links in a kinematic chain, is known as ***inversion of the mechanism***.

It may be noted that the relative motions between the various links is not changed in any manner through the process of inversion, but their absolute motions (those measured with respect to the fixed link) may be changed drastically.

Note: The part of a mechanism which initially moves with respect to the frame or fixed link is called ***driver*** and that part of the mechanism to which motion is transmitted is called ***follower***. Most of the mechanisms are reversible, so that same link can play the role of a driver and follower at different times. For example, in a reciprocating steam engine, the piston is the driver and flywheel is a follower while in a reciprocating air compressor, the flywheel is a driver.

5.16. Types of Kinematic Chains

The most important kinematic chains are those which consist of four lower pairs, each pair being a sliding pair or a turning pair. The following three types of kinematic chains with four lower pairs are important from the subject point of view :

1. Four bar chain or quadric cyclic chain,

2. Single slider crank chain, and

3. Double slider crank chain.

These kinematic chains are discussed, in detail, in the following articles.

5.17. Four Bar Chain or Quadric Cycle Chain

We have already discussed that the kinematic chain is a combination of four or more kinematic pairs, such that the relative motion between the links or elements is completely constrained. The simplest and the basic kinematic chain is a four bar chain or quadric cycle chain, as shown in Fig. 5.18. It consists of four links, each of them forms a turning pair at *A, B, C* and *D*. The four links may be of different lengths. According to **Grashof 's law** for a four bar mechanism, the sum of the shortest and longest link lengths should not be greater than the sum of the remaining two link lengths if there is to be continuous relative motion between the two links.

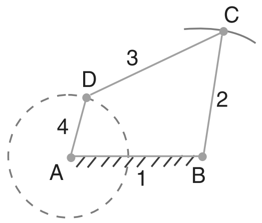
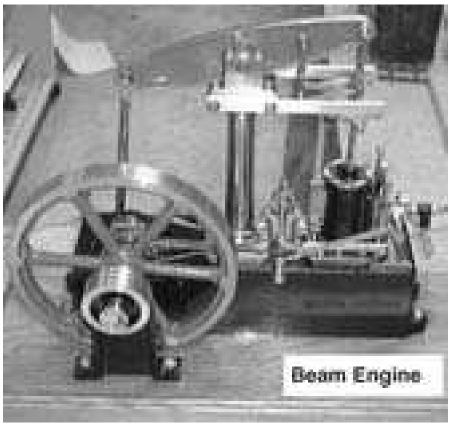


Fig. 5.18. Four bar chain.

A very important consideration in designing a mechanism is to ensure that the input crank makes a complete revolution relative to the other links. The mechanism in which no link makes a complete revolution will not be useful. In a four bar chain, one of the links, in particular the shortest link, will make a complete revolution relative to the other three links, if it satisfies the Grashof's law. Such a link is known as ***crank*** or ***driver***. In Fig. 5.18, AD (link 4) is a crank. The link BC (link 2) which makes a partial rotation or oscillates is known as ***lever*** or ***rocker*** or ***follower*** and the link CD (link 3) which connects the crank and lever is called ***connecting rod*** or ***coupler.*** The fixed link AB (link 1) is known as ***frame*** of the mechanism.

When the crank (link 4) is the driver, the mechanism is transforming rotary motion into oscillating motion.



5.18. Inversions of Four Bar Chain

Though there are many inversions of the four bar chain, yet the following are important from the subject point of view :

1. ***Beam engine*** **(*crank and lever mechanism*).** A part of the mechanism of a beam engine (also known as crank and lever mechanism) which consists of four links, is shown in Fig. 5.19. In this mechanism, when the crank rotates about the fixed centre A, the lever oscillates about a fixed centre D. The end E of the lever CDE is connected to a piston rod which reciprocates due to the rotation of the crank. In other words, the purpose of this mechanism is to convert rotary motion into reciprocating motion.

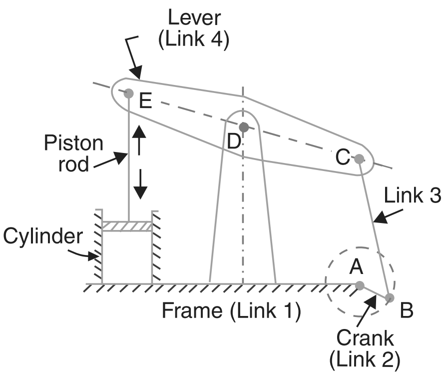


Fig. 5.19. Beam engine.

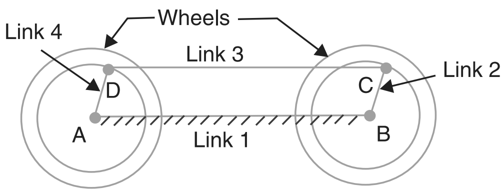


Fig. 5.20. Coupling rod of a locomotive.

2. ***Coupling rod of a locomotive*** **(*Double crank mechanism*).** The mechanism of a coupling rod of a locomotive (also known as double crank mechanism) which consists of four links, is shown in Fig. 5.20.

In this mechanism, the links AD and BC (having equal length) act as cranks and are connected to the respective wheels. The link CD acts as a coupling rod and the link AB is fixed in order to maintain a constant centre to centre distance between them. This mechanism is meant for transmitting rotary motion from one wheel to the other wheel.

3. ***Watt's indicator mechanism*** **(*Double lever mechanism*).** A *Watt's indicator mechanism (also known as Watt's straight line mechanism or double lever mechanism) which consists of four

* Refer Chapter 9, Art. 9.6

links, is shown in Fig. 5.21. The four links are : fixed link at A, link AC, link CE and link BFD. It may be noted that BF and FD form one link because these two parts have no relative motion between them. The links CE and BFD act as levers. The displacement of the link BFD is directly proportional to the pressure of gas or steam which acts on the indicator plunger. On any small displacement of the mechanism, the tracing point E at the end of the link CE traces out approximately a straight line.

The initial position of the mechanism is shown in Fig. 5.21 by full lines whereas the dotted lines show the position of the mechanism when the gas or steam pressure acts on the indicator plunger.

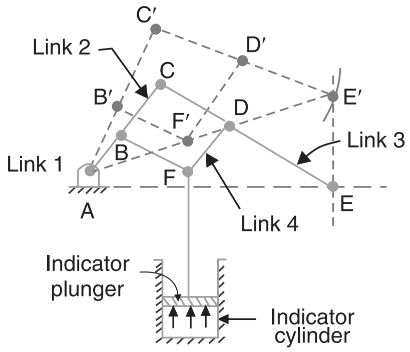


Fig. 5.21. Watt's indicator mechanism.

5.19. Single Slider Crank Chain

A single slider crank chain is a modification of the basic four bar chain. It consist of one sliding pair and three turning pairs. It is,usually, found in reciprocating steam engine mechanism. This type of mechanism converts rotary motion into reciprocating motion and vice versa.

In a single slider crank chain, as shown in Fig. 5.22, the links 1 and 2, links 2 and 3, and links 3 and 4 form three turning pairs while the links 4 and 1 form a sliding pair.

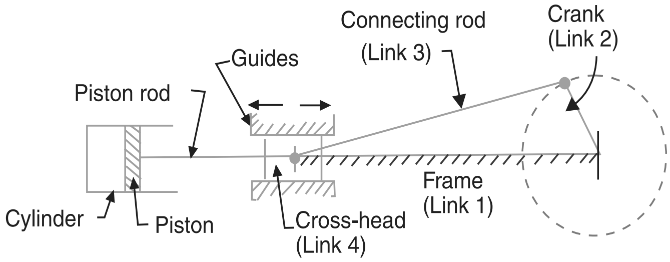


Fig. 5.22. Single slider crank chain.

The link 1 corresponds to the frame of the engine, which is fixed. The link 2 corresponds to the crank ; link 3 corresponds to the connecting rod and link 4 corresponds to cross-head. As the crank rotates, the cross-head reciprocates in the guides and thus the piston reciprocates in the cylinder.

5.20. Inversions of Single Slider Crank Chain

We have seen in the previous article that a single slider crank chain is a four-link mechanism. We know that by fixing, in turn, different links in a kinematic chain, an inversion is obtained and we can obtain as many mechanisms as the links in a kinematic chain. It is thus obvious, that four inversions of a single slider crank chain are possible. These inversions are found in the following mechanisms.

1. ***Pendulum pump or Bull engine***. In this mechanism, the inversion is obtained by fixing the cylinder or link 4 (*i.e.* sliding pair), as shown in Fig. 5.23. In this case, when the crank (link 2) rotates, the connecting rod (link 3) oscillates about a pin pivoted to the fixed link 4 at A and the piston attached to the piston rod (link 1) reciprocates. The duplex pump which is used to supply feed water to boilers have two pistons attached to link 1, as shown in Fig. 5.23.

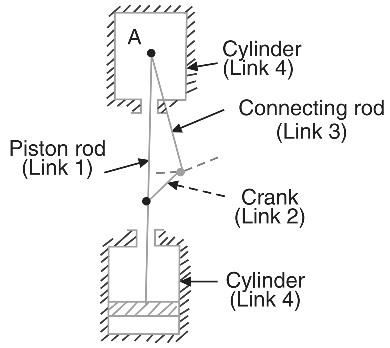


Fig. 5.23. Pendulum pump.

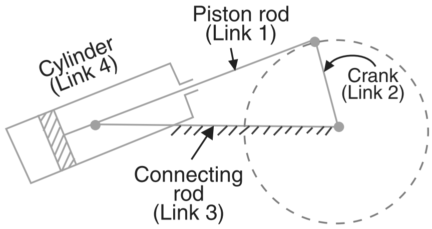


Fig. 5.24. Oscillating cylinder engine.

2. ***Oscillating cylinder engine*****.** The arrangement of oscillating cylinder engine mechanism, as shown in Fig. 5.24, is used to convert reciprocating motion into rotary motion. In this mechanism, the link 3 forming the turning pair is fixed. The link 3 corresponds to the connecting rod of a reciprocating steam engine mechanism. When the crank (link 2) rotates, the piston attached to piston rod (link 1) reciprocates and the cylinder (link 4) oscillates about a pin pivoted to the fixed link at *A*.

Rotary engine

3. ***Rotary internal combustion engine or Gnome engine*****.** Sometimes back, rotary internal combustion engines were used in aviation. But now-a-days gas turbines are used in its place. It consists of seven cylinders in one plane and all revolves about fixed centre *D*, as shown in Fig. 5.25, while the crank (link 2) is fixed. In this mechanism, when the connecting rod (link 4) rotates, the piston (link 3) reciprocates inside the cylinders forming link 1.

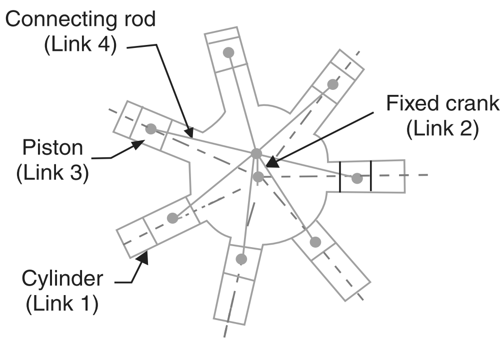


Fig. 5.25. Rotary internal combustion engine.

4. ***Crank and slotted lever quick return motion mechanism*****.** This mechanism is mostly used in shaping machines, slotting machines and in rotary internal combustion engines.

In this mechanism, the link AC (*i.e.* link 3) forming the turning pair is fixed, as shown in Fig. 5.26. The link 3 corresponds to the connecting rod of a reciprocating steam engine. The driving crank CB revolves with uniform angular speed about the fixed centre C. A sliding block attached to the crank pin at B slides along the slotted bar AP and thus causes AP to oscillate about the pivoted point A. A short link PR transmits the motion from AP to the ram which carries the tool and reciprocates along the line of stroke R_1R_2. The line of stroke of the ram (*i.e.* R_1R_2) is perpendicular to AC produced.

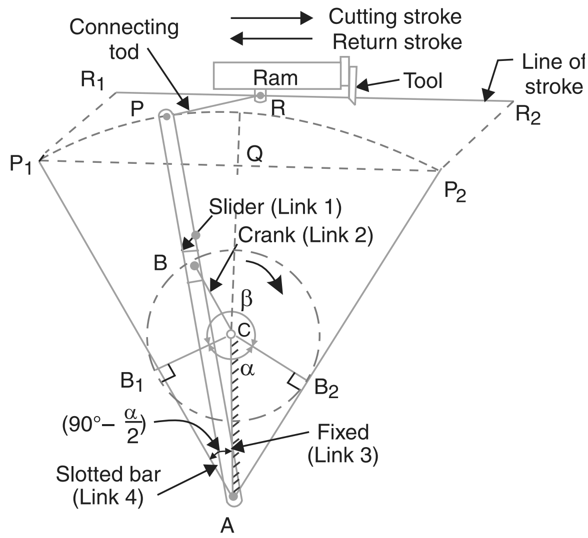
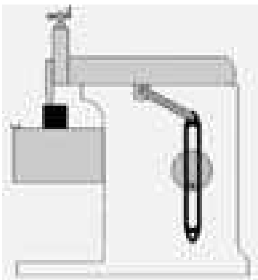


Fig. 5.26. Crank and slotted lever quick return motion mechanism.

In the extreme positions, AP_1 and AP_2 are tangential to the circle and the cutting tool is at the end of the stroke. The forward or cutting stroke occurs when the crank rotates from the position CB_1 to CB_2 (or through an angle β) in the clockwise direction. The return stroke occurs when the crank rotates from the position CB_2 to CB_1 (or through angle α) in the clockwise direction. Since the crank has uniform angular speed, therefore,



The Shaping Machine

$$\frac{\text{Time of cutting stroke}}{\text{Time of return stroke}} = \frac{\beta}{\alpha} = \frac{\beta}{360° - \beta} \quad \text{or} \quad \frac{360° - \alpha}{\alpha}$$

Since the tool travels a distance of $R_1 R_2$ during cutting and return stroke, therefore travel of the tool or length of stroke

$$= R_1R_2 = P_1P_2 = 2P_1Q = 2AP_1 \sin \angle P_1AQ$$

$$= 2AP_1 \sin\left(90° - \frac{\alpha}{2}\right) = 2AP \cos\frac{\alpha}{2} \qquad \text{..... } (\because AP_1 = AP)$$

$$= 2AP \times \frac{CB_1}{AC} \qquad \text{... } \left(\because \cos\frac{\alpha}{2} = \frac{CB_1}{AC}\right)$$

$$= 2AP \times \frac{CB}{AC} \qquad \text{... } (\because CB_1 = CB)$$

Note: From Fig. 5.26, we see that the angle β made by the forward or cutting stroke is greater than the angle α described by the return stroke. Since the crank rotates with uniform angular speed, therefore the return stroke is completed within shorter time. Thus it is called quick return motion mechanism.

5. *Whitworth quick return motion mechanism.* This mechanism is mostly used in shaping and slotting machines. In this mechanism, the link *CD* (link 2) forming the turning pair is fixed, as shown in Fig. 5.27. The link 2 corresponds to a crank in a reciprocating steam engine. The driving crank *CA* (link 3) rotates at a uniform angular speed. The slider (link 4) attached to the crank pin at *A* slides along the slotted bar *PA* (link 1) which oscillates at a pivoted point *D*. The connecting rod *PR* carries the ram at *R* to which a cutting tool is fixed. The motion of the tool is constrained along the line *RD* produced, *i.e.* along a line passing through *D* and perpendicular to *CD*.

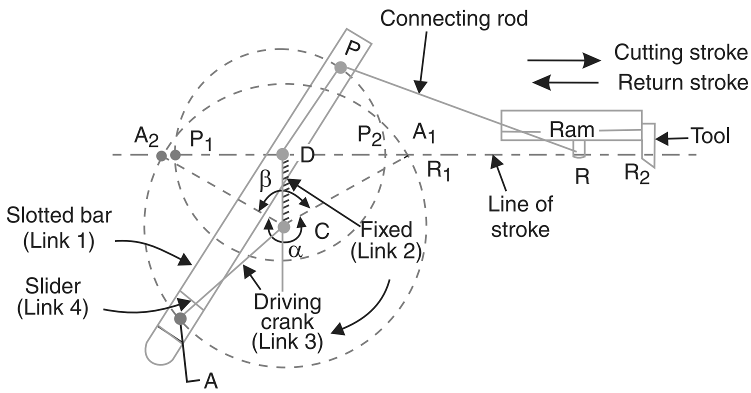


Fig. 5.27. Whitworth quick return motion mechanism.

When the driving crank *CA* moves from the position CA_1 to CA_2 (or the link *DP* from the position DP_1 to DP_2) through an angle α in the clockwise direction, the tool moves from the left hand end of its stroke to the right hand end through a distance 2 *PD*.

Now when the driving crank moves from the position CA_2 to CA_1 (or the link *DP* from DP_2 to DP_1) through an angle β in the clockwise direction, the tool moves back from right hand end of its stroke to the left hand end.

A little consideration will show that the time taken during the left to right movement of the ram (*i.e.* during forward or cutting stroke) will be equal to the time taken by the driving crank to move from CA_1 to CA_2. Similarly, the time taken during the right to left movement of the ram (or during the idle or return stroke) will be equal to the time taken by the driving crank to move from CA_2 to CA_1.

Since the crank link *CA* rotates at uniform angular velocity therefore time taken during the cutting stroke (or forward stroke) is more than the time taken during the return stroke. In other words, the mean speed of the ram during cutting stroke is less than the mean speed during the return stroke. The ratio between the time taken during the cutting and return strokes is given by

$$\frac{\text{Time of cutting stroke}}{\text{Time of return stroke}} = \frac{\alpha}{\beta} = \frac{\alpha}{360° - \alpha} \quad \text{or} \quad \frac{360° - \beta}{\beta}$$

Note. In order to find the length of effective stroke $R_1 R_2$, mark $P_1 R_1 = P_2 R_2 = PR$. The length of effective stroke is also equal to 2 *PD*.

Example 5.1. *A crank and slotted lever mechanism used in a shaper has a centre distance of 300 mm between the centre of oscillation of the slotted lever and the centre of rotation of the crank. The radius of the crank is 120 mm. Find the ratio of the time of cutting to the time of return stroke.*

Solution. Given : *A C* = 300 mm ; CB_1 = 120 mm

The extreme positions of the crank are shown in Fig. 5.28. We know that

$$\sin \angle CAB_1 = \sin(90° - \alpha/2)$$

$$= \frac{CB_1}{AC} = \frac{120}{300} = 0.4$$

$$\therefore \qquad \angle CAB_1 = 90° - \alpha/2$$

$$= \sin^{-1} 0.4 = 23.6°$$

or $$\alpha/2 = 90° - 23.6° = 66.4°$$

and $$\alpha = 2 \times 66.4 = 132.8°$$

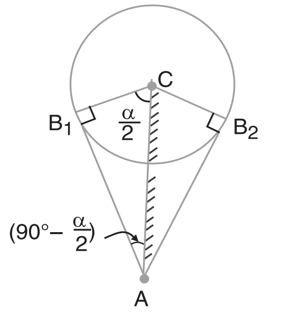


Fig. 5.28

We know that

$$\frac{\text{Time of cutting stroke}}{\text{Time of return stroke}} = \frac{360° - \alpha}{\alpha} = \frac{360° - 132.8°}{132.8°} = 1.72 \text{ Ans.}$$

Example 5.2. *In a crank and slotted lever quick return motion mechanism, the distance between the fixed centres is 240 mm and the length of the driving crank is 120 mm. Find the inclination of the slotted bar with the vertical in the extreme position and the time ratio of cutting stroke to the return stroke.*

If the length of the slotted bar is 450 mm, find the length of the stroke if the line of stroke passes through the extreme positions of the free end of the lever.

Solution. Given : $AC = 240$ mm ; $CB_1 = 120$ mm ; $AP_1 = 450$ mm

Inclination of the slotted bar with the vertical

Let $\angle CAB_1$ = Inclination of the slotted bar with the vertical.

The extreme positions of the crank are shown in Fig. 5.29. We know that

$$\sin \angle CAB_1 = \sin\left(90° - \frac{\alpha}{2}\right)$$

$$= \frac{B_1C}{AC} = \frac{120}{240} = 0.5$$

$$\therefore \ \angle CAB_1 = 90° - \frac{\alpha}{2}$$

$$= \sin^{-1} 0.5 = 30° \text{ Ans.}$$

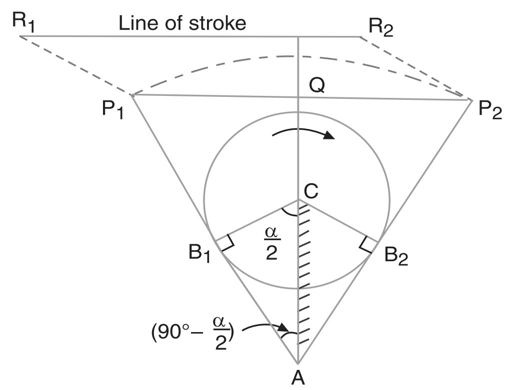


Fig. 5.29

Time ratio of cutting stroke to the return stroke

We know that

$$90° - \alpha/2 = 30°$$

$$\therefore \qquad \alpha/2 = 90° - 30° = 60°$$

or $$\alpha = 2 \times 60° = 120°$$

$$\therefore \qquad \frac{\text{Time of cutting stroke}}{\text{Time of return stroke}} = \frac{360° - \alpha}{\alpha} = \frac{360° - 120°}{120°} = 2 \text{ Ans.}$$

Length of the stroke

We know that length of the stroke,

$$R_1R_2 = P_1P_2 = 2\,P_1Q = 2\,AP_1 \sin(90° - \alpha/2)$$
$$= 2 \times 450 \sin(90° - 60°) = 900 \times 0.5 = 450 \text{ mm } \textbf{Ans.}$$

Example 5.3. *Fig. 5.30 shows the lay out of a quick return mechanism of the oscillating link type, for a special purpose machine. The driving crank BC is 30 mm long and time ratio of the working stroke to the return stroke is to be 1.7. If the length of the working stroke of R is 120 mm, determine the dimensions of AC and AP.*

Solution. Given : $BC = 30$ mm ; $R_1R_2 = 120$ mm ; Time ratio of working stroke to the return stroke = 1.7

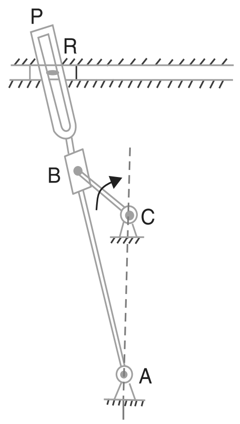


Fig. 5.30

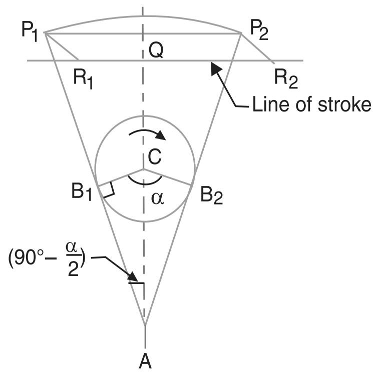


Fig. 5.31

We know that

$$\frac{\text{Time of working stroke}}{\text{Time of return stroke}} = \frac{360 - \alpha}{\alpha} \quad \text{or} \quad 1.7 = \frac{360 - \alpha}{\alpha}$$

$\therefore \quad \alpha = 133.3° \quad \text{or} \quad \alpha/2 = 66.65°$

The extreme positions of the crank are shown in Fig. 5.31. From right angled triangle AB_1C, we find that

$$\sin(90° - \alpha/2) = \frac{B_1C}{AC} \quad \text{or} \quad AC = \frac{B_1C}{\sin(90° - \alpha/2)} = \frac{BC}{\cos \alpha/2}$$

$$\ldots (\because B_1C = BC)$$

$$\therefore \quad AC = \frac{30}{\cos 66.65°} = \frac{30}{0.3963} = 75.7 \text{ mm } \textbf{Ans.}$$

We know that length of stroke,

$$R_1R_2 = P_1P_2 = 2P_1Q = 2\,AP_1 \sin(90° - \alpha/2) = 2\,AP_1 \cos \alpha/2$$
$$120 = 2\,AP \cos 66.65° = 0.7926\,AP \qquad \ldots (\because AP_1 = AP)$$

$$\therefore \quad AP = 120/0.7926 = 151.4 \text{ mm } \textbf{Ans.}$$

Example 5.4. *In a Whitworth quick return motion mechanism, as shown in Fig. 5.32, the distance between the fixed centers is 50 mm and the length of the driving crank is 75 mm. The length of the slotted lever is 150 mm and the length of the connecting rod is 135 mm. Find the ratio of the time of cutting stroke to the time of return stroke and also the effective stroke.*

Solution. Given : $CD = 50$ mm ; $CA = 75$ mm ; $PA = 150$ mm ; $PR = 135$ mm

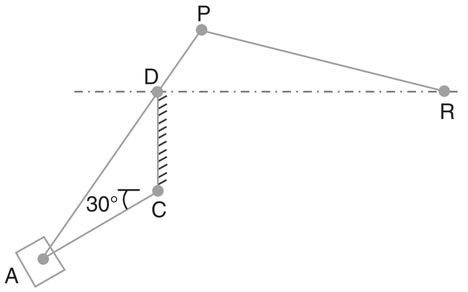


Fig. 5.32

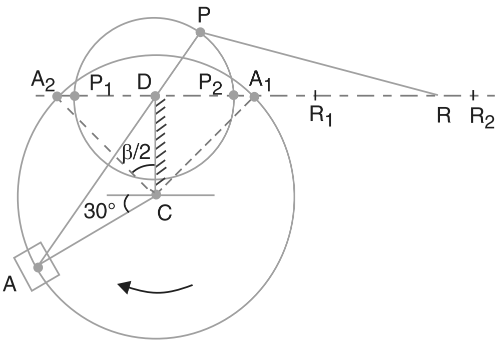


Fig. 5.33

The extreme positions of the driving crank are shown in Fig. 5.33. From the geometry of the figure,

$$\cos\beta/2 = \frac{CD}{CA_2} = \frac{50}{75} = 0.667 \qquad \ldots (\because CA_2 = CA)$$

$$\therefore \qquad \beta/2 = 48.2° \quad \text{or} \quad \beta = 96.4°$$

Ratio of the time of cutting stroke to the time of return stroke

We know that

$$\frac{\text{Time of cutting stroke}}{\text{Time of return stroke}} = \frac{360 - \beta}{\beta} = \frac{360 - 96.4}{96.4} = 2.735 \text{ Ans.}$$

Length of effective stroke

In order to find the length of effective stroke (*i.e.* R_1R_2), draw the space diagram of the mechanism to some suitable scale, as shown in Fig. 5.33. Mark $P_1R_2 = P_2R_2 = PR$. Therefore by measurement we find that,

Length of effective stroke = R_1R_2 = 87.5 mm **Ans.**

5.21. Double Slider Crank Chain

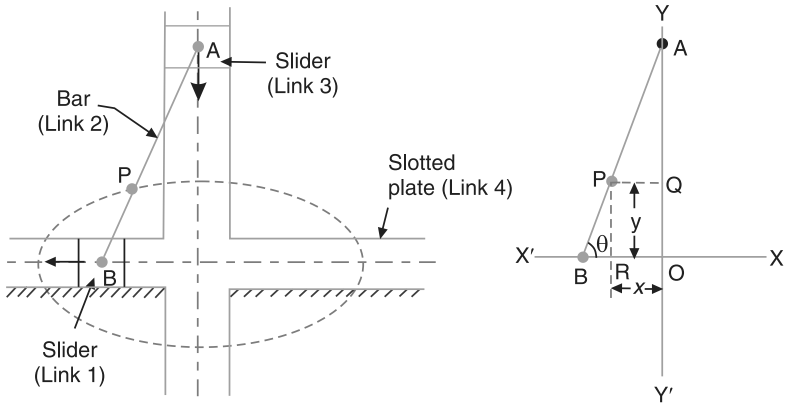
A kinematic chain which consists of two turning pairs and two sliding pairs is known as ***double slider crank chain***, as shown in Fig. 5.34. We see that the link 2 and link 1 form one turning pair and link 2 and link 3 form the second turning pair. The link 3 and link 4 form one sliding pair and link 1 and link 4 form the second sliding pair.

5.22. Inversions of Double Slider Crank Chain

The following three inversions of a double slider crank chain are important from the subject point of view :

1. *Elliptical trammels*. It is an instrument used for drawing ellipses. This inversion is obtained by fixing the slotted plate (link 4), as shown in Fig. 5.34. The fixed plate or link 4 has two straight grooves cut in it, at right angles to each other. The link 1 and link 3, are known as sliders and form sliding pairs with link 4. The link *A B* (link 2) is a bar which forms turning pair with links 1 and 3.

When the links 1 and 3 slide along their respective grooves, any point on the link 2 such as *P* traces out an ellipse on the surface of link 4, as shown in Fig. 5.34 (*a*). A little consideration will show that *AP* and *BP* are the semi-major axis and semi-minor axis of the ellipse respectively. This can be proved as follows :



(*a*) (*b*)

Fig. 5.34. Elliptical trammels.

Let us take OX and OY as horizontal and vertical axes and let the link BA is inclined at an angle θ with the horizontal, as shown in Fig. 5.34 (*b*). Now the co-ordinates of the point P on the link BA will be

$$x = PQ = AP \cos\theta; \text{ and } y = PR = BP \sin\theta$$

or $$\frac{x}{AP} = \cos\theta; \text{ and } \frac{y}{BP} = \sin\theta$$

Squaring and adding,

$$\frac{x^2}{(AP)^2} + \frac{y^2}{(BP)^2} = \cos^2\theta + \sin^2\theta = 1$$

This is the equation of an ellipse. Hence the path traced by point P is an ellipse whose semi-major axis is AP and semi-minor axis is BP.

Note : If P is the mid-point of link BA, then $AP = BP$. The above equation can be written as

$$\frac{x^2}{(AP)^2} + \frac{y^2}{(AP)^2} = 1 \qquad \text{or} \qquad x^2 + y^2 = (AP)^2$$

This is the equation of a circle whose radius is AP. Hence if P is the mid-point of link BA, it will trace a circle.

2. ***Scotch yoke mechanism***. This mechanism is used for converting rotary motion into a reciprocating motion. The inversion is obtained by fixing either the link 1 or link 3. In Fig. 5.35, link 1 is fixed. In this mechanism, when the link 2 (which corresponds to crank) rotates about B as centre, the link 4 (which corresponds to a frame) reciprocates. The fixed link 1 guides the frame.

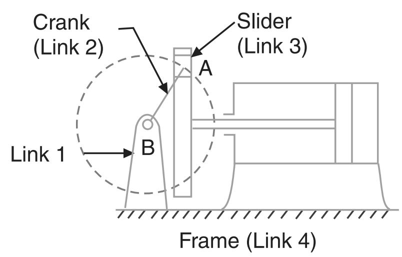


Fig. 5.35. Scotch yoke mechanism.

3. ***Oldham's coupling***. An oldham's coupling is used for connecting two parallel shafts whose axes are at a small distance apart. The shafts are coupled in such a way that if one shaft rotates, the other shaft also rotates at the same speed. This inversion is obtained by fixing the link 2, as shown in Fig. 5.36 (*a*). The shafts to be connected have two flanges (link 1 and link 3) rigidly fastened at their ends by forging.

The link 1 and link 3 form turning pairs with link 2. These flanges have diametrical slots cut in their inner faces, as shown in Fig. 5.36 (*b*). The intermediate piece (link 4) which is a circular disc, have two tongues (*i.e.* diametrical projections) T_1 and T_2 on each face at right angles to each other, as shown in Fig. 5.36 (*c*). The tongues on the link 4 closely fit into the slots in the two flanges (link 1 and link 3). The link 4 can slide or reciprocate in the slots in the flanges.

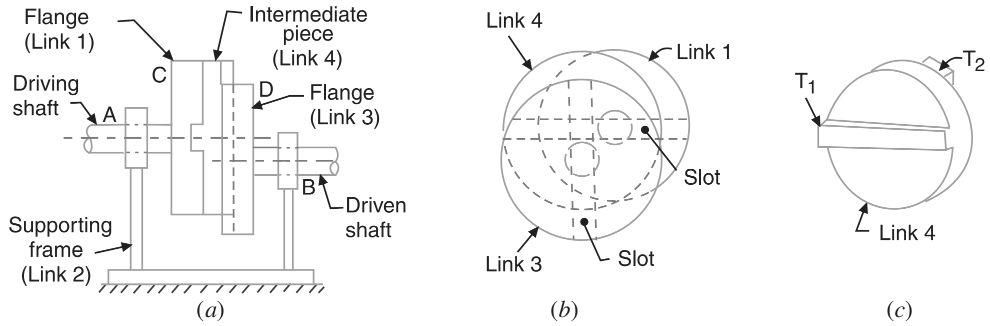


Fig. 5.36. Oldham's coupling.

When the driving shaft *A* is rotated, the flange *C* (link 1) causes the intermediate piece (link 4) to rotate at the same angle through which the flange has rotated, and it further rotates the flange *D* (link 3) at the same angle and thus the shaft *B* rotates. Hence links 1, 3 and 4 have the same angular velocity at every instant. A little consideration will show, that there is a sliding motion between the link 4 and each of the other links 1 and 3.

If the distance between the axes of the shafts is constant, the centre of intermediate piece will describe a circle of radius equal to the distance between the axes of the two shafts. Therefore, the maximum sliding speed of each tongue along its slot is equal to the peripheral velocity of the centre of the disc along its circular path.

Let ω = Angular velocity of each shaft in rad/s, and

r = Distance between the axes of the shafts in metres.

∴ Maximum sliding speed of each tongue (in m/s),

$$v = \omega.r$$

OBJECTIVE TYPE QUESTIONS

1. In a reciprocating steam engine, which of the following forms a kinematic link ?
(*a*) cylinder and piston (*b*) piston rod and connecting rod
(*c*) crank shaft and flywheel (*d*) flywheel and engine frame

2. The motion of a piston in the cylinder of a steam engine is an example of
(*a*) completely constrained motion (*b*) incompletely constrained motion
(*c*) successfully constrained motion (*d*) none of these

3. The motion transmitted between the teeth of gears in mesh is
(*a*) sliding (*b*) rolling
(*c*) may be rolling or sliding depending upon the shape of teeth
(*d*) partly sliding and partly rolling

4. The cam and follower without a spring forms a
(*a*) lower pair (*b*) higher pair
(*c*) self closed pair (*d*) force closed pair

5. A ball and a socket joint forms a
(*a*) turning pair (*b*) rolling pair (*c*) sliding pair (*d*) spherical pair

6. The lead screw of a lathe with nut forms a
(*a*) sliding pair (*b*) rolling pair (*c*) screw pair (*d*) turning pair

7. When the elements of the pair are kept in contact by the action of external forces, the pair is said to be a
(*a*) lower pair (*b*) higher pair
(*c*) self closed pair (*d*) force closed pair

8. Which of the following is a turning pair ?
(*a*) Piston and cylinder of a reciprocating steam engine
(*b*) Shaft with collars at both ends fitted in a circular hole
(*c*) Lead screw of a lathe with nut
(*d*) Ball and socket joint

9. A combination of kinematic pairs, joined in such a way that the relative motion between the links is completely constrained, is called a
(*a*) structure (*b*) mechanism
(*c*) kinematic chain (*d*) inversion

10. The relation between the number of pairs (p) forming a kinematic chain and the number of links (l) is
(*a*) $l = 2p - 2$ (*b*) $l = 2p - 3$ (*c*) $l = 2p - 4$ (*d*) $l = 2p - 5$

11. The relation between the number of links (l) and the number of binary joints (j) for a kinematic chain having constrained motion is given by $j = \frac{3}{2} l - 2.$ If the left hand side of this equation is greater than right hand side, then the chain is
(*a*) locked chain (*b*) completely constrained chain
(*c*) successfully constrained chain (*d*) incompletely constrained chain

12. In a kinematic chain, a quaternary joint is equivalent to
(*a*) one binary joint (*b*) two binary joints (*c*) three binary joints (*d*) four binary joints

13. If n links are connected at the same joint, the joint is equivalent to
(*a*) $(n - 1)$ binary joints (*b*) $(n - 2)$ binary joints (*c*) $(2n - 1)$ binary joints (*d*) none of these

14. In a 4 – bar linkage, if the lengths of shortest, longest and the other two links are denoted by s, l, p and q, then it would result in Grashof's linkage provided that
(*a*) $l + p < s + q$ (*b*) $l + s < p + q$ (*c*) $l + p = s + q$ (*d*) none of these

15. A kinematic chain is known as a mechanism when
(*a*) none of the links is fixed (*b*) one of the links is fixed
(*c*) two of the links are fixed (*d*) all of the links are fixed

16. The Grubler's criterion for determining the degrees of freedom (n) of a mechanism having plane motion is
(*a*) $n = (l - 1) - j$ (*b*) $n = 2\,(l - 1) - 2j$ (*c*) $n = 3\,(l - 1) - 2j$ (*d*) $n = 4\,(l - 1) - 3j$
where l = Number of links, and j = Number of binary joints.

17. The mechanism forms a structure, when the number of degrees of freedom (n) is equal to
(*a*) 0 (*b*) 1 (*c*) 2 (*d*) – 1

18. In a four bar chain or quadric cycle chain
(*a*) each of the four pairs is a turning pair (*b*) one is a turning pair and three are sliding pairs
(*c*) three are turning pairs and one is sliding pair (*d*) each of the four pairs is a sliding pair.

19. Which of the following is an inversion of single slider crank chain ?
(*a*) Beam engine (*b*) Watt's indicator mechanism
(*c*) Elliptical trammels (*d*) Whitworth quick return motion mechanism

20. Which of the following is an inversion of double slider crank chain ?
(*a*) Coupling rod of a locomotive (*b*) Pendulum pump
(*c*) Elliptical trammels (*d*) Oscillating cylinder engine

ANSWERS

1. (*c*)	**2.** (*a*)	**3.** (*d*)	**4.** (*c*)	**5.** (*d*)
6. (*c*)	**7.** (*d*)	**8.** (*b*)	**9.** (*c*)	**10.** (*c*)
11. (*a*)	**12.** (*c*)	**13.** (*a*)	**14.** (*b*)	**15.** (*b*)
16. (*c*)	**17.** (*a*)	**18.** (*a*)	**19.** (*d*)	**20.** (*c*)

Velocity in Mechanisms (Instantaneous Centre Method)

Features

1. *Introduction.*
2. *Space and Body Centrodes.*
3. *Methods for Determining the Velocity of a Point on a Link.*
4. *Velocity of a Point on a Link by Instantaneous Centre Method.*
5. *Properties of the Instantaneous Centre.*
6. *Number of Instantaneous Centres in a Mechanism.*
7. *Types of Instantaneous Centres.*
8. *Location of Instantaneous Centres.*
9. *Aronhold Kennedy (or Three Centres-in-Line) Theorem.*
10. *Method of Locating Instantaneous Centres in a Mechanism.*

6.1. Introduction

Sometimes, a body has simultaneously a motion of rotation as well as translation, such as wheel of a car, a sphere rolling (but not slipping) on the ground. Such a motion will have the combined effect of rotation and translation.

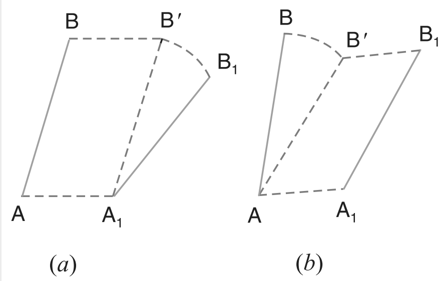
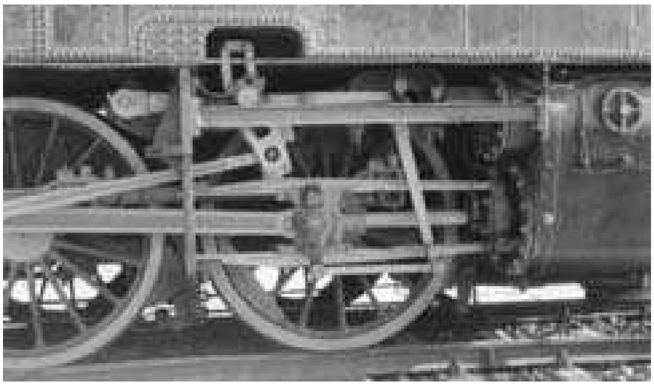


Fig. 6.1. Motion of a link.

Consider a rigid link AB, which moves from its initial position AB to $A_1 B_1$ as shown in Fig. 6.1 (*a*). A little consideration will show that the link neither has wholly a motion of translation nor wholly rotational, but a combination of the two motions. In Fig. 6.1 (*a*), the link has first the motion of translation from AB to A_1B' and then the motion of rotation about A_1, till it occupies the final position $A_1 B_1$. In Fig. 6.1 (*b*), the link AB has first the motion of rotation from AB to $A B'$ about A and then the motion of translation from $A B'$ to

$A_1 B_1$. Such a motion of link AB to $A_1 B_1$ is an example of combined motion of rotation and translation, it being immaterial whether the motion of rotation takes first, or the motion of translation.


Mechanisms on a steam automobile engine.

In actual practice, the motion of link AB is so gradual that it is difficult to see the two separate motions. But we see the two separate motions, though the point B moves faster than the point A. Thus, this combined motion of rotation and translation of the link AB may be assumed to be a motion of pure rotation about some centre I, known as the ***instantaneous centre of rotation*** (***also called centro or virtual centre***). The position of instantaneous centre may be located as discussed below:

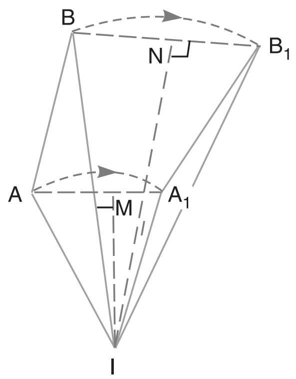


Fig. 6.2. Instantaneous centre of rotation.

Since the points A and B of the link has moved to A_1 and B_1 respectively under the motion of rotation (as assumed above), therefore the position of the centre of rotation must lie on the intersection of the right bisectors of chords $A A_1$ and $B B_1$. Let these bisectors intersect at I as shown in Fig. 6.2, which is the instantaneous centre of rotation or virtual centre of the link AB.

From above, we see that the position of the link AB goes on changing, therefore the centre about which the motion is assumed to take place (*i.e.* the instantaneous centre of rotation) also goes on changing. Thus the instantaneous centre of a moving body may be defined as ***that centre which goes on changing from one instant to another***. The locus of all such instantaneous centres is known as ***centrode***. A line drawn through an instantaneous centre and perpendicular to the plane of motion is called ***instantaneous axis***. The locus of this axis is known as ***axode***.

6.2. Space and Body Centrodes

A rigid body in plane motion relative to a second rigid body, supposed fixed in space, may be assumed to be rotating about an instantaneous centre at that particular moment. In other words, the instantaneous centre is a point in the body which may be considered fixed at any particular moment. The locus of the instantaneous centre in space during a definite motion of the body is called the ***space centrode*** and the locus of the instantaneous centre relative to the body itself is called the ***body centrode***. These two centrodes have the instantaneous centre as a common point at any instant and during the motion of the body, the body centrode rolls without slipping over the space centrode.

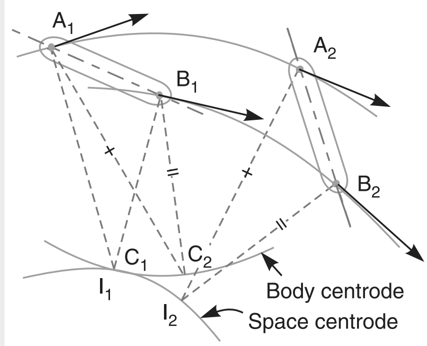


Fig. 6.3. Space and body centrode.

Let I_1 and I_2 be the instantaneous centres for the two different positions $A_1 B_1$ and $A_2 B_2$ of the link $A_1 B_1$ after executing a plane motion as shown in Fig. 6.3. Similarly, if the number of positions of the link $A_1 B_1$ are considered and a curve is drawn passing through these instantaneous centres (I_1, I_2....), then the curve so obtained is called the space centrode.

Now consider a point C_1 to be attached to the body or link $A_1 B_1$ and moves with it in such a way that C_1 coincides with I_1 when the body is in position $A_1 B_1$. Let C_2 be the position of the point C_1 when the link $A_1 B_1$ occupies the position $A_2 B_2$. A little consideration will show that the point C_2 will coincide with I_2 (when the link is in position $A_2 B_2$) only if triangles $A_1 B_1 C_1$ and $A_2 B_2 C_2$ are identical.

$\therefore \quad A_1 C_2 = A_2 I_2 \quad \text{and} \quad B_1 C_2 = B_2 I_2$

In the similar way, the number of positions of the point C_1 can be obtained for different positions of the link $A_1 B_1$. The curve drawn through these points (C_1, C_2....) is called the body centrode.

6.3. Methods for Determining the Velocity of a Point on a Link

Though there are many methods for determining the velocity of any point on a link in a mechanism whose direction of motion (*i.e.* path) and velocity of some other point on the same link is known in magnitude and direction, yet the following two methods are important from the subject point of view.

1. Instantaneous centre method, and **2.** Relative velocity method.

The instantaneous centre method is convenient and easy to apply in simple mechanisms, whereas the relative velocity method may be used to any configuration diagram. We shall discuss the relative velocity method in the next chapter.

6.4. Velocity of a Point on a Link by Instantaneous Centre Method

The instantaneous centre method of analysing the motion in a mechanism is based upon the concept (as discussed in Art. 6.1) that any displacement of a body (or a rigid link) having motion in one plane, can be considered as a pure rotational motion of a rigid link as a whole about some centre, known as instantaneous centre or virtual centre of rotation.

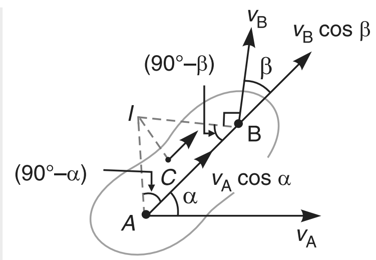
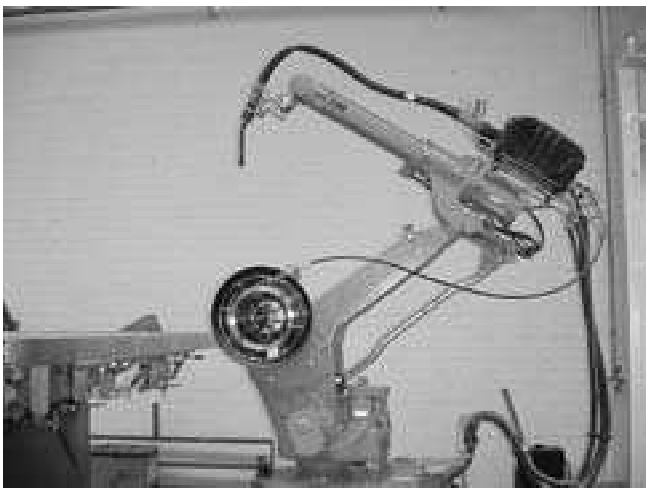


Fig. 6.4. Velocity of a point on a link.

Consider two points A and B on a rigid link. Let v_A and v_B be the velocities of points A and B, whose directions are given by angles α and β as shown in Fig. 6.4. If v_A is known in magnitude and direction and v_B in direction only, then the magnitude of v_B may be determined by the instantaneous centre method as discussed below :



Robots use various mechanisms to perform jobs.

Draw AI and BI perpendiculars to the directions v_A and v_B respectively. Let these lines intersect at I, which is known as instantaneous centre or virtual centre of the link. The complete rigid link is to rotate or turn about the centre I.

Since A and B are the points on a rigid link, therefore there cannot be any relative motion between them along the line AB.

Now resolving the velocities along AB,

$$v_A \cos \alpha = v_B \cos \beta$$

or
$$\frac{v_A}{v_B} = \frac{\cos \beta}{\cos \alpha} = \frac{\sin(90° - \beta)}{\sin(90° - \alpha)} \qquad ...(i)$$

Applying Lami's theorem to triangle ABI,

$$\frac{AI}{\sin(90° - \beta)} = \frac{BI}{\sin(90° - \alpha)}$$

or
$$\frac{AI}{BI} = \frac{\sin(90° - \beta)}{\sin(90° - \alpha)} \qquad ...(ii)$$

From equation (*i*) and (*ii*),

$$\frac{v_A}{v_B} = \frac{AI}{BI} \quad \text{or} \quad \frac{v_A}{AI} = \frac{v_B}{BI} = \omega \qquad ...(iii)$$

where ω = Angular velocity of the rigid link.

If C is any other point on the link, then

$$\frac{v_A}{AI} = \frac{v_B}{BI} = \frac{v_C}{CI} \qquad ...(iv)$$

From the above equation, we see that

1. If v_A is known in magnitude and direction and v_B in direction only, then velocity of point B or any other point C lying on the same link may be determined in magnitude and direction.

2. The magnitude of velocities of the points on a rigid link is inversely proportional to the distances from the points to the instantaneous centre and is perpendicular to the line joining the point to the instantaneous centre.

6.5. Properties of the Instantaneous Centre

The following properties of the instantaneous centre are important from the subject point of view :

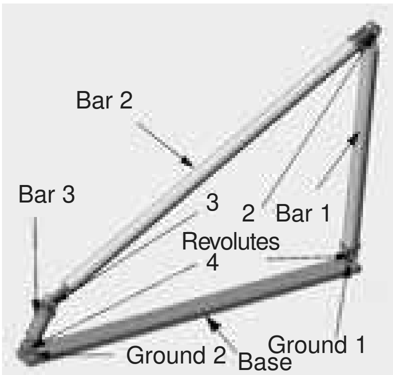
1. A rigid link rotates instantaneously relative to another link at the instantaneous centre for the configuration of the mechanism considered.

2. The two rigid links have no linear velocity relative to each other at the instantaneous centre. At this point (*i.e.* instantaneous centre), the two rigid links have the same linear velocity relative to the third rigid link. In other words, the velocity of the instantaneous centre relative to any third rigid link will be same whether the instantaneous centre is regarded as a point on the first rigid link or on the second rigid link.

6.6. Number of Instantaneous Centres in a Mechanism

The number of instantaneous centres in a constrained kinematic chain is equal to the number of possible combinations of two links. The number of pairs of links or the number of instantaneous centres is the number of combinations of n links taken two at a time. Mathematically, number of instantaneous centres,

$$N = \frac{n(n-1)}{2}, \text{ where } n = \text{Number of links.}$$



Four bar mechanisms.

6.7. Types of Instantaneous Centres

The instantaneous centres for a mechanism are of the following three types :

1. Fixed instantaneous centres, **2.** Permanent instantaneous centres, and **3.** Neither fixed nor permanent instantaneous centres.

The first two types *i.e.* fixed and permanent instantaneous centres are together known as ***primary instantaneous centres*** and the third type is known as ***secondary instantaneous centres***.

Consider a four bar mechanism *ABCD* as shown in Fig. 6.5. The number of instantaneous centres (N) in a four bar mechanism is given by

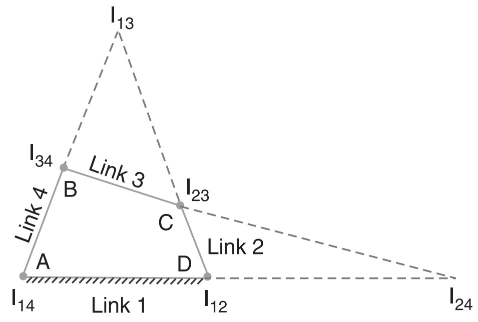


Fig. 6.5. Types of instantaneous centres.

$$N = \frac{n(n-1)}{2} = \frac{4(4-1)}{2} = 6 \qquad \ldots (\because n = 4)$$

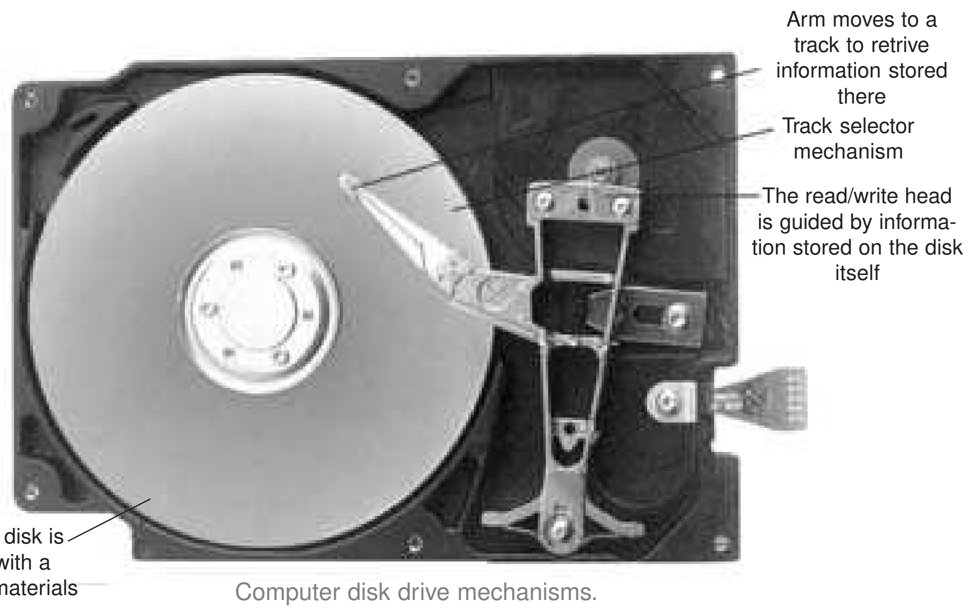
The instantaneous centres I_{12} and I_{14} are called the ***fixed instantaneous centres*** as they remain in the same place for all configurations of the mechanism. The instantaneous centres I_{23} and I_{34} are the ***permanent instantaneous centres*** as they move when the mechanism moves, but the joints are of permanent nature. The instantaneous centres I_{13} and I_{24} are ***neither fixed nor permanent instantaneous centres*** as they vary with the configuration of the mechanism.

Note: The instantaneous centre of two links such as link 1 and link 2 is usually denoted by I_{12} and so on. It is read as *I* one two and not *I* twelve.

6.8. Location of Instantaneous Centres

The following rules may be used in locating the instantaneous centres in a mechanism :

1. When the two links are connected by a pin joint (or pivot joint), the instantaneous centre



Computer disk drive mechanisms.

Note : This picture is given as additional information and is not a direct example of the current chapter.

lies on the centre of the pin as shown in Fig. 6.6 (*a*). Such a instantaneous centre is of permanent nature, but if one of the links is fixed, the instantaneous centre will be of fixed type.

2. When the two links have a pure rolling contact (*i.e.* link 2 rolls without slipping upon the fixed link 1 which may be straight or curved), the instantaneous centre lies on their point of contact, as shown in Fig. 6.6 (*b*). The velocity of any point A on the link 2 relative to fixed link 1 will be perpendicular to $I_{12}A$ and is proportional to $I_{12}A$. In other words

$$\frac{v_A}{v_B} = \frac{I_{12}A}{I_{12}B}$$

3. When the two links have a sliding contact, the instantaneous centre lies on the common normal at the point of contact. We shall consider the following three cases :

(*a*) When the link 2 (slider) moves on fixed link 1 having straight surface as shown in Fig. 6.6 (*c*), the instantaneous centre lies at infinity and each point on the slider have the same velocity.

(*b*) When the link 2 (slider) moves on fixed link 1 having curved surface as shown in Fig. 6.6 (*d*),the instantaneous centre lies on the centre of curvature of the curvilinear path in the configuration at that instant.

(*c*) When the link 2 (slider) moves on fixed link 1 having constant radius of curvature as shown in Fig. 6.6 (*e*), the instantaneous centre lies at the centre of curvature *i.e.* the centre of the circle, for all configuration of the links.

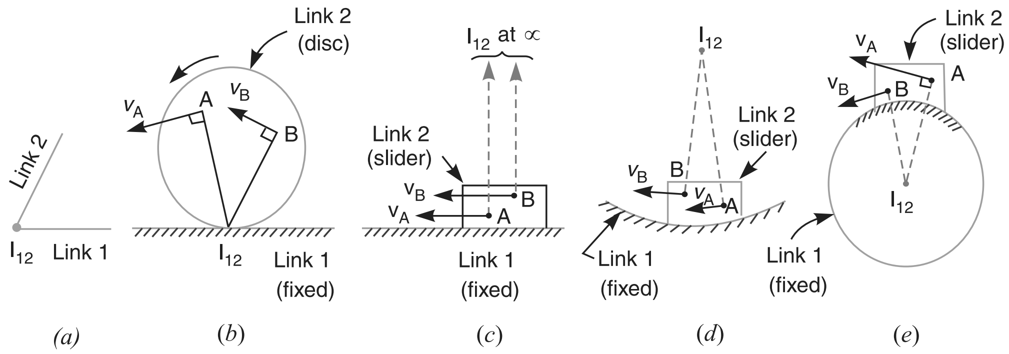


Fig. 6.6. Location of instantaneous centres.

6.9. Aronhold Kennedy (or Three Centres in Line) Theorem

The Aronhold Kennedy's theorem states that ***if three bodies move relatively to each other, they have three instantaneous centres and lie on a straight line.***

Consider three kinematic links A, B and C having relative plane motion. The number of instantaneous centres (N) is given by

$$N = \frac{n(n-1)}{2} = \frac{3(3-1)}{2} = 3$$

where n = Number of links = 3

The two instantaneous centres at the pin joints of B with A, and C with A (*i.e.* I_{ab} and I_{ac}) are the permanent instantaneous centres. According to Aronhold Kennedy's theorem, the third instantaneous centre I_{bc} must lie on the line joining I_{ab} and I_{ac}. In order to prove this,

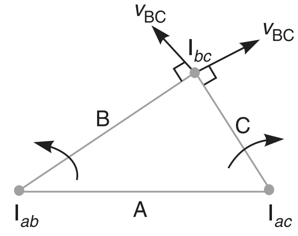
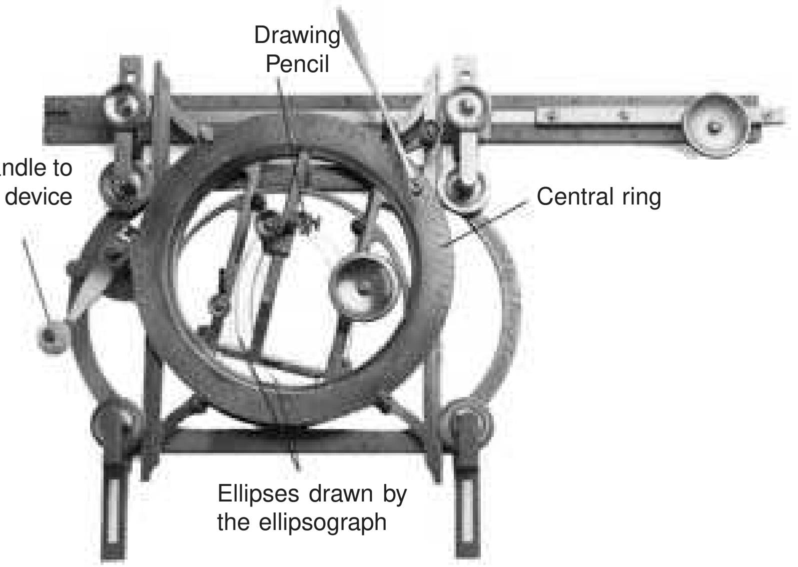


Fig. 6.7. Aronhold Kennedy's theorem.

let us consider that the instantaneous centre I_{bc} lies outside the line joining I_{ab} and I_{ac} as shown in Fig. 6.7. The point I_{bc} belongs to both the links B and C. Let us consider the point I_{bc} on the link B. Its velocity v_{BC} must be perpendicular to the line joining I_{ab} and I_{bc}. Now consider the point I_{bc} on the link C. Its velocity v_{BC} must be perpendicular to the line joining I_{ac} and I_{bc}.

We have already discussed in Art. 6.5, that the velocity of the instantaneous centre is same whether it is regarded as a point on the first link or as a point on the second link. Therefore, the velocity of the point I_{bc} cannot be perpendicular to both lines $I_{ab}\,I_{bc}$ and $I_{ac}\,I_{bc}$ unless the point I_{bc} lies on the line joining the points I_{ab} and I_{ac}. Thus the three instantaneous centres (I_{ab}, I_{ac} and I_{bc}) must lie on the same straight line. The exact location of I_{bc} on line $I_{ab}\,I_{ac}$ depends upon the directions and magnitudes of the angular velocities of B and C relative to A.



The above picture shows ellipsograph which is used to draw ellipses.

Note : This picture is given as additional information and is not a direct example of the current chapter.

6.10. Method of Locating Instantaneous Centres in a Mechanism

Consider a pin jointed four bar mechanism as shown in Fig. 6.8 (*a*). The following procedure is adopted for locating instantaneous centres.

1. First of all, determine the number of instantaneous centres (N) by using the relation

$$N = \frac{n(n-1)}{2}, \text{ where } n = \text{Number of links.}$$

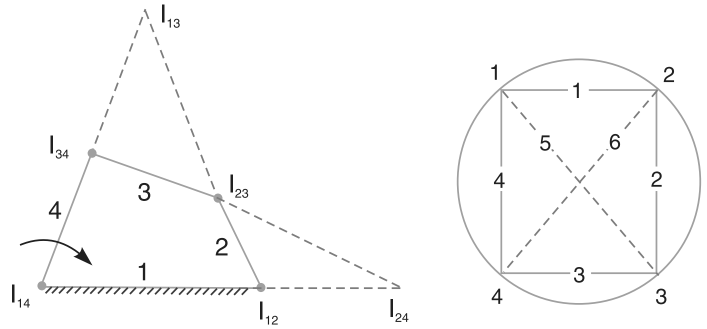
In the present case, $\quad N = \frac{4(4-1)}{2} = 6 \qquad ...(\because n = 4)$

2. Make a list of all the instantaneous centres in a mechanism. Since for a four bar mechanism, there are six instantaneous centres, therefore these centres are listed as shown in the following table (known as book-keeping table).

Links	1	2	3	4
Instantaneous	12	23	34	–
centres	13	24		
(6 in number)	14			

3. Locate the fixed and permanent instantaneous centres by inspection. In Fig. 6.8 (*a*), I_{12} and I_{14} are fixed instantaneous centres and I_{23} and I_{34} are permanent instantaneous centres.

Note. The four bar mechanism has four turning pairs, therefore there are four primary (*i.e.* fixed and permanent) instantaneous centres and are located at the centres of the pin joints.



(*a*) Four bar mechanism. (*b*) Circle diagram.

Fig. 6.8. Method of locating instantaneous centres.

4. Locate the remaining neither fixed nor permanent instantaneous centres (or secondary centres) by Kennedy's theorem. This is done by circle diagram as shown in Fig. 6.8 (*b*). Mark points on a circle equal to the number of links in a mechanism. In the present case, mark 1, 2, 3, and 4 on the circle.

5. Join the points by solid lines to show that these centres are already found. In the circle diagram [Fig. 6.8 (*b*)] these lines are 12, 23, 34 and 14 to indicate the centres I_{12}, I_{23}, I_{34} and I_{14}.

6. In order to find the other two instantaneous centres, join two such points that the line joining them forms two adjacent triangles in the circle diagram. The line which is responsible for completing two triangles, should be a common side to the two triangles. In Fig. 6.8 (*b*), join 1 and 3 to form the triangles 123 and 341 and the instantaneous centre* I_{13} will lie on the intersection of I_{12} I_{23} and I_{14} I_{34}, produced if necessary, on the mechanism. Thus the instantaneous centre I_{13} is located. Join 1 and 3 by a dotted line on the circle diagram and mark number 5 on it. Similarly the instantaneous centre I_{24} will lie on the intersection of I_{12} I_{14} and I_{23} I_{34}, produced if necessary, on the mechanism. Thus I_{24} is located. Join 2 and 4 by a dotted line on the circle diagram and mark 6 on it. Hence all the six instantaneous centres are located.

Note: Since some of the neither fixed nor permanent instantaneous centres are not required in solving problems, therefore they may be omitted.

Example 6.1. *In a pin jointed four bar mechanism, as shown in Fig. 6.9, AB = 300 mm, BC = CD = 360 mm, and AD = 600 mm. The angle BAD = 60°. The crank AB rotates uniformly at 100 r.p.m. Locate all the instantaneous centres and find the angular velocity of the link BC.*

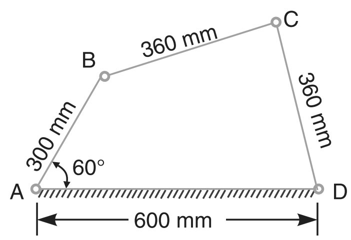


Fig. 6.9

Solution. Given : N_{AB} = 100 r.p.m or

$\omega_{AB} = 2\,\pi \times 100/60 = 10.47$ rad/s

Since the length of crank AB = 300 mm = 0.3 m, therefore velocity of point B on link AB,

* We may also say as follows: Considering links 1, 2 and 3, the instantaneous centres will be I_{12}, I_{23} and I_{13}. The centres I_{12} and I_{23} have already been located. Similarly considering links 1, 3 and 4, the instantaneous centres will be I_{13}, I_{34} and I_{14}, from which I_{14} and I_{34} have already been located. Thus we see that the centre I_{13} lies on the intersection of the lines joining the points I_{12} I_{23} and I_{14} I_{34}.

$$v_B = \omega_{AB} \times AB = 10.47 \times 0.3 = 3.141 \text{ m/s}$$

Location of instantaneous centres

The instantaneous centres are located as discussed below:

1. Since the mechanism consists of four links (*i.e.* $n = 4$), therefore number of instantaneous centres,

$$N = \frac{n(n-1)}{2} = \frac{4(4-1)}{2} = 6$$

2. For a four bar mechanism, the book keeping table may be drawn as discussed in Art. 6.10.

3. Locate the fixed and permanent instantaneous centres by inspection. These centres are I_{12}, I_{23}, I_{34} and I_{14}, as shown in Fig. 6.10.

4. Locate the remaining neither fixed nor permanent instantaneous centres by Aronhold Kennedy's theorem. This is done by circle diagram as shown in Fig. 6.11. Mark four points (equal to the number of links in a mechanism) 1, 2, 3, and 4 on the circle.

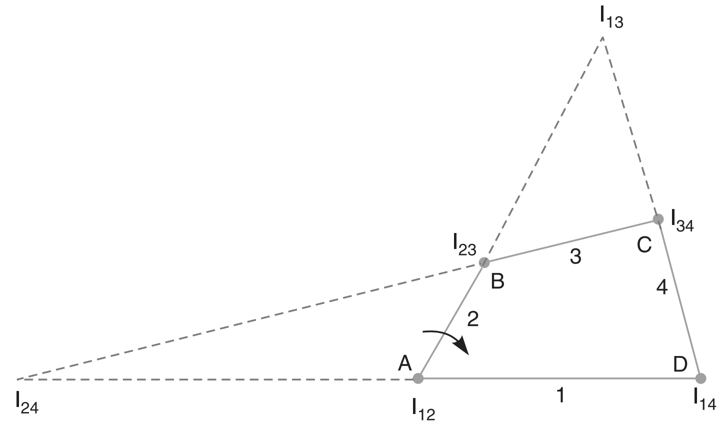


Fig. 6.10

5. Join points 1 to 2, 2 to 3, 3 to 4 and 4 to 1 to indicate the instantaneous centres already located *i.e.* I_{12}, I_{23}, I_{34} and I_{14}.

6. Join 1 to 3 to form two triangles 1 2 3 and 3 4 1. The side 13, common to both triangles, is responsible for completing the two triangles. Therefore the instantaneous centre I_{13} lies on the intersection of the lines joining the points I_{12} I_{23} and I_{34} I_{14} as shown in Fig. 6.10. Thus centre I_{13} is located. Mark number 5 (because four instantaneous centres have already been located) on the dotted line 1 3.

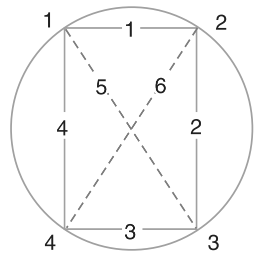


Fig. 6.11

7. Now join 2 to 4 to complete two triangles 2 3 4 and 1 2 4. The side 2 4, common to both triangles, is responsible for completing the two triangles. Therefore centre I_{24} lies on the intersection of the lines joining the points I_{23} I_{34} and I_{12} I_{14} as shown in Fig. 6.10. Thus centre I_{24} is located. Mark number 6 on the dotted line 2 4. Thus all the six instantaneous centres are located.

Angular velocity of the link BC

Let ω_{BC} = Angular velocity of the link BC.

Since B is also a point on link BC, therefore velocity of point B on link BC,

$$v_B = \omega_{BC} \times I_{13} B$$

By measurement, we find that $I_{13}B = 500 \text{ mm} = 0.5 \text{ m}$

$$\therefore \quad \omega_{BC} = \frac{v_B}{I_{13}B} = \frac{3.141}{0.5} = 6.282 \text{ rad/s} \quad \textbf{Ans.}$$

Example 6.2. *Locate all the instantaneous centres of the slider crank mechanism as shown in Fig. 6.12. The lengths of crank OB and connecting rod AB are 100 mm and 400 mm respectively. If the crank rotates clockwise with an angular velocity of 10 rad/s, find:* **1.** *Velocity of the slider A, and* **2.** *Angular velocity of the connecting rod AB.*

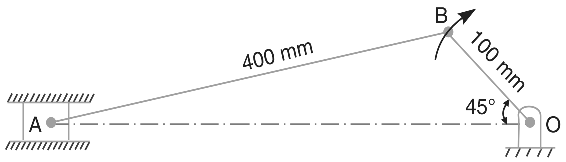


Fig. 6.12

Solution. Given : $\omega_{OB} = 10$ rad/ s; $OB = 100$ mm $= 0.1$ m

We know that linear velocity of the crank OB,

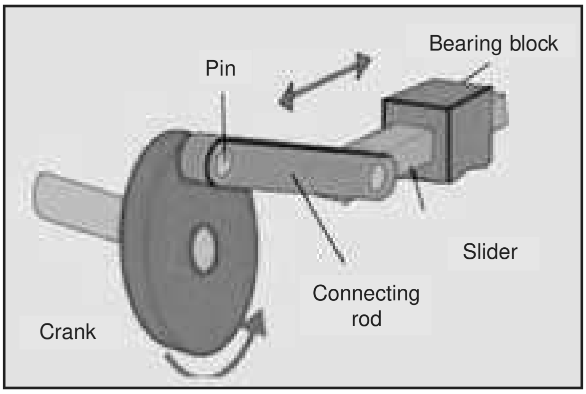
$$v_{OB} = v_B = \omega_{OB} \times OB = 10 \times 0.1 = 1 \text{ m/s}$$

Location of instantaneous centres

The instantaneous centres in a slider crank mechanism are located as discussed below:

1. Since there are four links (*i.e.* $n = 4$), therefore the number of instantaneous centres,

$$N = \frac{n(n-1)}{2} = \frac{4(4-1)}{2} = 6$$



Slider crank mechanism.

2. For a four link mechanism, the book keeping table may be drawn as discussed in Art. 6.10.

3. Locate the fixed and permanent instantaneous centres by inspection. These centres are I_{12}, I_{23} and I_{34} as shown in Fig. 6.13. Since the slider (link 4) moves on a straight surface (link 1), therefore the instantaneous centre I_{14} will be at infinity.

Note: Since the slider crank mechanism has three turning pairs and one sliding pair, therefore there will be three primary (*i.e.* fixed and permanent) instantaneous centres.

4. Locate the other two remaining neither fixed nor permanent instantaneous centres, by Aronhold Kennedy's theorem. This is done by circle diagram as shown in Fig. 6.14. Mark four points 1, 2, 3 and 4 (equal to the number of links in a mechanism) on the circle to indicate I_{12}, I_{23}, I_{34} and I_{14}.

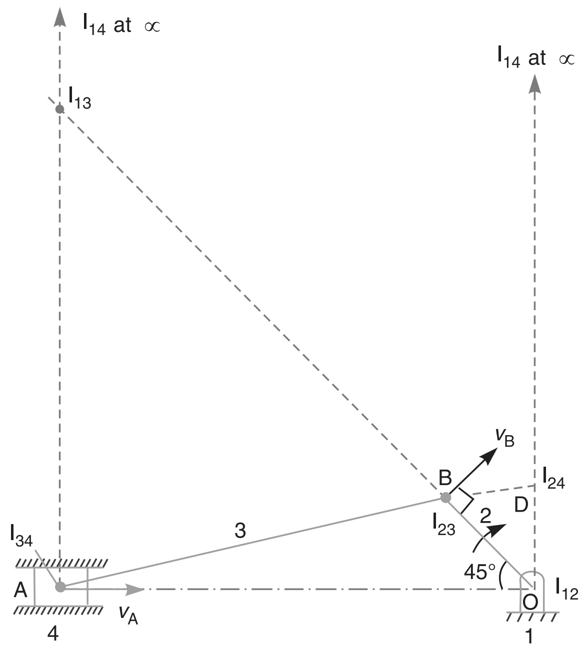


Fig. 6.13

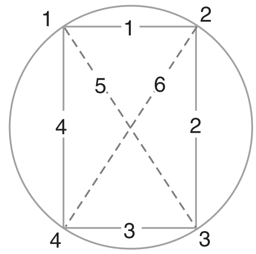


Fig. 6.14

5. Join 1 to 3 to form two triangles 1 2 3 and 3 4 1 in the circle diagram. The side 1 3, common to both triangles, is responsible for completing the two triangles. Therefore the centre I_{13} will lie on the intersection of $I_{12}\,I_{23}$ and $I_{14}\,I_{34}$, produced if necessary. Thus centre I_{13} is located. Join 1 to 3 by a dotted line and mark number 5 on it.

6. Join 2 to 4 by a dotted line to form two triangles 2 3 4 and 1 2 4. The side 2 4, common to both triangles, is responsible for completing the two triangles. Therefore the centre I_{24} lies on the intersection of $I_{23}\,I_{34}$ and $I_{12}\,I_{14}$. Join 2 to 4 by a dotted line on the circle diagram and mark number 6 on it. Thus all the six instantaneous centres are located.

By measurement, we find that

$$I_{13}A = 460 \text{ mm} = 0.46 \text{ m} \;;\text{ and } I_{13}B = 560 \text{ mm} = 0.56 \text{ m}$$

1. ***Velocity of the slider A***

Let v_A = Velocity of the slider A.

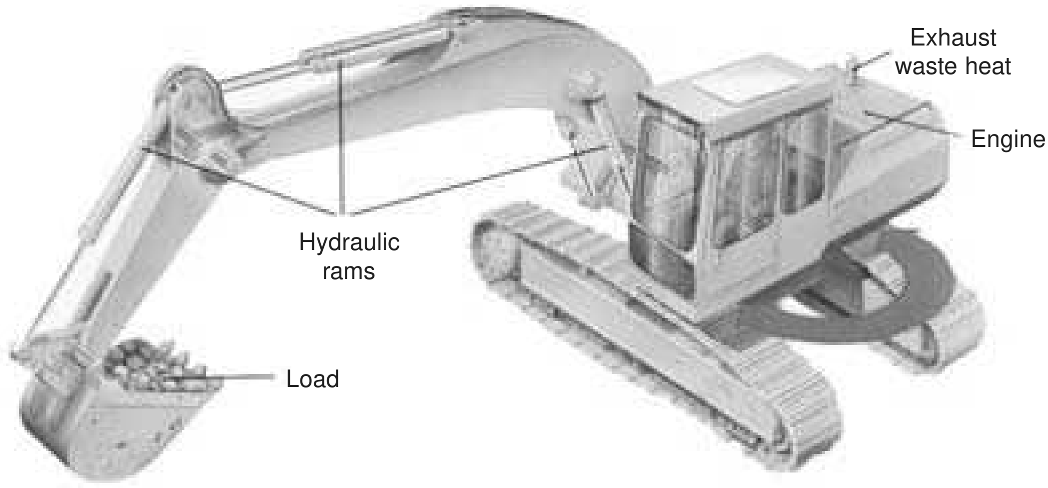
We know that $$\frac{v_A}{I_{13}A} = \frac{v_B}{I_{13}B}$$

or $$v_A = v_B \times \frac{I_{13}A}{I_{13}B} = 1 \times \frac{0.46}{0.56} = 0.82 \text{ m/s} \quad \textbf{Ans.}$$

2. ***Angular velocity of the connecting rod AB***

Let ω_{AB} = Angular velocity of the connecting rod AB.

We know that $$\frac{v_A}{I_{13}A} = \frac{v_B}{I_{13}B} = \omega_{AB}$$



The above picture shows a digging machine.

Note : This picture is given as additional information and is not a direct example of the current chapter.

$$\therefore \qquad \omega_{AB} = \frac{v_B}{I_{13} B} = \frac{1}{0.56} = 1.78 \text{ rad/s} \quad \textbf{Ans.}$$

Note: The velocity of the slider A and angular velocity of the connecting rod AB may also be determined as follows :

From similar triangles $I_{13} I_{23} I_{34}$ and $I_{12} I_{23} I_{24}$,

$$\frac{I_{12} I_{23}}{I_{13} I_{23}} = \frac{I_{23} I_{24}}{I_{23} I_{34}} \qquad ...(i)$$

and

$$\frac{I_{13} I_{34}}{I_{34} I_{23}} = \frac{I_{12} I_{24}}{I_{23} I_{24}} \qquad ...(ii)$$

We know that

$$\omega_{AB} = \frac{v_B}{I_{13} B} = \frac{\omega_{OB} \times OB}{I_{13} B} \qquad ...(\because \; v_B = \omega_{OB} \times OB)$$

$$= \omega_{OB} \times \frac{I_{12} I_{23}}{I_{13} I_{23}} = \omega_{OB} \times \frac{I_{23} I_{24}}{I_{23} I_{34}} \qquad ...[\text{From equation } (i)] \; ...(iii)$$

Also

$$v_A = \omega_{AB} \times I_{13} A = \omega_{OB} \times \frac{I_{23} I_{24}}{I_{23} I_{34}} \times I_{13} I_{34}. \qquad ...[\text{From equation } (iii)]$$

$$= \omega_{OB} \times I_{12} I_{24} = \omega_{OB} \times OD \qquad ...[\text{From equation } (ii)]$$

DO YOU KNOW ?

1. What do you understand by the instantaneous centre of rotation (centro) in kinematic of machines? Answer briefly.
2. Explain, with the help of a neat sketch, the space centrode and body centrode.
3. Explain with sketch the instantaneous centre method for determination of velocities of links and mechanisms.
4. Write the relation between the number of instantaneous centres and the number of links in a mechanism.
5. Discuss the three types of instantaneous centres for a mechanism.
6. State and prove the 'Aronhold Kennedy's Theorem' of three instantaneous centres.

OBJECTIVE TYPE QUESTIONS

1. The total number of instantaneous centres for a mechanism consisting of n links are

   (*a*) $\dfrac{n}{2}$ (*b*) n

   (*c*) $\dfrac{n-1}{2}$ (*d*) $\dfrac{n(n-1)}{2}$

2. According to Aronhold Kennedy's theorem, if three bodies move relatively to each other, their instantaneous centres will lie on a

   (*a*) straight line (*b*) parabolic curve

   (*c*) ellipse (*d*) none of these

3. In a mechanism, the fixed instantaneous centres are those centres which
   (*a*) remain in the same place for all configurations of the mechanism
   (*b*) vary with the configuration of the mechanism
   (*c*) moves as the mechanism moves, but joints are of permanent nature
   (*d*) none of the above
4. The instantaneous centres which vary with the configuration of the mechanism, are called
   (*a*) permanent instantaneous centres
   (*b*) fixed instantaneous centres
   (*c*) neither fixed nor permanent instantaneous centres
   (*d*) none of these
5. When a slider moves on a fixed link having curved surface, their instantaneous centre lies
   (*a*) on their point of contact
   (*b*) at the centre of curvature
   (*c*) at the centre of circle
   (*d*) at the pin joint

ANSWERS

1. (*d*) 2. (*a*) 3. (*a*) 4. (*c*) 5. (*b*)

Velocity in Mechanisms
(Relative Velocity Method)

Features

7.1. Introduction

We have discussed, in the previous chapter, the instantaneous centre method for finding the velocity of various points in the mechanisms. In this chapter, we shall discuss the relative velocity method for determining the velocity of different points in the mechanism. The study of velocity analysis is very important for determining the acceleration of points in the mechanisms which is discussed in the next chapter.

7.2. Relative Velocity of Two Bodies Moving in Straight Lines

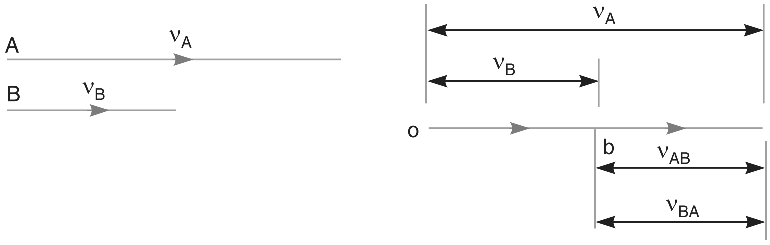
Here we shall discuss the application of vectors for the relative velocity of two bodies moving along parallel lines and inclined lines, as shown in Fig. 7.1 (*a*) and 7.2 (*a*) respectively.

Consider two bodies *A* and *B* moving along parallel lines in the same direction with absolute velocities v_A and v_B such that $v_A > v_B$, as shown in Fig. 7.1 (*a*). The relative velocity of *A* with respect to *B*,

$$v_{AB} = \text{Vector difference of } v_A \text{ and } v_B = \overline{v_A} - \overline{v_B} \qquad \ldots(i)$$

From Fig. 7.1 (*b*), the relative velocity of *A* with respect to *B* (*i.e.* v_{AB}) may be written in the vector form as follows :

$$\overline{ba} = \overline{oa} - \overline{ob}$$



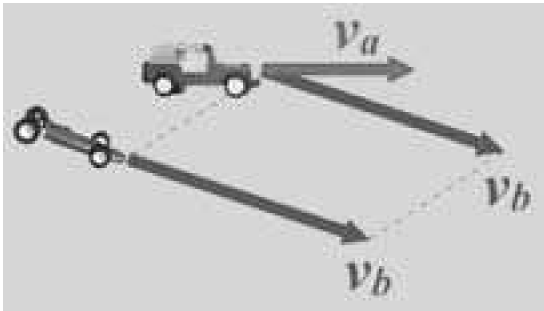
(*a*) (*b*)

Fig. 7.1. Relative velocity of two bodies moving along parallel lines.

Similarly, the relative velocity of *B* with respect to *A*,

v_{BA} = Vector difference of v_B and $v_A = \overline{v_B} - \overline{v_A}$ ...(*ii*)

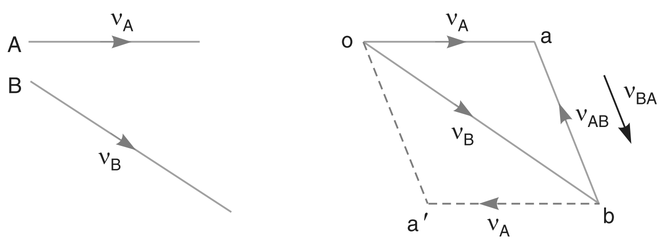
or $\overline{ab} = \overline{ob} - \overline{oa}$



Now consider the body *B* moving in an inclined direction as shown in Fig. 7.2 (*a*). The relative velocity of *A* with respect to *B* may be obtained by the law of parallelogram of velocities or triangle law of velocities. Take any fixed point *o* and draw vector *oa* to represent v_A in magnitude and direction to some suitable scale. Similarly, draw vector *ob* to represent v_B in magnitude and direction to the same scale. Then vector *ba* represents the relative velocity of *A* with respect to *B* as shown in Fig. 7.2 (*b*). In the similar way as discussed above, the relative velocity of *A* with respect to *B*,

v_{AB} = Vector difference of v_A and $v_B = \overline{v_A} - \overline{v_B}$

or $\overline{ba} = \overline{oa} - \overline{ob}$



(*a*) (*b*)

Fig. 7.2. Relative velocity of two bodies moving along inclined lines.

Similarly, the relative velocity of *B* with respect to *A*,

v_{BA} = Vector difference of v_B and $v_A = \overline{v_B} - \overline{v_A}$

or $\overline{ab} = \overline{ob} - \overline{oa}$

From above, we conclude that the relative velocity of point *A* with respect to *B* (v_{AB}) and the relative velocity of point *B* with respect *A* (v_{BA}) are equal in magnitude but opposite in direction, *i.e.*

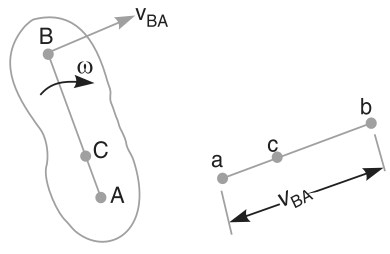
$$v_{AB} = -v_{BA} \quad \text{or} \quad \overline{ba} = -\overline{ab}$$

Note: It may be noted that to find v_{AB}, start from point *b* towards *a* and for v_{BA}, start from point *a* towards *b*.

7.3. Motion of a Link

Consider two points *A* and *B* on a rigid link *AB*, as shown in Fig. 7.3 (*a*). Let one of the extremities (*B*) of the link move relative to *A*, in a clockwise direction. Since the distance from *A* to *B* remains the same, therefore there can be no relative motion between *A* and *B*, along the line *AB*. It is thus obvious, that the relative motion of *B* with respect to *A* must be perpendicular to *AB*.

Hence ***velocity of any point on a link with respect to another point on the same link is always perpendicular to the line joining these points on the configuration (or space) diagram.***



(*a*) (*b*)

Fig. 7.3. Motion of a Link.

The relative velocity of *B* with respect to *A* (*i.e.* v_{BA}) is represented by the vector *ab* and is perpendicular to the line *AB* as shown in Fig. 7.3 (*b*).

Let ω = Angular velocity of the link *AB* about *A*.

We know that the velocity of the point *B* with respect to *A*,

$$v_{BA} = \overline{ab} = \omega.AB \qquad ...(i)$$

Similarly, the velocity of any point *C* on *AB* with respect to *A*,

$$v_{CA} = \overline{ac} = \omega.AC \qquad ...(ii)$$

From equations (*i*) and (*ii*),

$$\frac{v_{CA}}{v_{BA}} = \frac{\overline{ac}}{\overline{ab}} = \frac{\omega.AC}{\omega.AB} = \frac{AC}{AB} \qquad ...(iii)$$

Thus, we see from equation (*iii*), that the point *c* on the vector *ab* divides it in the same ratio as *C* divides the link *AB*.

Note: The relative velocity of *A* with respect to *B* is represented by *ba*, although *A* may be a fixed point. The motion between *A* and *B* is only relative. Moreover, it is immaterial whether the link moves about *A* in a clockwise direction or about *B* in a clockwise direction.

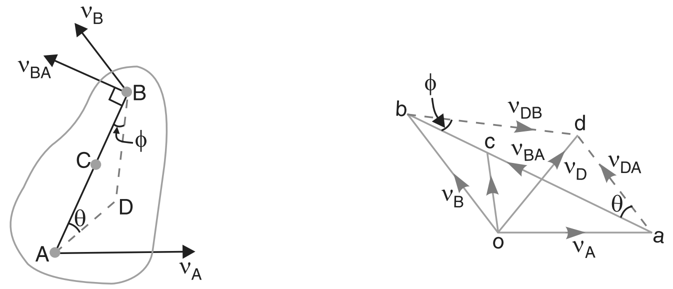
7.4. Velocity of a Point on a Link by Relative Velocity Method

The relative velocity method is based upon the relative velocity of the various points of the link as discussed in Art. 7.3.

Consider two points *A* and *B* on a link as shown in Fig. 7.4 (*a*). Let the absolute velocity of the point *A* *i.e.* v_A is known in magnitude and direction and the absolute velocity of the point *B* *i.e.* v_B is known in direction only. Then the velocity of *B* may be determined by drawing the velocity diagram as shown in Fig. 7.4 (*b*). The velocity diagram is drawn as follows :

1. Take some convenient point *o*, known as the pole.
2. Through *o*, draw *oa* parallel and equal to v_A, to some suitable scale.
3. Through *a*, draw a line perpendicular to *AB* of Fig. 7.4 (*a*). This line will represent the velocity of *B* with respect to *A*, *i.e.* v_{BA}.
4. Through *o*, draw a line parallel to v_B intersecting the line of v_{BA} at *b*.

5. Measure ob, which gives the required velocity of point B (v_B), to the scale.



(*a*) Motion of points on a link. (*b*) Velocity diagram.

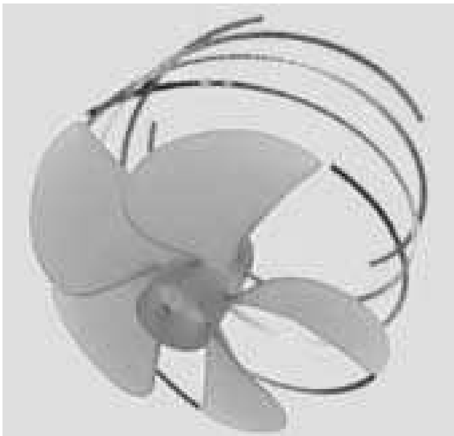
Fig. 7.4

Notes : 1. The vector ab which represents the velocity of B with respect to A (v_{BA}) is known as velocity of image of the link AB.

2. The absolute velocity of any point C on AB may be determined by dividing vector ab at c in the same ratio as C divides AB in Fig. 7.4 (*a*).

In other words

$$\frac{ac}{ab} = \frac{AC}{AB}$$



Join oc. The *vector oc represents the absolute velocity of point C (v_C) and the vector ac represents the velocity of C with respect to A *i.e.* v_{CA}.

3. The absolute velocity of any other point D outside AB, as shown in Fig. 7.4 (*a*), may also be obtained by completing the velocity triangle abd and similar to triangle ABD, as shown in Fig. 7.4 (*b*).

4. The angular velocity of the link AB may be found by dividing the relative velocity of B with respect to A (*i.e.* v_{BA}) to the length of the link AB. Mathematically, angular velocity of the link AB,

$$\omega_{AB} = \frac{v_{BA}}{AB} = \frac{ab}{AB}$$

7.5. Velocities in Slider Crank Mechanism

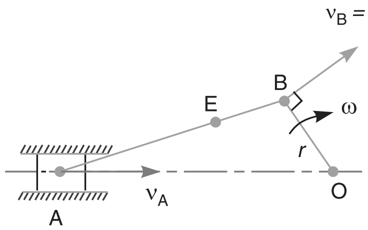
In the previous article, we have discused the relative velocity method for the velocity of any point on a link, whose direction of motion and velocity of some other point on the same link is known. The same method may also be applied for the velocities in a slider crank mechanism.

A slider crank mechanism is shown in Fig. 7.5 (*a*). The slider A is attached to the connecting rod AB. Let the radius of crank OB be r and let it rotates in a clockwise direction, about the point O with uniform angular velocity ω rad/s. Therefore, the velocity of B *i.e.* v_B is known in magnitude and direction. The slider reciprocates along the line of stroke AO.

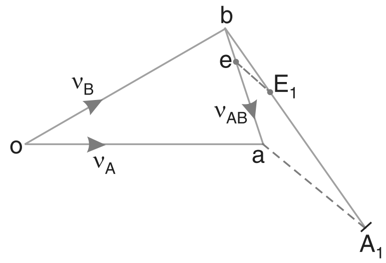
The velocity of the slider A (*i.e.* v_A) may be determined by relative velocity method as discussed below :

1. From any point o, draw vector ob parallel to the direction of v_B (or perpendicular to OB) such that $ob = v_B = \omega.r$, to some suitable scale, as shown in Fig. 7.5 (*b*).

* The absolute velocities of the points are measured from the pole (*i.e.* fixed points) of the velocity diagram.



(*a*) Slider crank mechanism.



(*b*) Velocity diagram.

Fig. 7.5

2. Since AB is a rigid link, therefore the velocity of A relative to B is perpendicular to AB. Now draw vector *ba* perpendicular to AB to represent the velocity of A with respect to B *i.e.* v_{AB}.

3. From point *o*, draw vector *oa* parallel to the path of motion of the slider A (which is along AO only). The vectors *ba* and *oa* intersect at *a*. Now *oa* represents the velocity of the slider A *i.e.* v_A, to the scale.

The angular velocity of the connecting rod AB (ω_{AB}) may be determined as follows:

$$\omega_{AB} = \frac{v_{BA}}{AB} = \frac{ab}{AB} \qquad \text{(Anticlockwise about A)}$$

The direction of vector *ab* (or *ba*) determines the sense of ω_{AB} which shows that it is anticlockwise.

Note : The absolute velocity of any other point E on the connecting rod AB may also be found out by dividing vector *ba* such that $be/ba = BE/BA$. This is done by drawing any line bA_1 equal in length of BA. Mark $bE_1 = BE$. Join aA_1. From E_1 draw a line E_1e parallel to aA_1. The vector *oe* now represents the velocity of E and vector *ae* represents the velocity of E with respect to A.

7.6. Rubbing Velocity at a Pin Joint

The links in a mechanism are mostly connected by means of pin joints. The rubbing velocity is defined as **the algebraic sum between the angular velocities of the two links which are connected by pin joints, multiplied by the radius of the pin.**

Consider two links OA and OB connected by a pin joint at O as shown in Fig. 7.6.

Let ω_1 = Angular velocity of the link OA or the angular velocity of the point A with respect to O.

ω_2 = Angular velocity of the link OB or the angular velocity of the point B with respect to O, and

r = Radius of the pin.

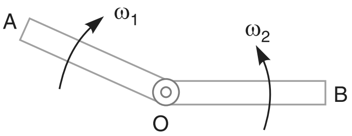


Fig. 7.6. Links connected by pin joints.

According to the definition,

Rubbing velocity at the pin joint O

$= (\omega_1 - \omega_2)\, r$, if the links move in the same direction

$= (\omega_1 + \omega_2)\, r$, if the links move in the opposite direction

Note : When the pin connects one sliding member and the other turning member, the angular velocity of the sliding member is zero. In such cases,

Rubbing velocity at the pin joint = $\omega.r$

where ω = Angular velocity of the turning member, and

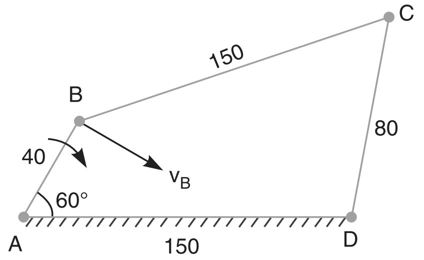
r = Radius of the pin.

Example 7.1. *In a four bar chain ABCD, AD is fixed and is 150 mm long. The crank AB is 40 mm long and rotates at 120 r.p.m. clockwise, while the link CD = 80 mm oscillates about D. BC and AD are of equal length. Find the angular velocity of link CD when angle BAD = 60°.*

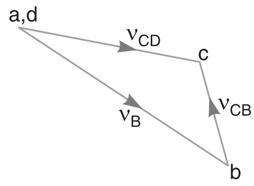
Solution. Given : N_{BA} = 120 r.p.m. or $\omega_{BA} = 2\,\pi \times 120/60 = 12.568$ rad/s

Since the length of crank AB = 40 mm = 0.04 m, therefore velocity of B with respect to A or velocity of B, (because A is a fixed point),

$$v_{BA} = v_B = \omega_{BA} \times AB = 12.568 \times 0.04 = 0.503 \text{ m/s}$$



(*a*) Space diagram (All dimensions in mm).



(*b*) Velocity diagram.

Fig. 7.7

First of all, draw the space diagram to some suitable scale, as shown in Fig. 7.7 (*a*). Now the velocity diagram, as shown in Fig. 7.7 (*b*), is drawn as discussed below :

1. Since the link AD is fixed, therefore points *a* and *d* are taken as one point in the velocity diagram. Draw vector *ab* perpendicular to BA, to some suitable scale, to represent the velocity of B with respect to A or simply velocity of B (*i.e.* v_{BA} or v_B) such that

$$\text{vector } ab = v_{BA} = v_B = 0.503 \text{ m/s}$$

2. Now from point *b*, draw vector *bc* perpendicular to *CB* to represent the velocity of *C* with respect to *B* (*i.e.* v_{CB}) and from point *d*, draw vector *dc* perpendicular to *CD* to represent the velocity of *C* with respect to *D* or simply velocity of *C* (*i.e.* v_{CD} or v_C). The vectors *bc* and *dc* intersect at *c*.

By measurement, we find that

$$v_{CD} = v_C = \text{vector } dc = 0.385 \text{ m/s}$$

We know that CD = 80 mm = 0.08 m

∴ Angular velocity of link *CD*,

$$\omega_{CD} = \frac{v_{CD}}{CD} = \frac{0.385}{0.08} = 4.8 \text{ rad/s (clockwise about } D) \textbf{ Ans.}$$

Example 7.2. *The crank and connecting rod of a theoretical steam engine are 0.5 m and 2 m long respectively. The crank makes 180 r.p.m. in the clockwise direction. When it has turned 45° from the inner dead centre position, determine :* **1.** *velocity of piston,* **2.** *angular velocity of connecting rod,* **3.** *velocity of point E on the connecting rod 1.5 m from the gudgeon pin,* **4.** *velocities of rubbing at the pins of the crank shaft, crank and crosshead when the diameters of their pins are 50 mm, 60 mm and 30 mm respectively,* **5.** *position and linear velocity of any point G on the connecting rod which has the least velocity relative to crank shaft.*

Solution. Given : $N_{BO} = 180$ r.p.m. or $\omega_{BO} = 2\pi \times 180/60 = 18.852$ rad/s

Since the crank length $OB = 0.5$ m, therefore linear velocity of B with respect to O or velocity of B (because O is a fixed point),

$$v_{BO} = v_B = \omega_{BO} \times OB = 18.852 \times 0.5 = 9.426 \text{ m/s}$$

. . . (Perpendicular to BO)

1. ***Velocity of piston***

First of all draw the space diagram, to some suitable scale, as shown in Fig. 7.8 (*a*). Now the velocity diagram, as shown in Fig. 7.8 (*b*), is drawn as discussed below :

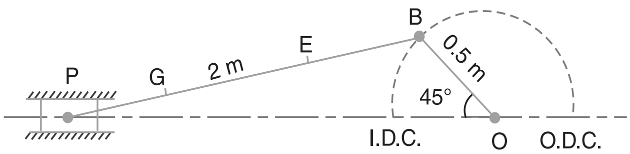
1. Draw vector ob perpendicular to BO, to some suitable scale, to represent the velocity of B with respect to O or velocity of B such that

$$\text{vector } ob = v_{BO} = v_B = 9.426 \text{ m/s}$$

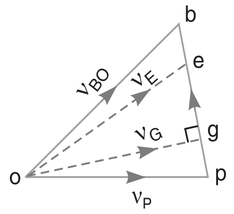
2. From point b, draw vector bp perpendicular to BP to represent velocity of P with respect to B (*i.e.* v_{PB}) and from point o, draw vector op parallel to PO to represent velocity of P with respect to O (*i.e.* v_{PO} or simply v_P). The vectors bp and op intersect at point p.

By measurement, we find that velocity of piston P,

$$v_P = \text{vector } op = 8.15 \text{ m/s} \quad \textbf{Ans.}$$



(*a*) Space diagram.



(*b*) Velocity diagram.

Fig. 7.8

2. ***Angular velocity of connecting rod***

From the velocity diagram, we find that the velocity of P with respect to B,

$$v_{PB} = \text{vector } bp = 6.8 \text{ m/s}$$

Since the length of connecting rod PB is 2 m, therefore angular velocity of the connecting rod,

$$\omega_{PB} = \frac{v_{PB}}{PB} = \frac{6.8}{2} = 3.4 \text{ rad/s (Anticlockwise)} \quad \textbf{Ans.}$$

3. ***Velocity of point E on the connecting rod***

The velocity of point E on the connecting rod 1.5 m from the gudgeon pin (*i.e.* $PE = 1.5$ m) is determined by dividing the vector bp at e in the same ratio as E divides PB in Fig. 7.8 (*a*). This is done in the similar way as discussed in Art 7.6. Join oe. The vector oe represents the velocity of E. By measurement, we find that velocity of point E,

$$v_E = \text{vector } oe = 8.5 \text{ m/s} \quad \textbf{Ans.}$$

Note : The point e on the vector bp may also be obtained as follows :

$$\frac{BE}{BP} = \frac{be}{bp} \quad \text{or} \quad be = \frac{BE \cdot bp}{BP}$$

4. ***Velocity of rubbing***

We know that diameter of crank-shaft pin at O,

$$d_O = 50 \text{ mm} = 0.05 \text{ m}$$

Diameter of crank-pin at *B*,

$$d_B = 60 \text{ mm} = 0.06 \text{ m}$$

and diameter of cross-head pin,

$$d_C = 30 \text{ mm} = 0.03 \text{ m}$$

We know that velocity of rubbing at the pin of crank-shaft

$$= \frac{d_O}{2} \cdot \omega_{BO} = \frac{0.05}{2} \times 18.85 = 0.47 \text{ m/s} \quad \textbf{Ans.}$$

Velocity of rubbing at the pin of crank

$$= \frac{d_B}{2}(\omega_{BO} + \omega_{PB}) = \frac{0.06}{2}(18.85 + 3.4) = 0.6675 \text{ m/s} \quad \textbf{Ans.}$$

...($\because$ ω_{BO} is clockwise and ω_{PB} is anticlockwise.)

and velocity of rubbing at the pin of cross-head

$$= \frac{d_C}{2} \cdot \omega_{PB} = \frac{0.03}{2} \times 3.4 = 0.051 \text{ m/s} \quad \textbf{Ans.}$$

...($\because$ At the cross-head, the slider does not rotate and only the connecting rod has angular motion.)

5. ***Position and linear velocity of point G on the connecting rod which has the least velocity relative to crank-shaft***

The position of point *G* on the connecting rod which has the least velocity relative to crank-shaft is determined by drawing perpendicular from *o* to vector *bp*. Since the length of *og* will be the least, therefore the point *g* represents the required position of *G* on the connecting rod.

By measurement, we find that

$$\text{vector } bg = 5 \text{ m/s}$$

The position of point *G* on the connecting rod is obtained as follows:

$$\frac{bg}{bp} = \frac{BG}{BP} \quad \text{or} \quad BG = \frac{bg}{bp} \cdot BP = \frac{5}{6.8} \times 2 = 1.47 \text{ m} \quad \textbf{Ans.}$$

By measurement, we find that the linear velocity of point *G*,

$$v_G = \text{vector } og = 8 \text{ m/s} \quad \textbf{Ans.}$$

7.8. Mechanical Advantage

It is defined as the ratio of the load to the effort. In a four bar mechanism, as shown in Fig. 7.24, the link DA is called the driving link and the link CB as the driven link. The force F_A acting at A is the effort and the force F_B at B will be the load or the resistance to overcome. We know from the principle of conservation of energy, neglecting effect of friction,

$$F_A \times v_A = F_B \times v_B \quad \text{or} \quad \frac{F_B}{F_A} = \frac{v_A}{v_B}$$

∴ Ideal mechanical advantage,

$$\text{M.A.}_{(ideal)} = \frac{F_B}{F_A} = \frac{v_A}{v_B}$$

If we consider the effect of friction, less resistance will be overcome with the given effort. Therefore the actual mechanical advantage will be less.

Let η = Efficiency of the mechanism.

∴ Actual mechanical advantage,

$$\text{M.A.}_{(actual)} = \eta \cdot \frac{F_B}{F_A} = \eta \cdot \frac{v_A}{v_B}$$

Note : The mechanical advantage may also be defined as the ratio of output torque to the input torque.

Let T_A = Driving torque,

T_B = Resisting torque,

ω_A and ω_B = Angular velocity of the driving and driven links respectively.

∴ Ideal mechanical advantage,

$$\text{M.A.}_{(ideal)} = \frac{T_B}{T_A} = \frac{\omega_A}{\omega_B} \qquad \text{. . . (Neglecting effect of friction)}$$

and actual mechanical advantage,

$$\text{M.A.}_{(actual)} = \eta \cdot \frac{T_B}{T_A} = \eta \cdot \frac{\omega_A}{\omega_B} \qquad \text{. . . (Considering the effect of friction)}$$

Example 7.10. *A four bar mechanism has the following dimensions :*

DA = 300 mm ; CB = AB = 360 mm ; DC = 600 mm. The link DC is fixed and the angle ADC is 60°. The driving link DA rotates uniformly at a speed of 100 r.p.m. clockwise and the constant driving torque has the magnitude of 50 N-m. Determine the velocity of the point B and angular velocity of the driven link CB. Also find the actual mechanical advantage and the resisting torque if the efficiency of the mechanism is 70 per cent.

Solution. Given : N_{AD} = 100 r.p.m. or $\omega_{AD} = 2\pi \times 100/60 = 10.47$ rad/s ; T_A = 50 N-m

Since the length of driving link, DA = 300 mm = 0.3 m, therefore velocity of A with respect to D or velocity of A (because D is a fixed point),

$$v_{AD} = v_A = \omega_{AD} \times DA = 10.47 \times 0.3 = 3.14 \text{ m/s}$$

. . . (Perpendicular to DA)

Velocity of point B

First of all draw the space diagram, to some suitable scale, as shown in Fig. 7.25 (*a*). Now the velocity diagram, as shown in Fig. 7.25 (*b*), is drawn as discussed below :

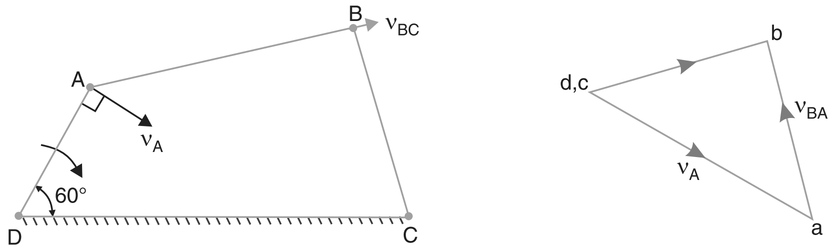
1. Since the link *DC* is fixed, therefore points *d* and *c* are taken as one point in the velocity diagram. Draw vector *da* perpendicular to *DA*, to some suitable scale, to represent the velocity of *A* with respect to *D* or simply velocity of *A* (*i.e.* v_{AD} or v_A) such that

$$\text{vector } da = v_{AD} = v_A = 3.14 \text{ m/s}$$

2. Now from point *a*, draw vector *ab* perpendicular to *A B* to represent the velocity of *B* with respect to *A* (*i.e.* v_{BA}), and from point *c* draw vector *cb* perpendicular to *CB* to represent the velocity of *B* with respect to *C* or simply velocity of *B* (*i.e.* v_{BC} or v_B). The vectors *ab* and *cb* intersect at *b*.

By measurement, we find that velocity of point *B*,

$$v_B = v_{BC} = \text{vector } cb = 2.25 \text{ m/s} \quad \textbf{Ans.}$$



(*a*) Space diagram. (*b*) Velocity diagram.

Fig. 7.25

Angular velocity of the driven link CB

Since *CB* = 360 mm = 0.36 m, therefore angular velocity of the driven link *CB*,

$$\omega_{BC} = \frac{v_{BC}}{BC} = \frac{2.25}{0.36} = 6.25 \text{ rad/s (Clockwise about } C) \quad \textbf{Ans.}$$

Actual mechanical advantage

We know that the efficiency of the mechanism,

$$\eta = 70\% = 0.7 \qquad \text{. . . (Given)}$$

∴ Actual mechanical advantage,

$$\text{M.A.}_{(actual)} = \eta \cdot \frac{\omega_A}{\omega_B} = 0.7 \cdot \frac{10.47}{6.25} = 1.17 \quad \textbf{Ans.}$$

Resisting torque

Let T_B = Resisting torque.

We know that efficiency of the mechanism (η),

$$0.7 = \frac{T_B \cdot \omega_B}{T_A \cdot \omega_A} = \frac{T_B \cdot 6.25}{50 \cdot 10.47} = 0.012 T_B$$

∴ T_B = 58.3 N–m **Ans.**

OBJECTIVE TYPE QUESTIONS

1. The direction of linear velocity of any point on a link with respect to another point on the same link is

   (*a*) parallel to the link joining the points (*b*) perpendicular to the link joining the points

   (*c*) at 45° to the link joining the points (*d*) none of these

2. The magnitude of linear velocity of a point B on a link AB relative to point A is

   (*a*) ωAB (*b*) $\omega (AB)^2$

   (*c*) $\omega^2 . AB$ (*d*) $(\omega . AB)^2$

   where ω = Angular velocity of the link AB.

3. The two links OA and OB are connected by a pin joint at O. If the link OA turns with angular velocity ω_1 rad/s in the clockwise direction and the link OB turns with angular velocity ω_2 rad/s in the anti-clockwise direction, then the rubbing velocity at the pin joint O is

   (*a*) $\omega_1.\omega_2.r$ (*b*) $(\omega_1 - \omega_2)\, r$

   (*c*) $(\omega_1 + \omega_2)\, r$ (*d*) $(\omega_1 - \omega_2)\, 2\, r$

   where r = Radius of the pin at O.

4. In the above question, if both the links OA and OB turn in clockwise direction, then the rubbing velocity at the pin joint O is

   (*a*) $\omega_1.\omega_2.r$ (*b*) $(\omega_1 - \omega_2)\, r$

   (*c*) $(\omega_1 + \omega_2)\, r$ (*d*) $(\omega_1 - \omega_2)\, 2\, r$

5. In a four bar mechanism, as shown in Fig. 7.43, if a force F_A is acting at point A in the direction of its velocity v_A and a force F_B is transmitted to the joint B in the direction of its velocity v_B, then the ideal mechanical advantage is equal to

   (*a*) $F_B.v_A$ (*b*) $F_A.v_B$

   (*c*) $\dfrac{F_B}{v_B}$ (*d*) $\dfrac{F_B}{F_A}$

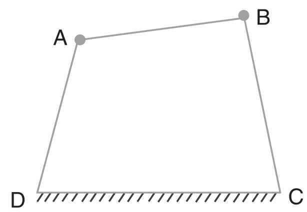


Fig. 7.43

ANSWERS

1. (*b*) **2.** (*a*) **3.** (*c*) **4.** (*b*) **5.** (*d*)

Acceleration in Mechanisms

Features

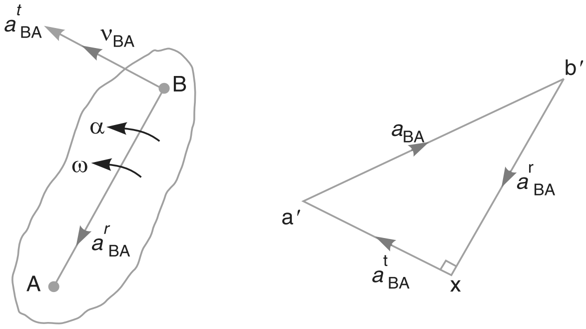
1. *Introduction.*
2. *Acceleration Diagram for a Link.*
3. *Acceleration of a Point on a Link.*
4. *Acceleration in the Slider Crank Mechanism.*
5. *Coriolis Component of Acceleration.*

8.1. Introduction

We have discussed in the previous chapter the velocities of various points in the mechanisms. Now we shall discuss the acceleration of points in the mechanisms. The acceleration analysis plays a very important role in the development of machines and mechanisms.

8.2. Acceleration Diagram for a Link

Consider two points *A* and *B* on a rigid link as shown in Fig. 8.1 (*a*). Let the point *B* moves with respect to *A*, with an angular velocity of ω rad/s and let α rad/s² be the angular acceleration of the link *AB*.



(*a*) Link. (*b*) Acceleration diagram.

Fig. 8.1. Acceleration for a link.

We have already discussed that acceleration of a particle whose velocity changes both in magnitude and direction at any instant has the following two components :

1. The ***centripetal or radial component,*** which is perpendicular to the velocity of the particle at the given instant.

2. The ***tangential component,*** which is parallel to the velocity of the particle at the given instant.

Thus for a link AB, the velocity of point B with respect to A (*i.e.* v_{BA}) is perpendicular to the link AB as shown in Fig. 8.1 (*a*). Since the point B moves with respect to A with an angular velocity of ω rad/s, therefore centripetal or radial component of the acceleration of B with respect to A,

$$a_{BA}^{r} = \omega^2 \cdot \text{Length of link } AB = \omega^2 \cdot AB = v_{BA}^2 / AB \qquad \ldots\left(\because \omega = \frac{v_{BA}}{AB}\right)$$

This radial component of acceleration acts perpendicular to the velocity v_{BA}, In other words, it acts ***parallel*** to the link AB.

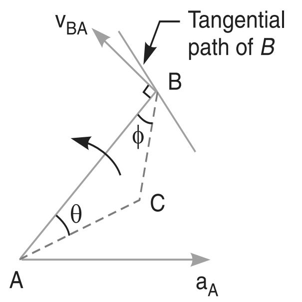
We know that tangential component of the acceleration of B with respect to A,

$$a_{BA}^{t} = \alpha \cdot \text{Length of the link } AB = \alpha \cdot AB$$

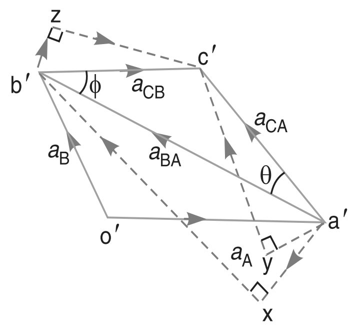
This tangential component of acceleration acts parallel to the velocity v_{BA}. In other words, it acts ***perpendicular*** to the link AB.

In order to draw the acceleration diagram for a link AB, as shown in Fig. 8.1 (*b*), from any point *b'*, draw vector ***b'x parallel to BA*** to represent the radial component of acceleration of B with respect to A *i.e.* a_{BA}^{r} and from point *x* draw vector *xa'* perpendicular to BA to represent the tangential component of acceleration of B with respect to A *i.e.* a_{BA}^{t} . ***Join b' a'.*** The vector *b' a'* (known as ***acceleration image*** of the link AB) represents the total acceleration of B with respect to A (*i.e.* a_{BA}) and it is the vector sum of radial component (a_{BA}^{r}) and tangential component (a_{BA}^{t}) of acceleration.

8.3. Acceleration of a Point on a Link



(*a*) Points on a Link.



(*b*) Acceleration diagram.

Fig. 8.2. Acceleration of a point on a link.

Consider two points A and B on the rigid link, as shown in Fig. 8.2 (*a*). Let the acceleration of the point A *i.e.* a_A is known in magnitude and direction and the direction of path of B is given. The acceleration of the point B is determined in magnitude and direction by drawing the acceleration diagram as discussed below.

1. From any point *o'*, draw vector *o'a'* parallel to the direction of absolute acceleration at point A *i.e.* a_A , to some suitable scale, as shown in Fig. 8.2 (*b*).

2. We know that the acceleration of B with respect to A *i.e.* a_{BA} has the following two components:

(*i*) Radial component of the acceleration of B with respect to A *i.e.* a^r_{BA}, and

(*ii*) Tangential component of the acceleration B with respect to A *i.e.* a^t_{BA}. These two components are mutually perpendicular.

3. Draw vector $a'x$ parallel to the link AB (because radial component of the acceleration of B with respect to A will pass through AB), such that

$$\text{vector } a'x = a^r_{BA} = v^2_{BA} / AB$$

where v_{BA} = Velocity of B with respect to A.

Note: The value of v_{BA} may be obtained by drawing the velocity diagram as discussed in the previous chapter.

4. From point x, draw vector xb' perpendicular to AB or vector $a'x$ (because tangential component of B with respect to A *i.e.* a^t_{BA}, is perpendicular to radial component a^r_{BA}) and through o' draw a line parallel to the path of B to represent the absolute acceleration of B *i.e.* a_B. The vectors xb' and $o'b'$ intersect at b'. Now the values of a_B and a^t_{BA} may be measured, to the scale.

A refracting telescope uses mechanisms to change directions.

Note : This picture is given as additional information and is not a direct example of the current chapter.

5. By joining the points a' and b' we may determine the total acceleration of B with respect to A *i.e.* a_{BA}. The vector $a'b'$ is known as ***acceleration image*** of the link AB.

6. For any other point C on the link, draw triangle $a'b'c'$ similar to triangle ABC. Now vector $b'c'$ represents the acceleration of C with respect to B *i.e.* a_{CB}, and vector $a'c'$ represents the acceleration of C with respect *to* A *i.e.* a_{CA}. As discussed above, a_{CB} and a_{CA} will each have two components as follows :

(*i*) a_{CB} has two components; a^r_{CB} and a^t_{CB} as shown by triangle $b'zc'$ in Fig. 8.2 (*b*), in which $b'z$ is parallel to BC and zc' is perpendicular to $b'z$ or BC.

(*ii*) a_{CA} has two components ; a^r_{CA} and a^t_{CA} as shown by triangle $a'yc'$ in Fig. 8.2 (*b*), in which $a'y$ is parallel to AC and yc' is perpendicular to $a'y$ or AC.

7. The angular acceleration of the link AB is obtained by dividing the tangential components of the acceleration of B with respect to A (a^t_{BA}) to the length of the link. Mathematically, angular acceleration of the link AB,

$$\alpha_{AB} = a^t_{BA} / AB$$

8.4. Acceleration in the Slider Crank Mechanism

A slider crank mechanism is shown in Fig. 8.3 (*a*). Let the crank OB makes an angle θ with the inner dead centre (*I.D.C*) and rotates in a clockwise direction about the fixed point O with uniform angular velocity ω_{BO} rad/s.

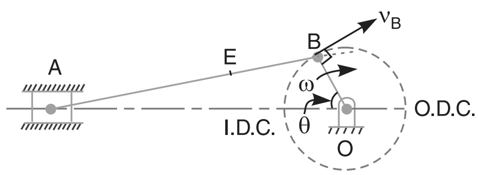
∴ Velocity of B with respect to O or velocity of B (because O is a fixed point),

$$v_{BO} = v_B = \omega_{BO} \cdot OB, \text{ acting tangentially at } B.$$

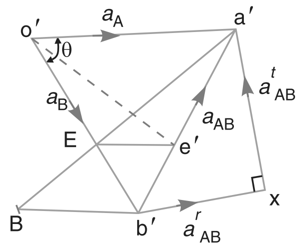
We know that centripetal or radial acceleration of B with respect to O or acceleration of B (because O is a fixed point),

$$a_{BO}^{r} = a_B = \omega_{BO}^2 \cdot OB = \frac{v_{BO}^2}{OB}$$

Note : A point at the end of a link which moves with constant angular velocity has no tangential component of acceleration.



(*a*) Slider crank mechanism.



(*b*) Acceleration diagram.

Fig. 8.3. Acceleration in the slider crank mechanism.

The acceleration diagram, as shown in Fig. 8.3 (*b*), may now be drawn as discussed below:

1. Draw vector $o'\,b'$ parallel to BO and set off equal in magnitude of $a_{BO}^{r} = a_B$, to some suitable scale.

2. From point b', draw vector $b'x$ parallel to BA. The vector $b'x$ represents the radial component of the acceleration of A with respect to B whose magnitude is given by :

$$a_{AB}^{r} = v_{AB}^2 / BA$$

Since the point B moves with constant angular velocity, therefore there will be ***no tangential*** component of the acceleration.

3. From point x, draw vector xa' perpendicular to $b'x$ (or AB). The vector xa' represents the tangential component of the acceleration of A with respect to B *i.e.* a_{AB}^{t}.

Note: When a point moves along a straight line, it has **no centripetal or radial** component of the acceleration.

4. Since the point A reciprocates along AO, therefore the acceleration must be parallel to velocity. Therefore from o', draw $o'\,a'$ parallel to AO, intersecting the vector xa' at a'.

Now the acceleration of the piston or the slider A (a_A) and a_{AB}^{t} may be measured to the scale.

5. The vector $b'\,a'$, which is the sum of the vectors $b'\,x$ and $x\,a'$, represents the total acceleration of A with respect to B *i.e.* a_{AB}. The vector $b'\,a'$ represents the acceleration of the connecting rod AB.

6. The acceleration of any other point on AB such as E may be obtained by dividing the vector $b'\,a'$ at e' in the same ratio as E divides AB in Fig. 8.3 (*a*). In other words

$$a'\,e' / a'\,b' = AE / AB$$

7. The angular acceleration of the connecting rod AB may be obtained by dividing the tangential component of the acceleration of A with respect to B $\left(a_{AB}^{t}\right)$ to the length of AB. In other words, angular acceleration of AB,

$$\alpha_{AB} = a_{AB}^{t} / AB \text{ (Clockwise about } B\text{)}$$

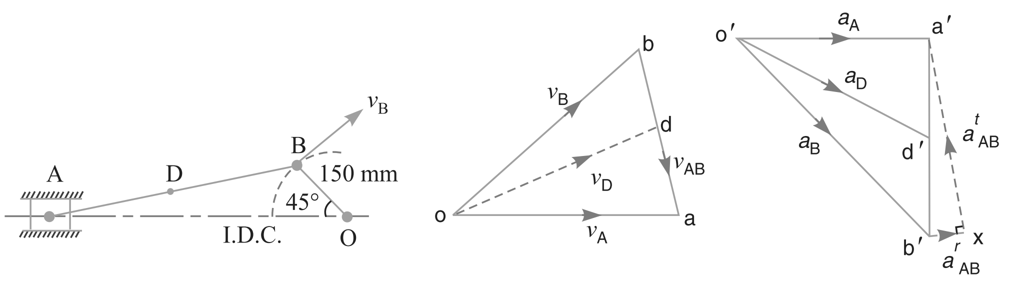
Example 8.1. *The crank of a slider crank mechanism rotates clockwise at a constant speed of 300 r.p.m. The crank is 150 mm and the connecting rod is 600 mm long. Determine :* **1.** *linear velocity and acceleration of the midpoint of the connecting rod, and* **2.** *angular velocity and angular acceleration of the connecting rod, at a crank angle of 45° from inner dead centre position.*

Solution. Given : N_{BO} = 300 r.p.m. or $\omega_{BO} = 2\,\pi \times 300/60 = 31.42$ rad/s; OB = 150 mm = 0.15 m ; BA = 600 mm = 0.6 m

We know that linear velocity of B with respect to O or velocity of B,

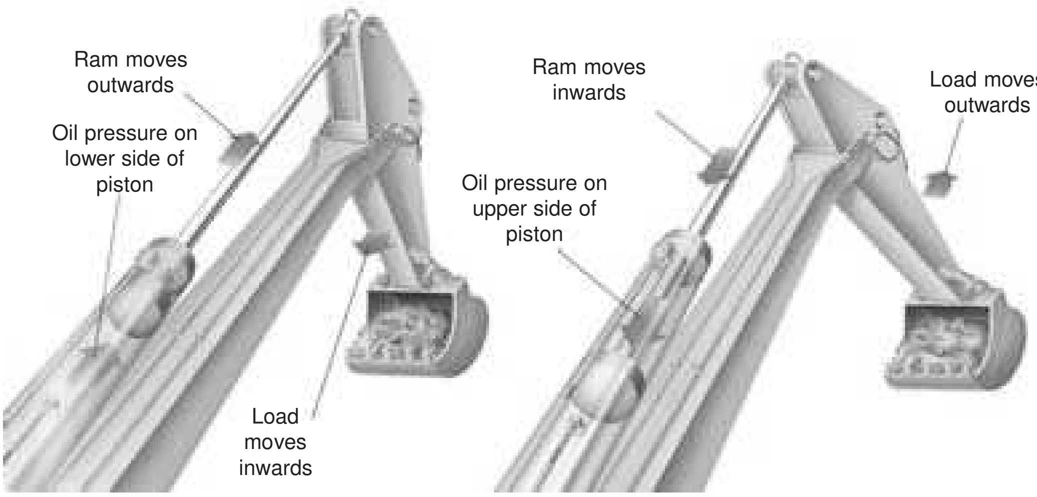
$$v_{BO} = v_B = \omega_{BO} \times OB = 31.42 \times 0.15 = 4.713 \text{ m/s}$$

...(Perpendicular to BO)



(*a*) Space diagram. (*b*) Velocity diagram. (*c*) Acceleration diagram.

Fig. 8.4



Pushing with fluids

Note : This picture is given as additional information and is not a direct example of the current chapter.

1. ***Linear velocity of the midpoint of the connecting rod***

First of all draw the space diagram, to some suitable scale; as shown in Fig. 8.4 (*a*). Now the velocity diagram, as shown in Fig. 8.4 (*b*), is drawn as discussed below:

1. Draw vector *ob* perpendicular to BO, to some suitable scale, to represent the velocity of B with respect to O or simply velocity of B *i.e.* v_{BO} or v_B, such that

$$\text{vector } ob = v_{BO} = v_B = 4.713 \text{ m/s}$$

2. From point *b*, draw vector *ba* perpendicular to BA to represent the velocity of A with respect to B *i.e.* v_{AB} , and from point *o* draw vector *oa* parallel to the motion of A (which is along AO) to represent the velocity of A *i.e.* v_A. The vectors *ba* and *oa* intersect at *a.*

By measurement, we find that velocity of A with respect to B,

$$v_{AB} = \text{vector } ba = 3.4 \text{ m/s}$$

and Velocity of A, $v_A = \text{vector } oa = 4 \text{ m/s}$

3. In order to find the velocity of the midpoint D of the connecting rod AB, divide the vector ba at d in the same ratio as D divides AB, in the space diagram. In other words,

$$bd / ba = BD/BA$$

Note: Since D is the midpoint of AB, therefore d is also midpoint of vector ba.

4. Join od. Now the vector od represents the velocity of the midpoint D of the connecting rod *i.e.* v_D.

By measurement, we find that

$$v_D = \text{vector } od = 4.1 \text{ m/s } \textbf{Ans.}$$

Acceleration of the midpoint of the connecting rod

We know that the radial component of the acceleration of B with respect to O or the acceleration of B,

$$a^r_{BO} = a_B = \frac{v^2_{BO}}{OB} = \frac{(4.713)^2}{0.15} = 148.1 \text{ m/s}^2$$

and the radial component of the acceleraiton of A with respect to B,

$$a^r_{AB} = \frac{v^2_{AB}}{BA} = \frac{(3.4)^2}{0.6} = 19.3 \text{ m/s}^2$$

Now the acceleration diagram, as shown in Fig. 8.4 (*c*) is drawn as discussed below:

1. Draw vector $o'b'$ parallel to BO, to some suitable scale, to represent the radial component of the acceleration of B with respect to O or simply acceleration of B *i.e.* a^r_{BO} or a_B, such that

$$\text{vector } o'b' = a^r_{BO} = a_B = 148.1 \text{ m/s}^2$$

Note: Since the crank OB rotates at a constant speed, therefore there will be no tangential component of the acceleration of B with respect to O.

2. The acceleration of A with respect to B has the following two components:

(*a*) The radial component of the acceleration of A with respect to B *i.e.* a^r_{AB}, and

(*b*) The tangential component of the acceleration of A with respect to B *i.e.* a^t_{AB}. These two components are mutually perpendicular.

Therefore from point b', draw vector $b'x$ parallel to AB to represent $a^r_{AB} = 19.3 \text{ m/s}^2$ and from point x draw vector xa' perpendicular to vector $b'x$ whose magnitude is yet unknown.

3. Now from o', draw vector $o'a'$ parallel to the path of motion of A (which is along AO) to represent the acceleration of A *i.e.* a_A. The vectors xa' and $o'a'$ intersect at a'. Join $a'b'$.

4. In order to find the acceleration of the midpoint D of the connecting rod AB, divide the vector $a'b'$ at d' in the same ratio as D divides AB. In other words

$$b'd' / b'a' = BD / BA$$

Note: Since D is the midpoint of AB, therefore d' is also midpoint of vector $b'a'$.

5. Join $o'd'$. The vector $o'd'$ represents the acceleration of midpoint D of the connecting rod *i.e.* a_D.

By measurement, we find that

$$a_D = \text{vector } o'd' = 117 \text{ m/s}^2 \textbf{ Ans.}$$

2. ***Angular velocity of the connecting rod***

We know that angular velocity of the connecting rod AB,

$$\omega_{AB} = \frac{v_{AB}}{BA} = \frac{3.4}{0.6} = 5.67 \text{ rad/s}^2 \text{ (Anticlockwise about } B) \textbf{ Ans.}$$

Angular acceleration of the connecting rod

From the acceleration diagram, we find that

$$a^t_{AB} = 103 \text{ m/s}^2 \qquad \text{...(By measurement)}$$

We know that angular acceleration of the connecting rod AB,

$$\alpha_{AB} = \frac{a^t_{AB}}{BA} = \frac{103}{0.6} = 171.67 \text{ rad/s}^2 \text{ (Clockwise about } B) \textbf{ Ans.}$$

Example 8.2. *An engine mechanism is shown in Fig. 8.5. The crank CB = 100 mm and the connecting rod BA = 300 mm with centre of gravity G, 100 mm from B. In the position shown, the crankshaft has a speed of 75 rad/s and an angular acceleration of 1200 rad/s². Find:* **1.** *velocity of G and angular velocity of AB, and* **2.** *acceleration of G and angular acceleration of AB.*

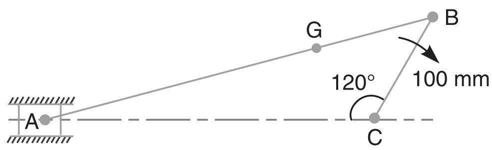


Fig. 8.5

Solution. Given : $\omega_{BC} = 75$ rad/s ; $\alpha_{BC} = 1200$ rad/s², $CB = 100$ mm $= 0.1$ m; $BA = 300$ mm $= 0.3$ m

We know that velocity of B with respect to C or velocity of B,

$$v_{BC} = v_B = \omega_{BC} \times CB = 75 \times 0.1 = 7.5 \text{ m/s} \qquad \text{...(Perpendicular to } BC)$$

Since the angular acceleration of the crankshaft, $\alpha_{BC} = 1200$ rad/s², therefore tangential component of the acceleration of B with respect to C,

$$a^t_{BC} = \alpha_{BC} \times CB = 1200 \times 0.1 = 120 \text{ m/s}^2$$

Note: When the angular acceleration is not given, then there will be no tangential component of the acceleration.

1. ***Velocity of G and angular velocity of AB***

First of all, draw the space diagram, to some suitable scale, as shown in Fig. 8.6 (*a*). Now the velocity diagram, as shown in Fig. 8.6 (*b*), is drawn as discussed below:

1. Draw vector cb perpendicular to CB, to some suitable scale, to represent the velocity of B with respect to C or velocity of B (*i.e.* v_{BC} *or* v_B), such that

$$\text{vector } cb = v_{BC} = v_B = 7.5 \text{ m/s}$$

2. From point b, draw vector ba perpendicular to BA to represent the velocity of A with respect to B *i.e.* v_{AB}, and from point c, draw vector ca parallel to the path of motion of A (which is along AC) to represent the velocity of A *i.e.* v_A. The vectors ba and ca intersect at a.

3. Since the point G lies on AB, therefore divide vector ab at g in the same ratio as G divides AB in the space diagram. In other words,

$$ag/ab = AG/AB$$

The vector cg represents the velocity of G.

By measurement, we find that velocity of G,

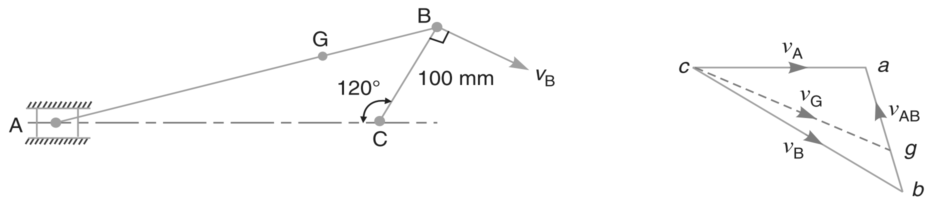
$$v_G = \text{vector } cg = 6.8 \text{ m/s} \textbf{ Ans.}$$

From velocity diagram, we find that velocity of A with respect to B,

$$v_{AB} = \text{vector } ba = 4 \text{ m/s}$$

We know that angular velocity of AB,

$$\omega_{AB} = \frac{v_{AB}}{BA} = \frac{4}{0.3} = 13.3 \text{ rad/s (Clockwise)} \textbf{ Ans.}$$



(*a*) Space diagram.

(*b*) Velocity diagram.

Fig. 8.6

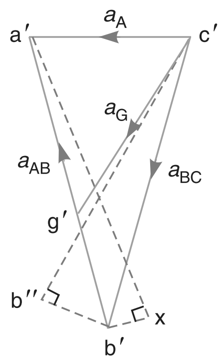
2. ***Acceleration of G and angular acceleration of AB***

We know that radial component of the acceleration of B with respect to C,

$$^{*}\ a^r_{BC} = \frac{v^2_{BC}}{CB} = \frac{(7.5)^2}{0.1} = 562.5 \text{ m/s}^2$$

and radial component of the acceleration of A with respect to B,

$$a^r_{AB} = \frac{v^2_{AB}}{BA} = \frac{4^2}{0.3} = 53.3 \text{ m/s}^2$$



(*c*) Acceleration diagram.

Fig. 8.6

Now the acceleration diagram, as shown in Fig. 8.6 (*c*), is drawn as discussed below:

1. Draw vector $c'b''$ parallel to CB, to some suitable scale, to represent the radial component of the acceleration of B with respect to C, *i.e.* a^r_{BC}, such that

$$\text{vector } c'b'' = a^r_{BC} = 562.5 \text{ m/s}^2$$

2. From point b'', draw vector $b''b'$ perpendicular to vector $c'b''$ or CB to represent the tangential component of the acceleration of B with respect to C *i.e.* a^t_{BC}, such that

$$\text{vector } b''b' = a^t_{BC} = 120 \text{ m/s}^2 \qquad \text{... (Given)}$$

3. Join $c'b'$. The vector $c'b'$ represents the total acceleration of B with respect to C *i.e.* a_{BC}.

4. From point b', draw vector $b'x$ parallel to BA to represent radial component of the acceleration of A with respect to B *i.e.* a^r_{AB} such that

$$\text{vector } b'x = a^r_{AB} = 53.3 \text{ m/s}^2$$

5. From point x, draw vector xa' perpendicular to vector $b'x$ or BA to represent tangential component of the acceleration of A with respect to B *i.e.* a^t_{AB}, whose magnitude is not yet known.

6. Now draw vector $c'a'$ parallel to the path of motion of A (which is along AC) to represent the acceleration of A *i.e.* a_A. The vectors xa' and $c'a'$ intersect at a'. Join $b'a'$. The vector $b'a'$ represents the acceleration of A with respect to B *i.e.* a_{AB}.

---

* When angular acceleration of the crank is not given, then there is no a^t_{BC}. In that case, $a^r_{BC} = a_{BC} = a_B$, as discussed in the previous example.

7. In order to find the acceleratio of G, divide vector $a'b'$ in g' in the same ratio as G divides BA in Fig. 8.6 (*a*). Join $c'g'$. The vector $c'g'$ represents the acceleration of G.

By measurement, we find that acceleration of G,

$$a_G = \text{vector } c'g' = 414 \text{ m/s}^2 \textbf{ Ans.}$$

From acceleration diagram, we find that tangential component of the acceleration of A with respect to B,

$$a^t_{AB} = \text{vector } xa' = 546 \text{ m/s}^2 \qquad \text{...(By measurement)}$$

∴ Angular acceleration of AB,

$$\alpha_{AB} = \frac{a^t_{AB}}{BA} = \frac{546}{0.3} = 1820 \text{ rad/s}^2 \text{ (Clockwise) } \textbf{Ans.}$$

Example 8.3. *In the mechanism shown in Fig. 8.7, the slider C is moving to the right with a velocity of 1 m/s and an acceleration of 2.5 m/s².*

The dimensions of various links are AB = 3 m inclined at 45° with the vertical and BC = 1.5 m inclined at 45° with the horizontal. Determine: **1.** *the magnitude of vertical and horizontal component of the acceleration of the point B, and* **2.** *the angular acceleration of the links AB and BC.*

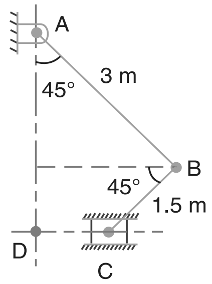


Fig. 8.7

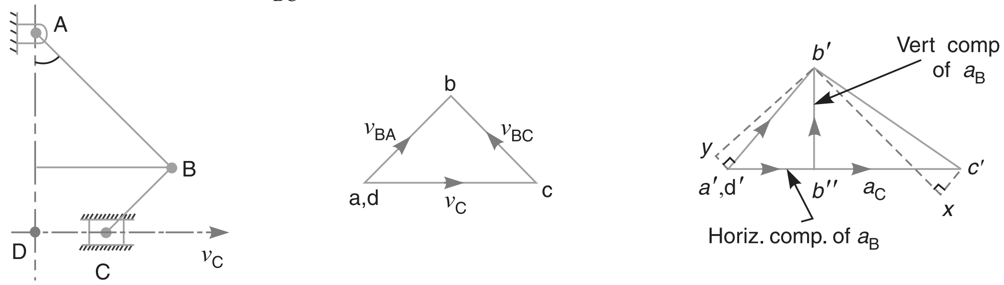
Solution. Given : $v_C = 1$ m/s ; $a_C = 2.5$ m/s²; $AB = 3$ m ; $BC = 1.5$ m

First of all, draw the space diagram, as shown in Fig. 8.8 (*a*), to some suitable scale. Now the velocity diagram, as shown in Fig. 8.8 (*b*), is drawn as discussed below:

1. Since the points A and D are fixed points, therefore they lie at one place in the velocity diagram. Draw vector dc parallel to DC, to some suitable scale, which represents the velocity of slider C with respect to D or simply velocity of C, such that

$$\text{vector } dc = v_{CD} = v_C = 1 \text{ m/s}$$

2. Since point B has two motions, one with respect to A and the other with respect to C, therefore from point a, draw vector ab perpendicular to AB to represent the velocity of B with respect to A, *i.e.* v_{BA} and from point c draw vector cb perpendicular to CB to represent the velocity of B with respect to C *i.e.* v_{BC}. The vectors ab and cb intersect at b.



(*a*) Space diagram. (*b*) Velocity diagram. (*c*) Acceleration diagram.

Fig. 8.8

By measurement, we find that velocity of B with respect to A,

$$v_{BA} = \text{vector } ab = 0.72 \text{ m/s}$$

and velocity of B with respect to C,

$$v_{BC} = \text{vector } cb = 0.72 \text{ m/s}$$

We know that radial component of acceleration of B with respect to C,

$$a_{BC}^{r} = \frac{v_{BC}^{2}}{CB} = \frac{(0.72)^2}{1.5} = 0.346 \text{ m/s}^2$$

and radial component of acceleration of B with respect to A,

$$a_{BA}^{r} = \frac{v_{BA}^{2}}{AB} = \frac{(0.72)^2}{3} = 0.173 \text{ m/s}^2$$

Now the acceleration diagram, as shown in Fig. 8.8 (*c*), is drawn as discussed below:

1. *Since the points A and D are fixed points, therefore they lie at one place in the acceleration diagram. Draw vector $d'c'$ parallel to DC, to some suitable scale, to represent the acceleration of C with respect to D or simply acceleration of C *i.e.* a_{CD} or a_C such that

$$\text{vector } d'c' = a_{CD} = a_C = 2.5 \text{ m/s}^2$$

2. The acceleration of B with respect to C will have two components, *i.e.* one radial component of B with respect to C $\left(a_{BC}^{r}\right)$ and the other tangential component of B with respect to C $\left(a_{BC}^{t}\right)$. Therefore from point c', draw vector $c'x$ parallel to CB to represent a_{BC}^{r} such that

$$\text{vector } c'x = a_{BC}^{r} = 0.346 \text{ m/s}^2$$

3. Now from point x, draw vector xb' perpendicular to vector $c'x$ or CB to represent a^t_{BC} whose magnitude is yet unknown.

4. The acceleration of B with respect to A will also have two components, *i.e.* one radial component of B with respect to A (a^r_{BA}) and other tangential component of B with respect to A (a^t_{BA}). Therefore from point a' draw vector $a'y$ parallel to AB to represent a^r_{BA}, such that

$$\text{vector } a'y = a^r_{BA} = 0.173 \text{ m/s}^2$$

5. From point y, draw vector yb' perpendicular to vector $a'y$ or AB to represent a_{BA}^{t}. The vector yb' intersect the vector xb' at b'. Join $a'b'$ and $c'b'$. The vector $a'b'$ represents the acceleration of point B (a_B) and the vector $c'b'$ represents the acceleration of B with respect to C.

1. ***Magnitude of vertical and horizontal component of the acceleration of the point B***

Draw $b'b''$ perpendicular to $a'c'$. The vector $b'b''$ is the vertical component of the acceleration of the point B and $a'b''$ is the horizontal component of the acceleration of the point B. By measurement,

vector $b'b'' = 1.13$ m/s^2 and vector $a'b'' = 0.9$ m/s^2 **Ans.**

2. Angular acceleration of AB and BC

By measurement from acceleration diagram, we find that tangential component of acceleration of the point B with respect to A,

$$a_{BA}^{t} = \text{vector } yb' = 1.41 \text{ m/s}^2$$

and tangential component of acceleration of the point B with respect to C,

$$a_{BC}^{t} = \text{vector } xb' = 1.94 \text{ m/s}^2$$

* If the mechanism consists of more than one fixed point, then all these points lie at the same place in the velocity and acceleration diagrams.

We know that angular acceleration of AB,

$$\alpha_{AB} = \frac{a^t_{BA}}{AB} = \frac{1.41}{3} = 0.47 \text{ rad/s}^2 \textbf{ Ans.}$$

and angular acceleration of BC,

$$\alpha_{BC} = \frac{a^t_{BA}}{CB} = \frac{1.94}{1.5} = 1.3 \text{ rad/s}^2 \textbf{ Ans.}$$

Example 8.4. *PQRS is a four bar chain with link PS fixed. The lengths of the links are PQ = 62.5 mm ; QR = 175 mm ; RS = 112.5 mm ; and PS = 200 mm. The crank PQ rotates at 10 rad/s clockwise. Draw the velocity and acceleration diagram when angle QPS = 60° and Q and R lie on the same side of PS. Find the angular velocity and angular acceleration of links QR and RS.*

Solution. Given : $\omega_{QP} = 10$ rad/s; $PQ = 62.5$ mm $= 0.0625$ m ; $QR = 175$ mm $= 0.175$ m ; $RS = 112.5$ mm $= 0.1125$ m ; $PS = 200$ mm $= 0.2$ m

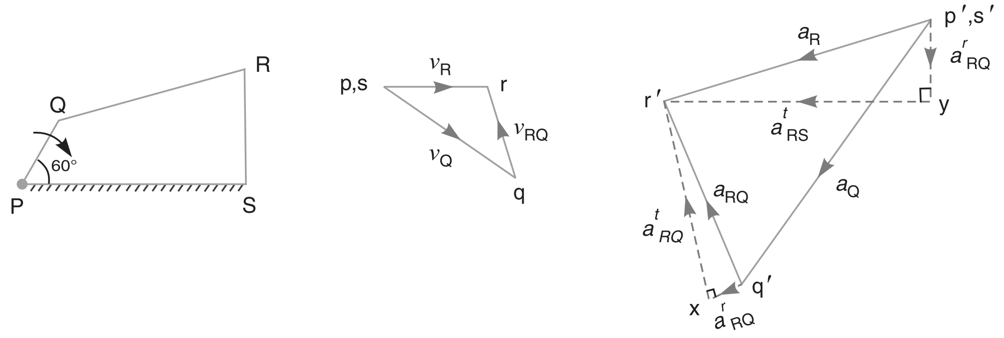
We know that velocity of Q with respect to P or velocity of Q,

$$v_{QP} = v_Q = \omega_{QP} \times PQ = 10 \times 0.0625 = 0.625 \text{ m/s}$$

...(Perpendicular to PQ)

Angular velocity of links QR and RS

First of all, draw the space diagram of a four bar chain, to some suitable scale, as shown in Fig. 8.9 (*a*). Now the velocity diagram as shown in Fig. 8.9 (*b*), is drawn as discussed below:



(*a*) Space diagram. (*b*) Velocity diagram. (*c*) Acceleration diagram.

Fig. 8.9

1. Since P and S are fixed points, therefore these points lie at one place in velocity diagram. Draw vector pq perpendicular to PQ, to some suitable scale, to represent the velocity of Q with respect to P or velocity of Q *i.e.* v_{QP} or v_Q such that

vector $pq = v_{QP} = v_Q = 0.625$ m/s

2. From point q, draw vector qr perpendicular to QR to represent the velocity of R with respect to Q (*i.e.* v_{RQ}) and from point s, draw vector sr perpendicular to SR to represent the velocity of R with respect to S or velocity of R (*i.e.* v_{RS} or v_R). The vectors qr and sr intersect at r. By measurement, we find that

v_{RQ} = vector $qr = 0.333$ m/s, and $v_{RS} = v_R$ = vector $sr = 0.426$ m/s

We know that angular velocity of link QR,

$$\omega_{QR} = \frac{v_{RQ}}{RQ} = \frac{0.333}{0.175} = 1.9 \text{ rad/s (Anticlockwise)} \textbf{ Ans.}$$

and angular velocity of link RS,

$$\omega_{RS} = \frac{v_{RS}}{SR} = \frac{0.426}{0.1125} = 3.78 \text{ rad/s (Clockwise)} \quad \textbf{Ans.}$$

Angular acceleration of links QR and RS

Since the angular acceleration of the crank PQ is not given, therefore there will be no tangential component of the acceleration of Q with respect to P.

We know that radial component of the acceleration of Q with respect to P (or the acceleration of Q),

$$a^r_{QP} = a_{QP} = a_Q = \frac{v^2_{QP}}{PQ} = \frac{(0.625)^2}{0.0625} = 6.25 \text{ m/s}^2$$

Radial component of the acceleration of R with respect to Q,

$$a^r_{RQ} = \frac{v^2_{RQ}}{QR} = \frac{(0.333)^2}{0.175} = 0.634 \text{ m/s}^2$$

and radial component of the acceleration of R with respect to S (or the acceleration of R),

$$a^r_{RS} = a_{RS} = a_R = \frac{v^2_{RS}}{SR} = \frac{(0.426)^2}{0.1125} = 1.613 \text{ m/s}^2$$

The acceleration diagram, as shown in Fig. 8.9 (*c*) is drawn as follows :

1. Since P and S are fixed points, therefore these points lie at one place in the acceleration diagram. Draw vector $p'q'$ parallel to PQ, to some suitable scale, to represent the radial component of acceleration of Q with respect to P or acceleration of Q *i.e* a^r_{QP} or a_Q such that

$$\text{vector } p'q' = a^r_{QP} = a_Q = 6.25 \text{ m/s}^2$$

2. From point q', draw vector $q'x$ parallel to QR to represent the radial component of acceleration of R with respect to Q *i.e.* a^r_{RQ} such that

$$\text{vector } q'x = a^r_{RQ} = 0.634 \text{ m/s}^2$$

3. From point x, draw vector xr' perpendicular to QR to represent the tangential component of acceleration of R with respect to Q *i.e* a^t_{RQ} whose magnitude is not yet known.

4. Now from point s', draw vector $s'y$ parallel to SR to represent the radial component of the acceleration of R with respect to S *i.e.* a^r_{RS} such that

$$\text{vector } s'y = a^r_{RS} = 1.613 \text{ m/s}^2$$

5. From point y, draw vector yr' perpendicular to SR to represent the tangential component of acceleration of R with respect to S *i.e.* a^t_{RS}.

6. The vectors xr' and yr' intersect at r'. Join $p'r$ and $q'r'$. By measurement, we find that

$$a^t_{RQ} = \text{vector } xr' = 4.1 \text{ m/s}^2 \text{ and } a^t_{RS} = \text{vector } yr' = 5.3 \text{ m/s}^2$$

We know that angular acceleration of link QR,

$$\alpha_{QR} = \frac{a^t_{RQ}}{QR} = \frac{4.1}{0.175} = 23.43 \text{ rad/s}^2 \text{ (Anticlockwise)} \quad \textbf{Ans.}$$

and angular acceleration of link RS,

$$\alpha_{RS} = \frac{a^t_{RS}}{SR} = \frac{5.3}{0.1125} = 47.1 \text{ rad/s}^2 \text{ (Anticlockwise)} \quad \textbf{Ans.}$$

Example 8.5. *The dimensions and configuration of the four bar mechanism, shown in Fig. 8.10, are as follows :*

P_1A = 300 mm; P_2B = 360 mm; AB = 360 mm, and P_1P_2 = 600 mm.

The angle $AP_1P_2 = 60°$. The crank P_1A has an angular velocity of 10 rad/s and an angular acceleration of 30 rad/s², both clockwise. Determine the angular velocities and angular accelerations of P_2B, and AB and the velocity and acceleration of the joint B.

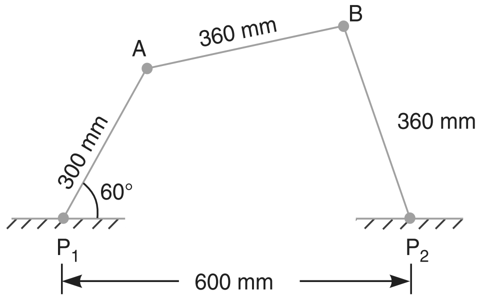


Fig. 8.10

Solution. Given : ω_{AP1} = 10 rad/s ; α_{AP1} = 30 rad/s²; P_1A = 300 mm = 0.3 m ; $P_2B = AB$ = 360 mm = 0.36 m

We know that the velocity of A with respect to P_1 or velocity of A,

$$v_{AP1} = v_A = \omega_{AP1} \times P_1A = 10 \times 0.3 = 3 \text{ m/s}$$

Velocity of B and angular velocitites of P_2B and AB

First of all, draw the space diagram, to some suitable scale, as shown in Fig. 8.11 (*a*). Now the velocity diagram, as shown in Fig. 8.11 (*b*), is drawn as discussed below:

1. Since P_1 and P_2 are fixed points, therefore these points lie at one place in velocity diagram. Draw vector $p_1\,a$ perpendicular to P_1A, to some suitable scale, to represent the velocity of A with respect to P_1 or velocity of A *i.e.* v_{AP1} or v_A, such that

$$\text{vector } p_1a = v_{A\,P1} = v_A = 3 \text{ m/s}$$

2. From point *a*, draw vector *ab* perpendicular to *AB* to represent velocity of *B* with respect to *A* (*i.e.* v_{BA}) and from point p_2 draw vector p_2b perpendicular to P_2B to represent the velocity of *B* with respect to P_2 or velocity of *B i.e.* v_{BP2} or v_B. The vectors *ab* and p_2b intersect at *b*.

By measurement, we find that

$$v_{BP2} = v_B = \text{vector } p_2b = 2.2 \text{ m/s} \quad \textbf{Ans.}$$

and $$v_{BA} = \text{vector } ab = 2.05 \text{ m/s}$$

We know that angular velocity of P_2B,

$$\omega_{P2B} = \frac{v_{BP_2}}{P_2B} = \frac{2.2}{0.36} = 6.1 \text{ rad/s (Clockwise)} \quad \textbf{Ans.}$$

and angular velocity of AB,

$$\omega_{AB} = \frac{v_{BA}}{AB} = \frac{2.05}{0.36} = 5.7 \text{ rad/s (Anticlockwise)} \quad \textbf{Ans.}$$

Acceleration of B and angular acceleration of P_2B and AB

We know that tangential component of the acceleration of A with respect to P_1,

$$a^t_{AP_1} = \alpha_{AP_1} \cdot P_1A = 30 \cdot 0.3 = 9 \text{ m/s}^2$$

Radial component of the acceleration of A with respect to P_1,

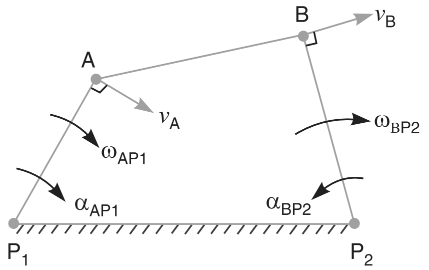
$$a^r_{AP_1} = \frac{v^2_{AP_1}}{P_1A} = \omega^2_{AP_1} \cdot P_1A = 10^2 \cdot 0.3 = 30 \text{ m/s}^2$$

Radial component of the acceleration of B with respect to A.

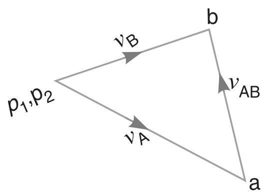
$$a_{BA}^{r} = \frac{v_{BA}^2}{AB} = \frac{(2.05)^2}{0.36} = 11.67 \text{ m/s}^2$$

and radial component of the acceleration of B with respect to P_2,

$$a_{BP_2}^{r} = \frac{v_{BP_2}^2}{P_2B} = \frac{(2.2)^2}{0.36} = 13.44 \text{ m/s}^2$$



(*a*) Space diagram.



(*b*) Velocity diagram.

Fig. 8.11

The acceleration diagram, as shown in Fig. 8.11 (*c*), is drawn as follows:

1. Since P_1 and P_2 are fixed points, therefore these points will lie at one place, in the acceleration diagram. Draw vector $p_1'x$ parallel to P_1A, to some suitable scale, to represent the radial component of the acceleration of A with respect to P_1, such that

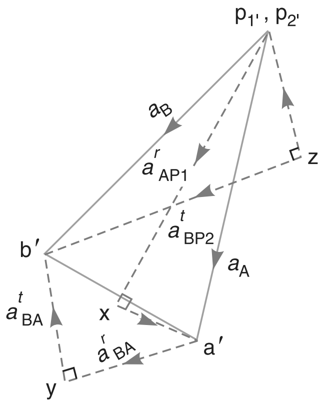
$$\text{vector } p_1'x = a_{AP_1}^{r} = 30 \text{ m/s}^2$$

2. From point x, draw vector xa' perpendicular to P_1A to represent the tangential component of the acceleration of A with respect to P_1, such that

$$\text{vector } xa' = a_{AP_1}^{t} = 9 \text{ m/s}^2$$

3. Join $p_1'a'$. The vector $p_1'a'$ represents the acceleration of A. By measurement, we find that the acceleration of A,

$$a_A = a_{AP_1} = 31.6 \text{ m/s}^2$$



(*c*) Acceleration diagram

Fig. 8.11

4. From point a', draw vector $a'y$ parallel to AB to represent the radial component of the acceleration of B with respect to A, such that

$$\text{vector } a'y = a_{BA}^{r} = 11.67 \text{ m/s}^2$$

5. From point y, draw vector yb' perpendicular to AB to represent the tangential component of the acceleration of B with respect to A (*i.e.* a_{BA}^{t}) whose magnitude is yet unknown.

6. Now from point p_2', draw vector $p_2'z$ parallel to P_2B to represent the radial component of the acceleration B with respect to P_2, such that

$$\text{vector } p_2'z = a_{BP_2}^{r} = 13.44 \text{ m/s}^2$$

7. From point z, draw vector zb' perpendicular to P_2B to represent the tangential component of the acceleration of B with respect to P_2 *i.e.* $a^t_{BP_2}$.

8. The vectors yb' and zb' intersect at b'. Now the vector $p_2'b'$ represents the acceleration of B with respect to P_2 or the acceleration of B *i.e.* a_{BP2} or a_B. By measurement, we find that

$$a_{BP2} = a_B = \text{vector } p_2'b' = 29.6 \text{ m/s}^2 \textbf{ Ans.}$$

Also vector $yb' = a^t_{BA} = 13.6 \text{ m/s}^2$, and vector $zb' = a^t_{BP_2} = 26.6 \text{ m/s}^2$

We know that angular acceleration of P_2B,

$$\alpha_{P2B} = \frac{a^t_{BP_2}}{P_2B} = \frac{26.6}{0.36} = 73.8 \text{ rad/s}^2 \text{ (Anticlockwise)} \textbf{ Ans.}$$

and angular acceleration of AB, $\alpha_{AB} = \dfrac{a^t_{BA}}{AB} = \dfrac{13.6}{0.36} = 37.8 \text{ rad/s}^2$ (Anticlockwise) **Ans.**

8.5. Coriolis Component of Acceleration

When a point on one link is sliding along another rotating link, such as in quick return motion mechanism, then the coriolis component of the acceleration must be calculated.

Consider a link OA and a slider B as shown in Fig. 8.26 (a). The slider B moves along the link OA. The point C is the coincident point on the link OA.

Let ω = Angular velocity of the link OA at time t seconds.

v = Velocity of the slider B along the link OA at time t seconds.

$\omega.r$ = Velocity of the slider B with respect to O (perpendicular to the link OA) at time t seconds, and

$(\omega + \delta\omega)$, $(v + \delta v)$ and $(\omega + \delta\omega)\,(r + \delta r)$

= Corresponding values at time $(t + \delta t)$ seconds.

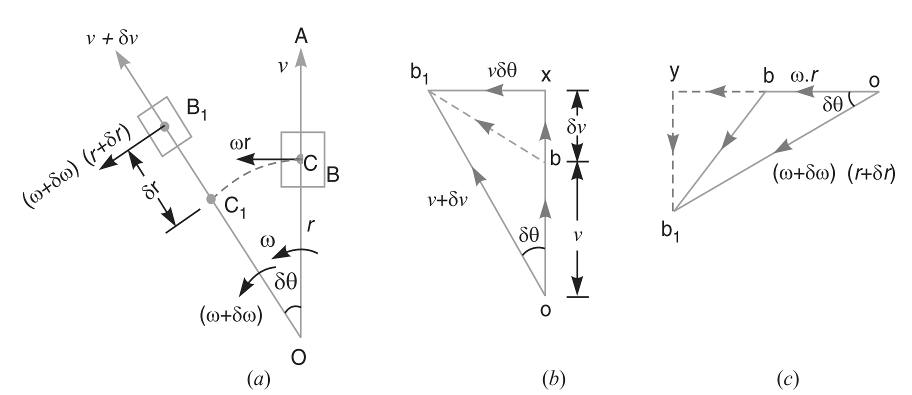


Fig. 8.26. Coriolis component of acceleration.

Let us now find out the acceleration of the slider *B* with respect to *O* and with respect to its coincident point *C* lying on the link *OA*.

Fig. 8.26 (*b*) shows the velocity diagram when their velocities v and $(v + \delta v)$ are considered. In this diagram, the vector bb_1 represents the change in velocity in time δt sec ; the vector bx represents the component of change of velocity bb_1 along OA (*i.e.* along radial direction) and vector xb_1 represents the component of change of velocity bb_1 in a direction perpendicular to OA (*i.e.* in tangential direction). Therefore

$$bx = ox - ob = (v + \delta v)\cos\delta\theta - v \uparrow$$

Since $\delta\theta$ is very small, therefore substituting $\cos\delta\theta = 1$, we have

$$bx = (v + \delta v - v) \uparrow = \delta v \uparrow$$

...(Acting radially outwards)

and $$xb_1 = (v + \delta v)\sin\delta\theta$$

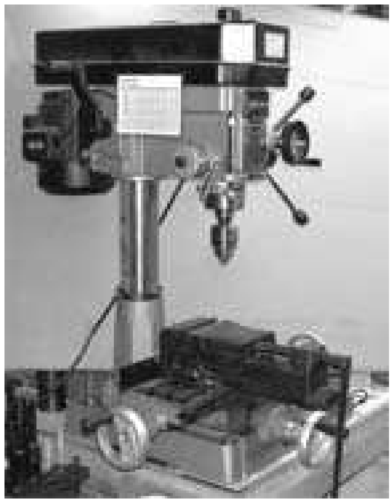
Since $\delta\theta$ is very small, therefore substituting $\sin\delta\theta = \delta\theta$, we have

$$xb_1 = (v + \delta v)\,\delta\theta = v.\delta\theta + \delta v.\delta\theta$$

Neglecting $\delta v.\delta\theta$ being very small, therefore

$$xb_1 = \overleftarrow{v.\delta\theta}$$

...(Perpendicular to *OA* and towards left)



A drill press has a pointed tool which is used for boring holes in hard materials usually by rotating abrasion or repeated bolows.

Note : This picture is given as additional information and is not a direct example of the current chapter.

Fig. 8.26 (*c*) shows the velocity diagram when the velocities $\omega.r$ and $(\omega + \delta\omega)\,(r + \delta r)$ are considered. In this diagram, vector bb_1 represents the change in velocity ; vector yb_1 represents the component of change of velocity bb_1 along OA (*i.e.* along radial direction) and vector by represents the component of change of velocity bb_1 in a direction perpendicular to OA (*i.e.* in a tangential direction). Therefore

$$yb_1 = (\omega + \delta\omega)\,(r + \delta r)\sin\delta\theta \downarrow$$

$$= (\omega.r + \omega.\delta r + \delta\omega.r + \delta\omega.\delta r)\sin\delta\theta$$

Since $\delta\theta$ is very small, therefore substituting $\sin \delta\theta = \delta\theta$ in the above expression, we have

$$yb_1 = \omega.r.\delta\theta + \omega.\delta r.\delta\theta + \delta\omega.r.\delta\theta + \delta\omega.\delta r.\delta\theta$$

$$= \omega.r.\delta\theta \downarrow, \text{ acting radially inwards} \qquad \text{...(Neglecting all other quantities)}$$

and $$by = oy - ob = (\omega + \delta\omega)\,(r + \delta r) \cos \delta\theta - \omega.r$$

$$= (\omega.r + \omega.\delta r + \delta\omega.r + \delta\omega.\delta r) \cos \delta\theta - \omega.r$$

Since $\delta\theta$ is small, therefore substituting $\cos \delta\theta = 1$, we have

$$by = \omega.r + \omega.\delta r + \delta\omega.r + \delta\omega.\delta r - \omega.r = \omega.\delta r + r.\delta\omega \qquad \text{...(Neglecting } \delta\omega.\delta r)$$

...(Perpendicular to OA and towards left)

Therefore, total component of change of velocity along radial direction

$$= bx - yb_1 = (\delta v - \omega.r.\delta\theta) \uparrow \qquad \text{...(Acting radially outwards from } O \text{ to } A)$$

$\therefore$ Radial component of the acceleration of the slider B with respect to O on the link OA, acting radially outwards from O to A,

$$a^r_{BO} = \text{Lt}\, \frac{\delta v - \omega.r.\delta\theta}{\delta t} = \frac{dv}{dt} - \omega.r \cdot \frac{d\theta}{dt} = \frac{dv}{dt} - \omega^2.r \uparrow \qquad \text{...}(i)$$

...($\because d\theta/dt = \omega$)

Also, the total component of change of velocity along tangential direction,

$$= xb_1 + by = v.\overleftarrow{\delta\theta} + (\omega.\overleftarrow{\delta r} + r.\delta\omega)$$

...(Perpendicular to OA and towards left)

$\therefore$ Tangential component of acceleration of the slider B with respect to O on the link OA, acting perpendicular to OA and towards left,

$$a^t_{BO} = \text{Lt}\, \frac{v.\delta\theta + (\omega.\delta r + r.\delta\omega)}{\delta t} = v\,\frac{d\theta}{dt} + \omega\,\frac{dr}{dt} + r\,\frac{d\omega}{dt}$$

$$= v.\omega + \omega.v + r.\alpha = (2\,v.\overleftarrow{\omega} + r.\alpha) \qquad \text{...}(ii)$$

...($\because dr/dt = v$, and $d\omega/dt = \alpha$)

Now radial component of acceleration of the coincident point C with respect to O, acting in a direction from C to O,

$$a^r_{CO} = \omega^2.r \uparrow \qquad \text{...}(iii)$$

and tangential component of acceleraiton of the coincident point C with respect to O, acting in a direction perpendicular to CO and towards left,

$$a^t_{CO} = \overleftarrow{\alpha}.r \uparrow \qquad \text{...}(iv)$$

Radial component of the slider B with respect to the coincident point C on the link OA, acting radially outwards,

$$a^r_{BC} = a^r_{BO} - a^r_{CO} = \left(\frac{dv}{dt} - \omega^2.r\right) - \left(-\,\omega^2.r\right) = \frac{dv}{dt} \uparrow$$

and tangential component of the slider B with respect to the coincident point C on the link OA acting in a direction perpendicular to OA and towards left,

$$a^t_{BC} = a^t_{BO} - a^t_{CO} = (2\,\omega.v + \alpha.r) - \alpha.r = 2\,\overleftarrow{\omega}.v$$

This tangential component of acceleration of the slider *B* with respect to the coincident point *C* on the link is known as ***coriolis component of acceleration*** and is always perpendicualr to the link.

∴ Coriolis component of the acceleration of *B* with respect of *C*,

$$a^c_{BC} = a^t_{BC} = 2\,\omega.v$$

where ω = Angular velocity of the link *OA*, and

v = Velocity of slider *B* with respect to coincident point *C*.

In the above discussion, the anticlockwise direction for ω and the radially outward direction for *v* are taken as ***positive***. It may be noted that the direction of coriolis component of acceleration changes sign, if either ω or *v* is reversed in direction. But the direction of coriolis component of acceleration will not be changed in sign if both ω and *v* are reversed in direction. It is concluded that the direction of coriolis component of acceleration is obtained by rotating *v*, at 90°, about its origin in the same direction as that of ω.

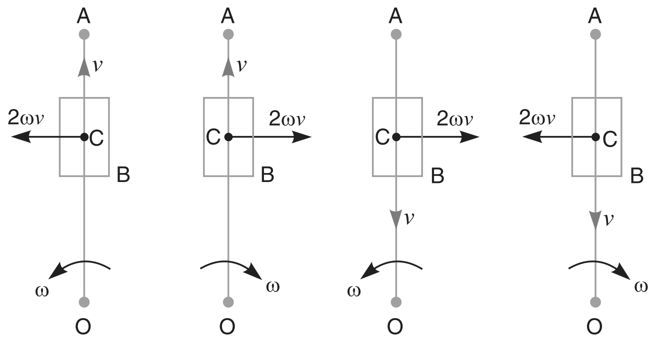


Fig. 8.27. Direction of coriolis component of acceleration.

The direction of coriolis component of acceleration (2 ω.*v*) for all four possible cases, is shown in Fig. 8.27. The directions of ω and *v* are given.

Example 8.13. *A mechanism of a crank and slotted lever quick return motion is shown in Fig. 8.28. If the crank rotates counter clockwise at 120 r.p.m., determine for the configuration shown, the velocity and acceleration of the ram D. Also determine the angular acceleration of the slotted lever.*

Crank, AB = 150 mm ; Slotted arm, OC = 700 mm and link CD = 200 mm.

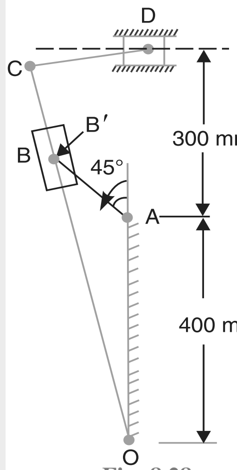


Fig. 8.28

Solution. Given : N_{BA} = 120 r.p.m or $\omega_{BA} = 2\pi \times 120/60$ = 12.57 rad/s ; *AB* = 150 mm = 0.15 m; *OC* = 700 mm = 0.7 m; *CD* = 200 mm = 0.2 m

We know that velocity of *B* with respect to *A* ,

$$v_{BA} = \omega_{BA} \cdot AB$$

$$= 12.57 \cdot 0.15 = 1.9 \text{ m/s}$$

...(Perpendicular to *AB*)

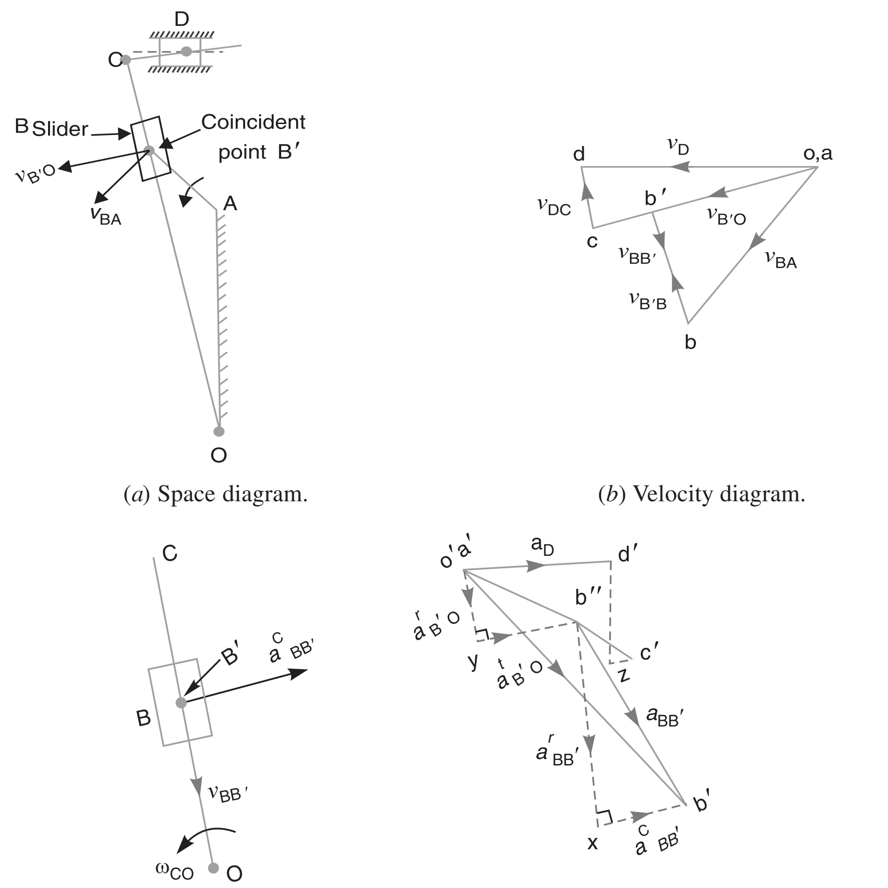
Velocity of the ram D

First of all draw the space diagram, to some suitable scale, as shown in Fig. 8.29 (*a*). Now the velocity diagram, as shown in Fig. 8.29

(*b*), is drawn as discussed below:

1. Since O and A are fixed points, therefore these points are marked as one point in velocity diagram. Now draw vector *ab* in a direction perpendicular to AB, to some suitable scale, to represent the velocity of slider B with respect to A *i.e.* v_{BA}, such that

$$\text{vector } ab = v_{BA} = 1.9 \text{ m/s}$$



(*a*) Space diagram. (*b*) Velocity diagram.

(*c*) Direction of coriolis component. (*d*) Acceleration diagram.

Fig. 8.29

2. From point *o*, draw vector *ob′* perpendicular to *OB′* to represent the velocity of coincident point *B′* (on the link *OC*) with respect to *O* *i.e.* $v_{B'O}$ and from point *b* draw vector *bb′* parallel to the path of motion of *B′* (which is along the link *OC*) to represent the velocity of coincident point *B′* with respect to the slider *B* *i.e.* $v_{B'B}$. The vectors *ob′* and *bb′* intersect at *b′*.

Note: Since we have to find the coriolis component of acceleration of the slider *B* with respect to the coincident point *B′*, therefore we require the velocity of *B* with respect to *B′* *i.e.* $v_{BB'}$. The vector *b′b* will represent $v_{BB'}$ as shown in Fig. 8.29 (*b*).

3. Since the point *C* lies on *OB′* produced, therefore, divide vector *ob′* at *c* in the same ratio as *C* divides *OB′* in the space diagram. In other words,

$$ob'/oc = OB'/OC$$

The vector *oc* represents the velocity of *C* with respect to *O* *i.e.* v_{CO}.

4. Now from point c, draw vector cd perpendicular to CD to represent the velocity of D with respect to C *i.e.* v_{DC}, and from point o draw vector od parallel to the path of motion of D (which is along the horizontal) to represent the velocity of D *i.e.* v_D. The vectors cd and od intersect at d.

By measurement, we find that velocity of the ram D,

$$v_D = \text{vector } od = 2.15 \text{ m/s} \quad \textbf{Ans.}$$

From velocity diagram, we also find that

Velocity of B with respect to B',

$$v_{BB'} = \text{vector } b'b = 1.05 \text{ m/s}$$

Velocity of D with respect to C,

$$v_{DC} = \text{vector } cd = 0.45 \text{ m/s}$$

Velocity of B' with respect to O

$$v_{B'O} = \text{vector } ob' = 1.55 \text{ m/s}$$

Velocity of C with respect to O,

$$v_{CO} = \text{vector } oc = 2.15 \text{ m/s}$$

$\therefore$ Angular velocity of the link OC or OB',

$$\omega_{CO} = \omega_{B'O} = \frac{v_{CO}}{OC} = \frac{2.15}{0.7} = 3.07 \text{ rad/s (Anticlockwise)}$$

Acceleration of the ram D

We know that radial component of the acceleration of B with respect to A,

$$a^r_{BA} = \omega^2_{BA} \times AB = (12.57)^2 \times 0.15 = 23.7 \text{ m/s}^2$$

Coriolis component of the acceleration of slider B with respect to the coincident point B',

$$a^c_{BB'} = 2\omega.v = 2\omega_{CO}.v_{BB'} = 2 \times 3.07 \times 1.05 = 6.45 \text{ m/s}^2$$

...($\because$ $\omega = \omega_{CO}$ and $v = v_{BB'}$)

Radial component of the acceleration of D with respect to C,

$$a^r_{DC} = \frac{v^2_{DC}}{CD} = \frac{(0.45)^2}{0.2} = 1.01 \text{ m/s}^2$$

Radial component of the acceleration of the coincident point B' with respect to O,

$$a^r_{B'O} = \frac{v^2_{B'O}}{B'O} = \frac{(1.55)^2}{0.52} = 4.62 \text{ m/s}^2 \qquad \text{...(By measurement } B'O = 0.52 \text{ m)}$$

Now the acceleration diagram, as shown in Fig. 8.29 (*d*), is drawn as discussed below:

1. Since O and A are fixed points, therefore these points are marked as one point in the acceleration diagram. Draw vector $a'b'$ parallel to AB, to some suitable scale, to represent the radial component of the acceleration of B with respect to A *i.e.* a^r_{BA} or a_B, such that

$$\text{vector } a'b' = a^r_{BA} = a_B = 23.7 \text{ m/s}^2$$

2. The acceleration of the slider B with respect to the coincident point B' has the following two components :

(*i*) Coriolis component of the acceleration of B with respect to B' *i.e.* $a^c_{BB'}$, and

(*ii*) Radial component of the acceleration of B with respect to B' *i.e.* $a^r_{BB'}$.

These two components are mutually perpendicular. Therefore from point b' draw vector $b'x$ perpendicular to $B'O$ *i.e.* in a direction as shown in Fig. 8.29 (*c*) to represent $a^c_{BB'} = 6.45 \text{ m/s}^2$. The direction of $a^c_{BB'}$ is obtained by rotating $v_{BB'}$ (represented by vector *b'b* in velocity diagram) through 90° in the same sense as that of link *OC* which rotates in the counter clockwise direction. Now from point *x*, draw vector *xb''* perpendicular to vector *b'x* (or parallel to *B'O*) to represent $a^r_{BB'}$ whose magnitude is yet unknown.

3. The acceleration of the coincident point *B'* with respect to *O* has also the following two components:

(*i*) Radial component of the acceleration of coincident point *B'* with respect to *O i.e.* $a^r_{B'O}$, and

(*ii*) Tangential component of the acceleration of coincident point *B'* with respect to *O*, *i.e.* $a^t_{B'O}$.

These two components are mutually perpendicular. Therefore from point *o'*, draw vector *o'y* parallel to *B'O* to represent $a^r_{B'O} = 4.62$ m/s^2 and from point *y* draw vector *yb''* perpendicular to vector *o'y* to represent $a^t_{B'O}$. The vectors *xb''* and *yb''* intersect at *b''*. Join *o'b''*. The vector *o'b''* represents the acceleration of *B'* with respect to *O*, *i.e.* $a_{B'O}$.

4. Since the point *C* lies on *OB'* produced, therefore divide vector *o'b''* at *c'* in the same ratio as *C* divides *OB'* in the space diagram. In other words,

$$o'b''/o'c' = OB'/OC$$

5. The acceleration of the ram *D* with respect to *C* has also the following two components:

(*i*) Radial component of the acceleration of *D* with respect to *C i.e.* a^r_{DC}, and

(*ii*) Tangential component of the acceleration of *D* with respect to *C*, *i.e.* a^t_{DC}.

The two components are mutually perpendicular. Therefore draw vector *c'z* parallel to *CD* to represent $a^r_{DC} = 1.01$ m/s^2 and from *z* draw *zd'* perpendicular to vector *zc'* to represent a^t_{DC}, whose magnitude is yet unknown.

6. From point *o',* draw vector *o'd'* in the direction of motion of the ram *D* which is along the horizontal. The vectors *zd'* and *o'd'* intersect at *d'*. The vector *o'd'* represents the acceleration of ram *D i.e.* a_D.

By measurement, we find that acceleration of the ram *D*,

$$a_D = \text{vector } o'd' = 8.4 \text{ m/s}^2 \textbf{ Ans.}$$

Angular acceleration of the slotted lever

By measurement from acceleration diagram, we find that tangential component of the coincident point *B'* with respect to *O*,

$$a^t_{B'O} = \text{vector } yb'' = 6.4 \text{ m/s}^2$$

We know that angular acceleration of the slotted lever,

$$= \frac{a^t_{B'O}}{OB'} = \frac{6.4}{0.52} = 12.3 \text{ rad/s}^2 \text{ (Anticlockwise) } \textbf{Ans.}$$

OBJECTIVE TYPE QUESTIONS

1. The component of the acceleration, parallel to the velocity of the particle, at the given instant is called

   (*a*) radial component (*b*) tangential component
   (*c*) coriolis component (*d*) none of these

2. A point B on a rigid link AB moves with respect to A with angular velocity ω rad/s. The radial component of the acceleration of B with respect to A,

   (*a*) $v_{BA} \times AB$ (*b*) $v^2_{BA} \times AB$ (*c*) $\frac{v_{BA}}{AB}$ (*d*) $\frac{v^2_{BA}}{AB}$

   where v_{BA} = Linear velocity of B with respect to $A = \omega \times AB$

3. A point B on a rigid link AB moves with respect to A with angular velocity ω rad/s. The angular acceleration of the link AB is

   (*a*) $\frac{a^r_{BA}}{AB}$ (*b*) $\frac{a^t_{BA}}{AB}$ (*c*) $v_{BA} \times AB$ (*d*) $\frac{v^2_{BA}}{AB}$

4. A point B on a rigid link AB moves with respect to A with angular velocity ω rad/s. The total acceleration of B with respect to A will be equal to

   (*a*) vector sum of radial component and coriolis component
   (*b*) vector sum of tangential component and coriolis component
   (*c*) vector sum of radial component and tangential component
   (*d*) vector difference of radial component and tangential component

5. The coriolis component of acceleration is taken into account for

   (*a*) slider crank mechanism (*b*) four bar chain mechanism
   (*c*) quick return motion mechanism (*d*) none of these

ANSWERS

1. (*b*) **2.** (*d*) **3.** (*b*) **4.** (*c*) **5.** (*c*)

MODULE-II

Gear and Gear Trains: Gear Terminology and definitions, Theory of shape and action of tooth properties and methods of generation of standard tooth profiles, Standard proportions, Force analysis, Interference and Undercutting, Methods for eliminating Interference, Minimum number of teeth to avoid interference. Analysis of mechanism Trains: Simple Train, Compound train, Reverted train, Epicyclic train and their applications.

Features

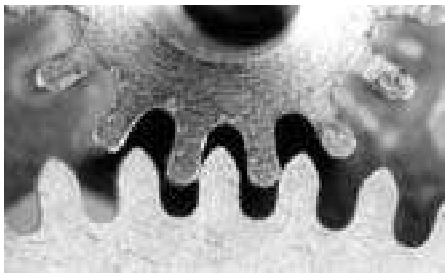
Toothed Gearing

12.1. Introduction

We have discussed in the previous chapter, that the slipping of a belt or rope is a common phenomenon, in the transmission of motion or power between two shafts. The effect of slipping is to reduce the velocity ratio of the system. In precision machines, in which a definite velocity ratio is of importance (as in watch mechanism), the only positive drive is by means of ***gears*** or ***toothed wheels***. A gear drive is also provided, when the distance between the driver and the follower is very small.

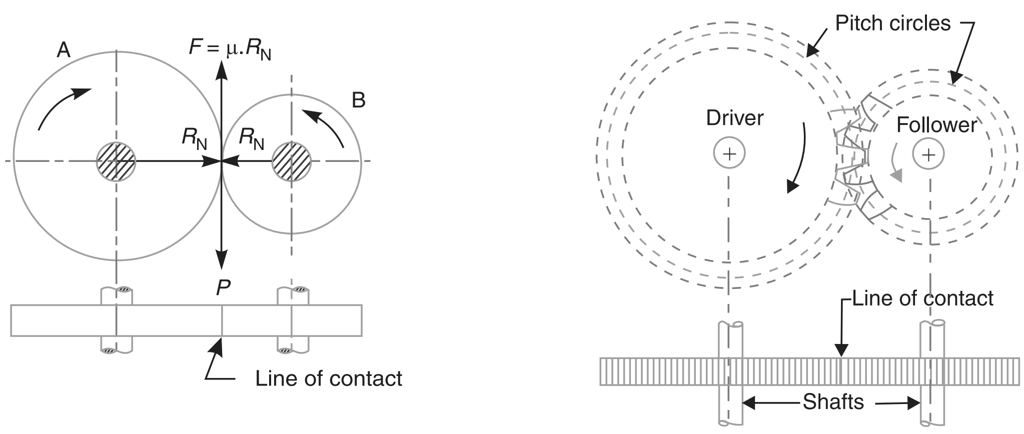
12.2. Friction Wheels

The motion and power transmitted by gears is kinematically equivalent to that transmitted by friction wheels or discs. In order to understand how the motion can be transmitted by two toothed wheels, consider two plain circular wheels *A* and *B* mounted on shafts, having sufficient rough surfaces and pressing against each other as shown in Fig. 12.1 (*a*).



Let the wheel A be keyed to the rotating shaft and the wheel B to the shaft, to be rotated. A little consideration will show, that when the wheel A is rotated by a rotating shaft, it will rotate the wheel B in the opposite direction as shown in Fig. 12.1 (*a*).

The wheel B will be rotated (by the wheel A) so long as the tangential force exerted by the wheel A does not exceed the maximum frictional resistance between the two wheels. But when the tangential force (P) exceeds the *frictional resistance (F), slipping will take place between the two wheels. Thus the friction drive is not a positive drive.



(*a*) Friction wheels. (*b*) Toothed wheels.

Fig. 12.1

In order to avoid the slipping, a number of projections (called teeth) as shown in Fig. 12.1 (*b*), are provided on the periphery of the wheel A, which will fit into the corresponding recesses on the periphery of the wheel B. A friction wheel with the teeth cut on it is known as ***toothed wheel* or *gear***. The usual connection to show the toothed wheels is by their **pitch circles.

Note : Kinematically, the friction wheels running without slip and toothed gearing are identical. But due to the possibility of slipping of wheels, the friction wheels can only be used for transmission of small powers.

12.3. Advantages and Disadvantages of Gear Drive

The following are the advantages and disadvantages of the gear drive as compared to belt, rope and chain drives :

Advantages

1. It transmits exact velocity ratio.
2. It may be used to transmit large power.
3. It has high efficiency.
4. It has reliable service.
5. It has compact layout.

Disadvantages

1. The manufacture of gears require special tools and equipment.
2. The error in cutting teeth may cause vibrations and noise during operation.

---

* The frictional force F is equal to $\mu . R_N$, where μ = Coefficient of friction between the rubbing surface of two wheels, and R_N = Normal reaction between the two rubbing surfaces.

** For details, please refer to Art. 12.4.

12.4. Classification of Toothed Wheels

The gears or toothed wheels may be classified as follows :

1. ***According to the position of axes of the shafts*****.** The axes of the two shafts between which the motion is to be transmitted, may be

(*a*) Parallel, (*b*) Intersecting, and (*c*) Non-intersecting and non-parallel.

The two parallel and co-planar shafts connected by the gears is shown in Fig. 12.1. These gears are called ***spur gears*** and the arrangement is known as ***spur gearing***. These gears have teeth parallel to the axis of the wheel as shown in Fig. 12.1. Another name given to the spur gearing is ***helical gearing***, in which the teeth are inclined to the axis. The single and double helical gears connecting parallel shafts are shown in Fig. 12.2 (*a*) and (*b*) respectively. The double helical gears are known as ***herringbone gears***. A pair of spur gears are kinematically equivalent to a pair of cylindrical discs, keyed to parallel shafts and having a line contact.

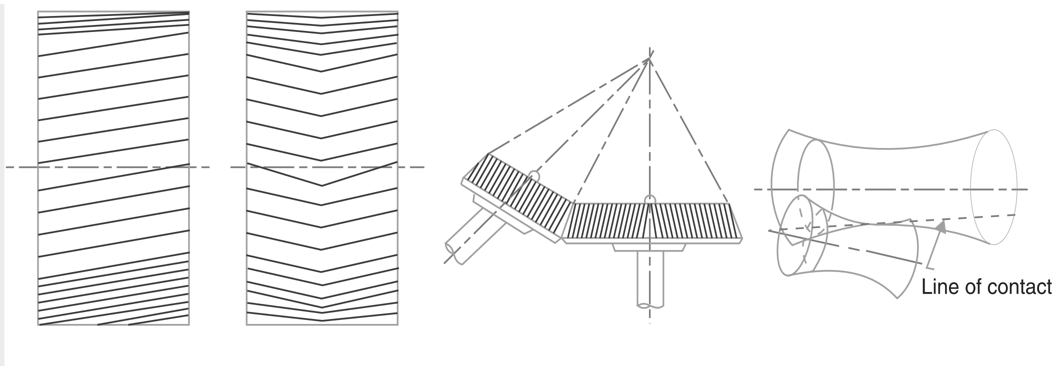
The two non-parallel or intersecting, but coplanar shafts connected by gears is shown in Fig. 12.2 (*c*). These gears are called ***bevel gears*** and the arrangement is known as ***bevel gearing***. The bevel gears, like spur gears, may also have their teeth inclined to the face of the bevel, in which case they are known as ***helical bevel gears***.

The two non-intersecting and non-parallel *i.e.* non-coplanar shaft connected by gears is shown in Fig. 12.2 (*d*). These gears are called ***skew bevel gears*** or ***spiral gears*** and the arrangement is known as ***skew bevel gearing*** or ***spiral gearing***. This type of gearing also have a line contact, the rotation of which about the axes generates the two pitch surfaces known as ***hyperboloids***.

Notes : (*a*) When equal bevel gears (having equal teeth) connect two shafts whose axes are mutually perpendicular, then the bevel gears are known as *mitres*.

(*b*) A hyperboloid is the solid formed by revolving a straight line about an axis (not in the same plane), such that every point on the line remains at a constant distance from the axis.

(*c*) The worm gearing is essentially a form of spiral gearing in which the shafts are usually at right angles.



(*a*) Single helical gear. (*b*) Double helical gear. (*c*) Bevel gear. (*d*) Spiral gear.

Fig. 12.2

2. ***According to the peripheral velocity of the gears*****.** The gears, according to the peripheral velocity of the gears may be classified as :

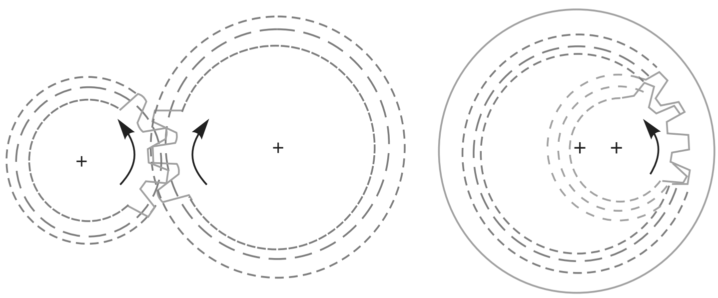
(*a*) Low velocity, (*b*) Medium velocity, and (*c*) High velocity.

The gears having velocity less than 3 m/s are termed as ***low velocity*** gears and gears having velocity between 3 and 15 m/s are known as ***medium velocity gears***. If the velocity of gears is more than 15 m/s, then these are called ***high speed gears***.

3. ***According to the type of gearing***. The gears, according to the type of gearing may be classified as :

(*a*) External gearing, (*b*) Internal gearing, and (*c*) Rack and pinion.

In ***external gearing***, the gears of the two shafts mesh externally with each other as shown in Fig. 12.3 (*a*). The larger of these two wheels is called ***spur wheel*** and the smaller wheel is called **pinion.** In an external gearing, the motion of the two wheels is always ***unlike***, as shown in Fig. 12.3 (*a*).



(*a*) External gearing. (*b*) Internal gearing.

Fig. 12.3

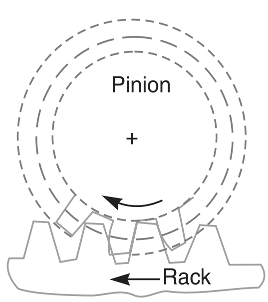


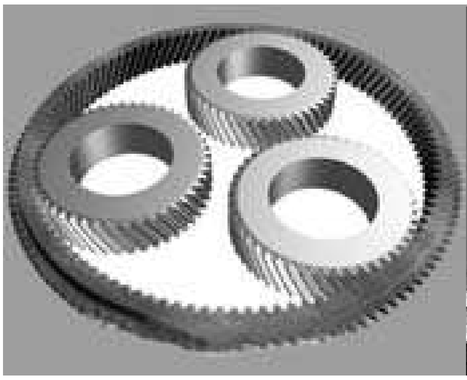
Fig. 12.4. Rack and pinion.

In ***internal gearing***, the gears of the two shafts mesh ***internally*** with each other as shown in Fig. 12.3 (*b*). The larger of these two wheels is called ***annular wheel*** and the smaller wheel is called ***pinion***. In an internal gearing, the motion of the two wheels is always ***like***, as shown in Fig. 12.3 (*b*).

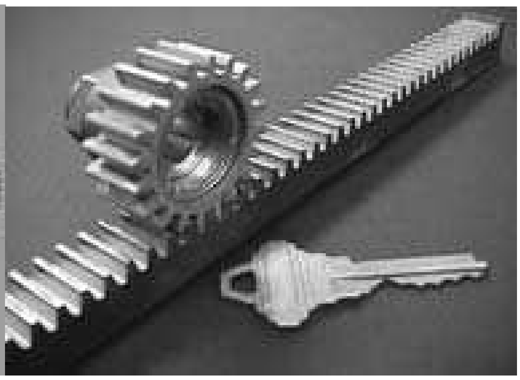
Sometimes, the gear of a shaft meshes externally and internally with the gears in a *straight line, as shown in Fig. 12.4. Such type of gear is called ***rack and pinion***. The straight line gear is called rack and the circular wheel is called pinion. A little consideration will show that with the help of a rack and pinion, we can convert linear motion into rotary motion and ***vice-versa*** as shown in Fig. 12.4.

4. ***According to position of teeth on the gear surface***. The teeth on the gear surface may be (*a*) straight, (*b*) inclined, and (*c*) curved.

We have discussed earlier that the spur gears have straight teeth where as helical gears have their teeth inclined to the wheel rim. In case of spiral gears, the teeth are curved over the rim surface.



Internal gears



Rack and pinion

12.5. Terms Used in Gears

The following terms, which will be mostly used in this chapter, should be clearly understood at this stage. These terms are illustrated in Fig. 12.5.

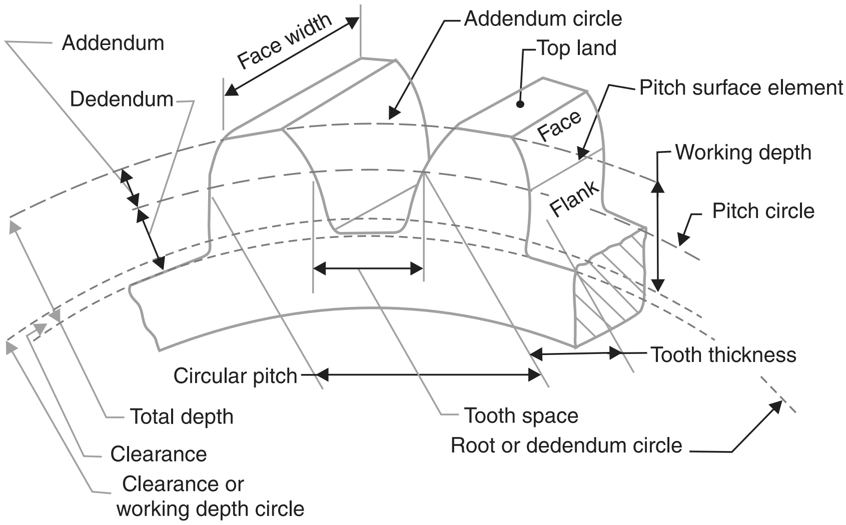


Fig. 12.5. Terms used in gears.

1. ***Pitch circle***. It is an imaginary circle which by pure rolling action, would give the same motion as the actual gear.

* A straight line may also be defined as a wheel of infinite radius.

2. *Pitch circle diameter*. It is the diameter of the pitch circle. The size of the gear is usually specified by the pitch circle diameter. It is also known as ***pitch diameter***.

3. *Pitch point*. It is a common point of contact between two pitch circles.

4. *Pitch surface*. It is the surface of the rolling discs which the meshing gears have replaced at the pitch circle.

5. *Pressure angle or angle of obliquity*. It is the angle between the common normal to two gear teeth at the point of contact and the common tangent at the pitch point. It is usually denoted by ϕ. The standard pressure angles are $14\frac{1}{2}°$ and 20°.

6. *Addendum*. It is the radial distance of a tooth from the pitch circle to the top of the tooth.

7. *Dedendum*. It is the radial distance of a tooth from the pitch circle to the bottom of the tooth.

8. *Addendum circle*. It is the circle drawn through the top of the teeth and is concentric with the pitch circle.

9. *Dedendum circle*. It is the circle drawn through the bottom of the teeth. It is also called root circle.

Note : Root circle diameter = Pitch circle diameter × cos ϕ, where ϕ is the pressure angle.

10. *Circular pitch*. It is the distance measured on the circumference of the pitch circle from a point of one tooth to the corresponding point on the next tooth. It is usually denoted by p_c. Mathematically,

Circular pitch, $p_c = \pi D/T$

where D = Diameter of the pitch circle, and

T = Number of teeth on the wheel.

A little consideration will show that the two gears will mesh together correctly, if the two wheels have the same circular pitch.

Note : If D_1 and D_2 are the diameters of the two meshing gears having the teeth T_1 and T_2 respectively, then for them to mesh correctly,

$$p_c = \frac{\pi D_1}{T_1} = \frac{\pi D_2}{T_2} \quad \text{or} \quad \frac{D_1}{D_2} = \frac{T_1}{T_2}$$

11. *Diametral pitch.* It is the ratio of number of teeth to the pitch circle diameter in millimetres. It is denoted by p_d. Mathematically,

$$\text{Diametral pitch,} \quad p_d = \frac{T}{D} = \frac{\pi}{p_c} \qquad \ldots\left(\because\ p_c = \frac{\pi D}{T}\right)$$

where T = Number of teeth, and

D = Pitch circle diameter.

12. *Module*. It is the ratio of the pitch circle diameter in millimeters to the number of teeth. It is usually denoted by m. Mathematically,

Module, $m = D/T$

Note : The recommended series of modules in Indian Standard are 1, 1.25, 1.5, 2, 2.5, 3, 4, 5, 6, 8, 10, 12, 16, and 20. The modules 1.125, 1.375, 1.75, 2.25, 2.75, 3.5, 4.5, 5.5, 7, 9, 11, 14 and 18 are of second choice.

13. *Clearance*. It is the radial distance from the top of the tooth to the bottom of the tooth, in a meshing gear. A circle passing through the top of the meshing gear is known as ***clearance circle***.

14. *Total depth*. It is the radial distance between the addendum and the dedendum circles of a gear. It is equal to the sum of the addendum and dedendum.

15. ***Working depth***. It is the radial distance from the addendum circle to the clearance circle. It is equal to the sum of the addendum of the two meshing gears.

16. ***Tooth thickness***. It is the width of the tooth measured along the pitch circle.

17. ***Tooth space*** . It is the width of space between the two adjacent teeth measured along the pitch circle.

18. ***Backlash***. It is the difference between the tooth space and the tooth thickness, as measured along the pitch circle. Theoretically, the backlash should be zero, but in actual practice some backlash must be allowed to prevent jamming of the teeth due to tooth errors and thermal expansion.

19. ***Face of tooth***. It is the surface of the gear tooth above the pitch surface.

20. ***Flank of tooth***. It is the surface of the gear tooth below the pitch surface.

21. ***Top land***. It is the surface of the top of the tooth.

22. ***Face width***. It is the width of the gear tooth measured parallel to its axis.

23. ***Profile***. It is the curve formed by the face and flank of the tooth.

24. ***Fillet radius***. It is the radius that connects the root circle to the profile of the tooth.

25. ***Path of contact***. It is the path traced by the point of contact of two teeth from the beginning to the end of engagement.

26. *****Length of the path of contact***. It is the length of the common normal cut-off by the addendum circles of the wheel and pinion.

27. ******Arc of contact***. It is the path traced by a point on the pitch circle from the beginning to the end of engagement of a given pair of teeth. The arc of contact consists of two parts, *i.e.*

(*a*) ***Arc of approach***. It is the portion of the path of contact from the beginning of the engagement to the pitch point.

(*b*) ***Arc of recess***. It is the portion of the path of contact from the pitch point to the end of the engagement of a pair of teeth.

Note : The ratio of the length of arc of contact to the circular pitch is known as ***contact ratio*** *i.e.* number of pairs of teeth in contact.

12.6. Gear Materials

The material used for the manufacture of gears depends upon the strength and service conditions like wear, noise etc. The gears may be manufactured from metallic or non-metallic materials. The metallic gears with cut teeth are commercially obtainable in cast iron, steel and bronze. The non-metallic materials like wood, raw hide, compressed paper and synthetic resins like nylon are used for gears, especially for reducing noise.

The cast iron is widely used for the manufacture of gears due to its good wearing properties, excellent machinability and ease of producing complicated shapes by casting method. The cast iron gears with cut teeth may be employed, where smooth action is not important.

The steel is used for high strength gears and steel may be plain carbon steel or alloy steel. The steel gears are usually heat treated in order to combine properly the toughness and tooth hardness.

The phosphor bronze is widely used for worm gears in order to reduce wear of the worms which will be excessive with cast iron or steel.

12.7. Condition for Constant Velocity Ratio of Toothed Wheels–Law of Gearing

Consider the portions of the two teeth, one on the wheel 1 (or pinion) and the other on the

* For details, see Art. 12.16.

** For details, see Art. 12.17.

wheel 2, as shown by thick line curves in Fig. 12.6. Let the two teeth come in contact at point Q, and the wheels rotate in the directions as shown in the figure.

Let $T\ T$ be the common tangent and MN be the common normal to the curves at the point of contact Q. From the centres O_1 and O_2, draw O_1M and O_2N perpendicular to MN. A little consideration will show that the point Q moves in the direction QC, when considered as a point on wheel 1, and in the direction QD when considered as a point on wheel 2.

Let v_1 and v_2 be the velocities of the point Q on the wheels 1 and 2 respectively. If the teeth are to remain in contact, then the components of these velocities along the common normal MN must be equal.

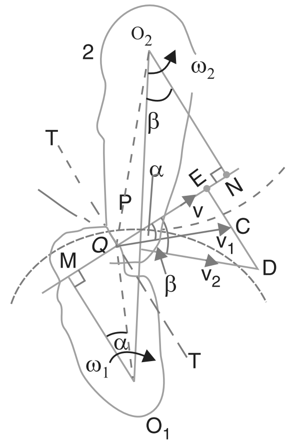


Fig. 12.6. Law of gearing.

$\therefore \qquad v_1 \cos \alpha = v_2 \cos \beta$

or $\qquad (\omega_1 \times O_1 Q) \cos \alpha = (\omega_2 \times O_2 Q) \cos \beta$

$$(\omega_1 \times O_1 Q) \frac{O_1 M}{O_1 Q} = (\omega_2 \times O_2 Q) \frac{O_2 N}{O_2 Q} \quad \text{or} \quad \omega_1 \times O_1 M = \omega_2 \times O_2 N$$

$$\therefore \qquad \frac{\omega_1}{\omega_2} = \frac{O_2 N}{O_1 M} \qquad \ldots(i)$$

Also from similar triangles O_1MP and O_2NP,

$$\frac{O_2 N}{O_1 M} = \frac{O_2 P}{O_1 P} \qquad \ldots(ii)$$

Combining equations (*i*) and (*ii*), we have

$$\frac{\omega_1}{\omega_2} = \frac{O_2 N}{O_1 M} = \frac{O_2 P}{O_1 P} \qquad \ldots(iii)$$

From above, we see that the angular velocity ratio is inversely proportional to the ratio of the distances of the point P from the centres O_1 and O_2, or the common normal to the two surfaces at the point of contact Q intersects the line of centres at point P which divides the centre distance inversely as the ratio of angular velocities.

Therefore in order to have a constant angular velocity ratio for all positions of the wheels, the point P must be the fixed point (called pitch point) for the two wheels. In other words, ***the common normal at the point of contact between a pair of teeth must always pass through the pitch point.*** This is the fundamental condition which must be satisfied while designing the profiles for the teeth of gear wheels. It is also known as ***law of gearing.***

Notes : 1. The above condition is fulfilled by teeth of involute form, provided that the root circles from which the profiles are generated are tangential to the common normal.

2. If the shape of one tooth profile is arbitrarily chosen and another tooth is designed to satisfy the above condition, then the second tooth is said to be conjugate to the first. The conjugate teeth are not in common use because of difficulty in manufacture, and cost of production.

3. If D_1 and D_2 are pitch circle diameters of wheels 1 and 2 having teeth T_1 and T_2 respectively, then velocity ratio,

$$\frac{\omega_1}{\omega_2} = \frac{O_2 P}{O_1 P} = \frac{D_2}{D_1} = \frac{T_2}{T_1}$$

12.8. Velocity of Sliding of Teeth

The sliding between a pair of teeth in contact at Q occurs along the common tangent $T\,T$ to the tooth curves as shown in Fig. 12.6. ***The velocity of sliding is the velocity of one tooth relative to its mating tooth along the common tangent at the point of contact***.

The velocity of point Q, considered as a point on wheel 1, along the common tangent $T\,T$ is represented by EC. From similar triangles QEC and O_1MQ,

$$\frac{EC}{MQ} = \frac{v}{O_1Q} = \omega_1 \quad \text{or} \quad EC = \omega_1 . MQ$$

Similarly, the velocity of point Q, considered as a point on wheel 2, along the common tangent $T\,T$ is represented by ED. From similar triangles QCD and $O_2\,NQ$,

$$\frac{ED}{QN} = \frac{v_2}{O_2Q} = \omega_2 \quad \text{or} \quad ED = \omega_2 . QN$$

Let $\quad v_S$ = Velocity of sliding at Q.

$\therefore$

$$\begin{aligned} v_S &= ED - EC = \omega_2 .\, QN - \omega_1 . MQ \\ &= \omega_2\,(QP + PN) - \omega_1\,(MP - QP) \\ &= (\omega_1 + \omega_2)\,QP + \omega_2 .\, PN - \omega_1 . MP \qquad \text{...}(i) \end{aligned}$$

Since $\dfrac{\omega_1}{\omega_2} = \dfrac{O_2P}{O_1P} = \dfrac{PN}{MP}$ or $\omega_1 .\, MP = \omega_2 . PN$, therefore equation (*i*) becomes

$$v_S = (\omega_1 + \omega_2)\,QP \qquad \text{...}(ii)$$

Notes : 1. We see from equation (*ii*), that the **velocity of sliding is proportional to the distance of the point of contact from the pitch point.**

2. Since the angular velocity of wheel 2 relative to wheel 1 is $(\omega_1 + \omega_2)$ and P is the instantaneous centre for this relative motion, therefore the value of v_s may directly be written as v_s $(\omega_1 + \omega_2)\,QP$, without the above analysis.

12.9. Forms of Teeth

We have discussed in Art. 12.7 (Note 2) that conjugate teeth are not in common use. Therefore, in actual practice following are the two types of teeth commonly used :

1. ***Cycloidal* teeth ;** and **2. Involute teeth.**

We shall discuss both the above mentioned types of teeth in the following articles. Both these forms of teeth satisfy the conditions as discussed in Art. 12.7.

12.10. Cycloidal Teeth

A ***cycloid*** is the curve traced by a point on the circumference of a circle which rolls without slipping on a fixed straight line. When a circle rolls without slipping on the outside of a fixed circle, the curve traced by a point on the circumference of a circle is known as ***epi-cycloid***. On the other hand, if a circle rolls without slipping on the inside of a fixed circle, then the curve traced by a point on the circumference of a circle is called ***hypo-cycloid***.

In Fig. 12.7 (*a*), the fixed line or pitch line of a rack is shown. When the circle *C* rolls without slipping above the pitch line in the direction as indicated in Fig. 12.7 (*a*), then the point *P* on the circle traces epi-cycloid *PA*. This represents the face of the cycloidal tooth profile. When the circle *D* rolls without slipping below the pitch line, then the point *P* on the circle *D* traces hypo-cycloid *PB*, which represents the flank of the cycloidal tooth. The profile *BPA* is one side of the cycloidal rack tooth. Similarly, the two curves *P′A′* and *P′B′* forming the opposite side of the tooth profile are traced by the point *P′* when the circles *C* and *D* roll in the opposite directions.

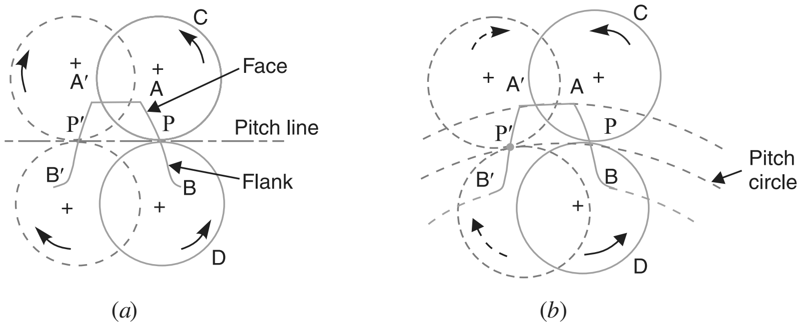


Fig. 12.7. Construction of cycloidal teeth of a gear.

In the similar way, the cycloidal teeth of a gear may be constructed as shown in Fig. 12.7 (*b*). The circle *C* is rolled without slipping on the outside of the pitch circle and the point *P* on the circle *C* traces epi-cycloid *PA*, which represents the face of the cycloidal tooth. The circle *D* is rolled on the inside of pitch circle and the point *P* on the circle *D* traces hypo-cycloid *PB*, which represents the flank of the tooth profile. The profile *BPA* is one side of the cycloidal tooth. The opposite side of the tooth is traced as explained above.

The construction of the two mating cycloidal teeth is shown in Fig. 12.8. A point on the circle *D* will trace the flank of the tooth T_1 when circle *D* rolls without slipping on the inside of pitch circle of wheel 1 and face of tooth T_2 when the circle *D* rolls without slipping on the outside of pitch circle of wheel 2. Similarly, a point on the circle *C* will trace the face of tooth T_1 and flank of tooth T_2. The rolling circles *C* and *D* may have unequal diameters, but if several wheels are to be interchangeable, they must have rolling circles of equal diameters.

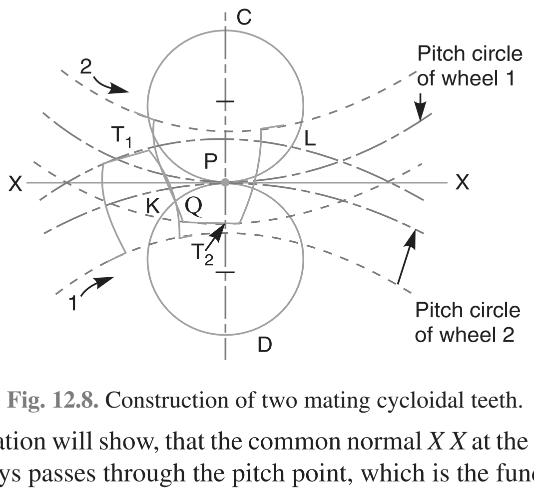


Fig. 12.8. Construction of two mating cycloidal teeth.

A little consideration will show, that the common normal *X X* at the point of contact between two cycloidal teeth always passes through the pitch point, which is the fundamental condition for a constant velocity ratio.

12.11. Involute Teeth

An involute of a circle is a plane curve generated by a point on a tangent, which rolls on the circle without slipping or by a point on a taut string which in unwrapped from a reel as shown in Fig. 12.9. In connection with toothed wheels, the circle is known as base circle. The involute is traced as follows :

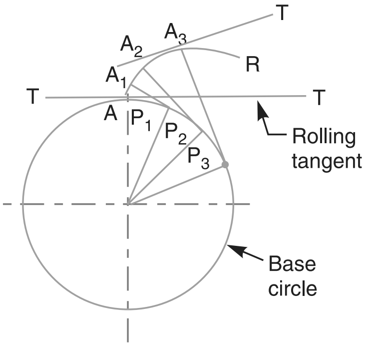


Fig. 12.9. Construction of involute.

Let A be the starting point of the involute. The base circle is divided into equal number of parts *e.g.* AP_1, P_1P_2, P_2P_3 etc. The tangents at P_1, P_2, P_3 etc. are drawn and the length P_1A_1, P_2A_2, P_3A_3 equal to the arcs AP_1, AP_2 and AP_3 are set off. Joining the points A, A_1, A_2, A_3 etc. we obtain the involute curve AR. A little consideration will show that at any instant A_3, the tangent A_3T to the involute is perpendicular to P_3A_3 and P_3A_3 is the normal to the involute. In other words, ***normal at any point of an involute is a tangent to the circle.***

Now, let O_1 and O_2 be the fixed centres of the two base circles as shown in Fig. 12.10 (*a*). Let the corresponding involutes AB and A_1B_1 be in contact at point Q. MQ and NQ are normals to the involutes at Q and are tangents to base circles. Since the normal of an involute at a given point is the tangent drawn from that point to the base circle, therefore the common normal MN at Q is also the common tangent to the two base circles. We see that the common normal MN intersects the line of centres O_1O_2 at the fixed point P (called pitch point). Therefore the involute teeth satisfy the fundamental condition of constant velocity ratio.

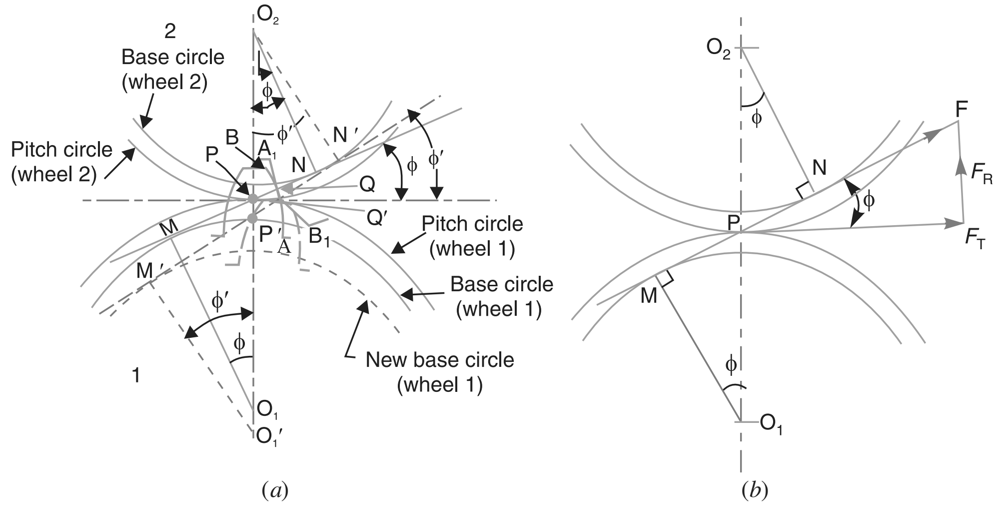


Fig. 12.10. Involute teeth.

From similar triangles O_2NP and O_1MP,

$$\frac{O_1M}{O_2N} = \frac{O_1P}{O_2P} = \frac{\omega_2}{\omega_1} \quad \ldots (i)$$

which determines the ratio of the radii of the two base circles. The radii of the base circles is given by

$$O_1M = O_1P\cos\phi, \text{ and } O_2N = O_2P\cos\phi$$

Also the centre distance between the base circles,

$$O_1O_2 = O_1P + O_2P = \frac{O_1M}{\cos\phi} + \frac{O_2N}{\cos\phi} = \frac{O_1M + O_2N}{\cos\phi}$$

where ϕ is the pressure angle or the angle of obliquity. It is the angle which the common normal to the base circles (*i.e.* MN) makes with the common tangent to the pitch circles.

When the power is being transmitted, the maximum tooth pressure (neglecting friction at the teeth) is exerted along the common normal through the pitch point. This force may be resolved into tangential and radial or normal components. These components act along and at right angles to the common tangent to the pitch circles.

If F is the maximum tooth pressure as shown in Fig. 12.10 (*b*), then

Tangential force, $F_T = F \cos \phi$

and radial or normal force, $F_R = F \sin \phi$.

$\therefore$ Torque exerted on the gear shaft

$= F_T \times r$, where r is the pitch circle radius of the gear.

Note : The tangential force provides the driving torque and the radial or normal force produces radial deflection of the rim and bending of the shafts.

12.12. Effect of Altering the Centre Distance on the Velocity Ratio for Involute Teeth Gears

In the previous article, we have seen that the velocity ratio for the involute teeth gears is given by

$$\frac{O_1 M}{O_2 N} = \frac{O_1 P}{O_2 P} = \frac{\omega_2}{\omega_1} \qquad \ldots(i)$$

Let, in Fig. 12.10 (*a*), the centre of rotation of one of the gears (say wheel 1) is shifted from O_1 to O_1'. Consequently the contact point shifts from Q to Q'. The common normal to the teeth at the point of contact Q' is the tangent to the base circle, because it has a contact between two involute curves and they are generated from the base circle. Let the tangent $M'N'$ to the base circles intersects $O_1' O_2$ at the pitch point P'. As a result of this, the wheel continues to work* correctly.

Now from similar triangles O_2NP and O_1MP,

$$\frac{O_1 M}{O_2 N} = \frac{O_1 P}{O_2 P} \qquad \ldots(ii)$$

and from similar triangles $O_2N'P'$ and $O_1'M'P'$,

$$\frac{O_1' M'}{O_2 N'} = \frac{O_1' P'}{O_2 P'} \qquad \ldots(iii)$$

But $O_2N = O_2N'$, and $O_1M = O_1'M'$. Therefore from equations (*ii*) and (*iii*),

$$\frac{O_1 P}{O_2 P} = \frac{O_1' P'}{O_2 P'} \qquad \ldots\text{[Same as equation } (i)]$$

Thus we see that if the centre distance is changed within limits, the velocity ratio remains unchanged. However, the pressure angle increases (from ϕ to ϕ') with the increase in the centre distance.

Example 12.1. *A single reduction gear of 120 kW with a pinion 250 mm pitch circle diameter and speed 650 r.p.m. is supported in bearings on either side. Calculate the total load due to the power transmitted, the pressure angle being 20°.*

Solution. Given : $P = 120$ kW $= 120 \times 10^3$ W ; $d = 250$ mm or $r = 125$ mm $= 0.125$ m ; $N = 650$ r.p.m. or $\omega = 2\pi \times 650/60 = 68$ rad/s ; $\phi = 20°$

* It is not the case with cycloidal teeth.

Let T = Torque transmitted in N-m.

We know that power transmitted (P),

$$120 \times 10^3 = T.\omega = T \times 68 \quad \text{or} \quad T = 120 \times 10^3/68 = 1765 \text{ N-m}$$

and tangential load on the pinion,

$$F_T = T/r = 1765 / 0.125 = 14\ 120 \text{ N}$$

∴ Total load due to power transmitted,

$$F = F_T / \cos\phi = 14\ 120 / \cos 20° = 15\ 026 \text{ N} = 15.026 \text{ kN}$$ **Ans.**

12.13. Comparison Between Involute and Cycloidal Gears

In actual practice, the involute gears are more commonly used as compared to cycloidal gears, due to the following advantages :

Advantages of involute gears

Following are the advantages of involute gears :

1. The most important advantage of the involute gears is that the centre distance for a pair of involute gears can be varied within limits without changing the velocity ratio. This is not true for cycloidal gears which requires exact centre distance to be maintained.

2. In involute gears, the pressure angle, from the start of the engagement of teeth to the end of the engagement, remains constant. It is necessary for smooth running and less wear of gears. But in cycloidal gears, the pressure angle is maximum at the beginning of engagement, reduces to zero at pitch point, starts decreasing and again becomes maximum at the end of engagement. This results in less smooth running of gears.

3. The face and flank of involute teeth are generated by a single curve where as in cycloidal gears, double curves (*i.e.* epi-cycloid and hypo-cycloid) are required for the face and flank respectively. Thus the involute teeth are easy to manufacture than cycloidal teeth. In involute system, the basic rack has straight teeth and the same can be cut with simple tools.

Note : The only disadvantage of the involute teeth is that the interference occurs (Refer Art. 12.19) with pinions having smaller number of teeth. This may be avoided by altering the heights of addendum and dedendum of the mating teeth or the angle of obliquity of the teeth.

Advantages of cycloidal gears

Following are the advantages of cycloidal gears :

1. Since the cycloidal teeth have wider flanks, therefore the cycloidal gears are stronger than the involute gears, for the same pitch. Due to this reason, the cycloidal teeth are preferred specially for cast teeth.

2. In cycloidal gears, the contact takes place between a convex flank and concave surface, whereas in involute gears, the convex surfaces are in contact. This condition results in less wear in cycloidal gears as compared to involute gears. However the difference in wear is negligible.

3. In cycloidal gears, the interference does not occur at all. Though there are advantages of cycloidal gears but they are outweighed by the greater simplicity and flexibility of the involute gears.

12.14. Systems of Gear Teeth

The following four systems of gear teeth are commonly used in practice :

1. $14\frac{1}{2}°$ Composite system, **2.** $14\frac{1}{2}°$ Full depth involute system, **3.** 20° Full depth involute system, and 4. 20° Stub involute system.

The $14\frac{1}{2}°$ ***composite system*** is used for general purpose gears. It is stronger but has no interchangeability. The tooth profile of this system has cycloidal curves at the top and bottom and involute curve at the middle portion. The teeth are produced by formed milling cutters or hobs. The tooth profile of the $14\frac{1}{2}°$ ***full depth involute system*** was developed for use with gear hobs for spur and helical gears.

The tooth profile of the **20°** ***full depth involute system*** may be cut by hobs. The increase of the pressure angle from $14\frac{1}{2}°$ to 20° results in a stronger tooth, because the tooth acting as a beam is wider at the base. The **20°** ***stub involute system*** has a strong tooth to take heavy loads.

12.15. Standard Proportions of Gear Systems

The following table shows the standard proportions in module (m) for the four gear systems as discussed in the previous article.

Table 12.1. Standard proportions of gear systems.

S. No.	*Particulars*	$14\frac{1}{2}°$ *composite or full depth involute system*	*20° full depth involute system*	*20° stub involute system*
1.	Addenddm	1 m	1 m	0.8 m
2.	Dedendum	1.25 m	1.25 m	1 m
3.	Working depth	2 m	2 m	1.60 m
4.	Minimum total depth	2.25 m	2.25 m	1.80 m
5.	Tooth thickness	1.5708 m	1.5708 m	1.5708 m
6.	Minimum clearance	0.25 m	0.25 m	0.2 m
7.	Fillet radius at root	0.4 m	0.4 m	0.4 m

12.16. Length of Path of Contact

Consider a pinion driving the wheel as shown in Fig. 12.11. When the pinion rotates in clockwise direction, the contact between a pair of involute teeth begins at K (on the flank near the base circle of pinion or the outer end of the tooth face on the wheel) and* ends at L (outer end of the tooth face on the pinion or on the flank near the base circle of wheel). MN is the common normal at the point of contacts and the common tangent to the base circles. The point K is the intersection of the addendum circle of wheel and the common tangent. The point L is the intersection of the addendum circle of pinion and common tangent.

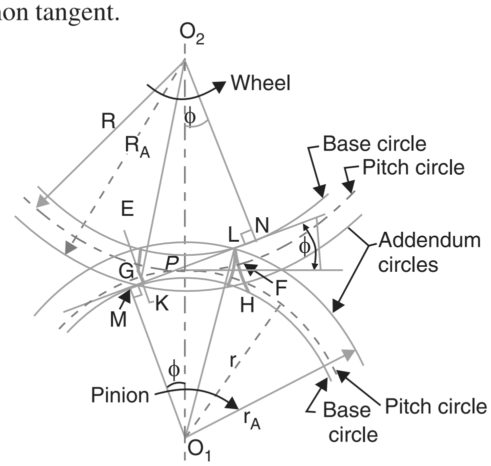
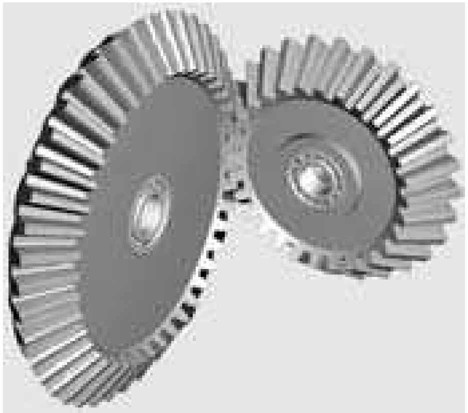


Fig. 12.11. Length of path of contact.

* If the wheel is made to act as a driver and the directions of motion are reversed, then the contact between a pair of teeth begins at L and ends at K.

We have discussed in Art. 12.4 that the length of path of contact is the length of common normal cut-off by the addendum circles of the wheel and the pinion. Thus the length of path of contact is KL which is the sum of the parts of the path of contacts KP and PL. The part of the path of contact KP is known as ***path of approach*** and the part of the path of contact PL is known as ***path of recess***.



Bevel gear

Let $r_A = O_1L$ = Radius of addendum circle of pinion,

$R_A = O_2K$ = Radius of addendum circle of wheel,

$r = O_1P$ = Radius of pitch circle of pinion, and

$R = O_2P$ = Radius of pitch circle of wheel.

From Fig. 12.11, we find that radius of the base circle of pinion,

$$O_1M = O_1P\cos\phi = r\cos\phi$$

and radius of the base circle of wheel,

$$O_2N = O_2P\cos\phi = R\cos\phi$$

Now from right angled triangle O_2KN,

$$KN = \sqrt{(O_2K)^2 - (O_2N)^2} = \sqrt{(R_A)^2 - R^2\cos^2\phi}$$

and $$PN = O_2P\sin\phi = R\sin\phi$$

∴ Length of the part of the path of contact, or the path of approach,

$$KP = KN - PN = \sqrt{(R_A)^2 - R^2\cos^2\phi} - R\sin\phi$$

Similarly from right angled triangle O_1ML,

$$ML = \sqrt{(O_1L)^2 - (O_1M)^2} = \sqrt{(r_A)^2 - r^2\cos^2\phi}$$

and $$MP = O_1P\sin\phi = r\sin\phi$$

∴ Length of the part of the path of contact, or path of recess,

$$PL = ML - MP = \sqrt{(r_A)^2 - r^2\cos^2\phi} - r\sin\phi$$

∴ Length of the path of contact,

$$KL = KP + PL = \sqrt{(R_A)^2 - R^2\cos^2\phi} + \sqrt{(r_A)^2 - r^2\cos^2\phi} - (R + r)\sin\phi$$

12.17. Length of Arc of Contact

We have already defined that the arc of contact is the path traced by a point on the pitch circle from the beginning to the end of engagement of a given pair of teeth. In Fig. 12.11, the arc of contact is EPF or GPH. Considering the arc of contact GPH, it is divided into two parts *i.e.* arc GP and arc PH. The arc GP is known as ***arc of approach*** and the arc PH is called ***arc of recess***. The angles subtended by these arcs at O_1 are called ***angle of approach*** and ***angle of recess*** respectively.

We know that the length of the arc of approach (arc *GP*)

$$= \frac{\text{Length of path of approach}}{\cos\phi} = \frac{KP}{\cos\phi}$$

and the length of the arc of recess (arc *PH*)

$$= \frac{\text{Length of path of recess}}{\cos\phi} = \frac{PL}{\cos\phi}$$

Since the length of the arc of contact *GPH* is equal to the sum of the length of arc of approach and arc of recess, therefore,

Length of the arc of contact

$$= \text{arc } GP + \text{arc } PH = \frac{KP}{\cos\phi} + \frac{PL}{\cos\phi} = \frac{KL}{\cos\phi}$$

$$= \frac{\text{Length of path of contact}}{\cos\phi}$$

12.18. Contact Ratio (or Number of Pairs of Teeth in Contact)

The contact ratio or the number of pairs of teeth in contact is defined as the **ratio of the length of the arc of contact to the circular pitch.** Mathematically,

Contact ratio or number of pairs of teeth in contact

$$= \frac{\text{Length of the arc of contact}}{p_c}$$

where p_c = Circular pitch = πm, and

m = Module.

Notes : 1. The contact ratio, usually, is not a whole number. For example, if the contact ratio is 1.6, it does not mean that there are 1.6 pairs of teeth in contact. It means that there are alternately one pair and two pairs of teeth in contact and on a time basis the average is 1.6.

2. The theoretical minimum value for the contact ratio is one, that is there must always be at least one pair of teeth in contact for continuous action.

3. Larger the contact ratio, more quietly the gears will operate.

Example 12.2. *The number of teeth on each of the two equal spur gears in mesh are 40. The teeth have 20° involute profile and the module is 6 mm. If the arc of contact is 1.75 times the circular pitch, find the addendum.*

Solution. Given : $T = t = 40$; $\phi = 20°$; $m = 6$ mm

We know that the circular pitch,

$$p_c = \pi\, m = \pi \times 6 = 18.85 \text{ mm}$$

∴ Length of arc of contact

$$= 1.75\, p_c = 1.75 \times 18.85 = 33 \text{ mm}$$

and length of path of contact

$$= \text{Length of arc of contact} \times \cos\phi = 33 \cos 20° = 31 \text{ mm}$$

Let $R_A = r_A$ = Radius of the addendum circle of each wheel.

We know that pitch circle radii of each wheel,

$$R = r = m.T / 2 = 6 \times 40/2 = 120 \text{ mm}$$

and length of path of contact

$$31 = \sqrt{(R_A)^2 - R^2 \cos^2 \phi} + \sqrt{(r_A)^2 - r^2 \cos^2 \phi} - (R + r) \sin \phi$$

$$= 2\left[\sqrt{(R_A)^2 - R^2 \cos^2 \phi} - R \sin \phi\right] \quad ...(\because R = r, \text{ and } R_A = r_A)$$

$$\frac{31}{2} = \sqrt{(R_A)^2 - (120)^2 \cos^2 20°} - 120 \sin 20°$$

$$15.5 = \sqrt{(R_A)^2 - 12\,715} - 41$$

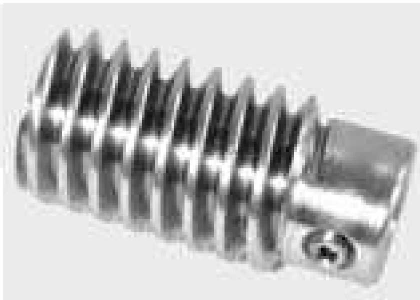
$$(15.5 + 41)^2 = (R_A)^2 - 12\,715$$

$$3192 + 12\,715 = (R_A)^2 \qquad \text{or} \qquad R_A = 126.12 \text{ mm}$$

We know that the addendum of the wheel,

$$= R_A - R = 126.12 - 120 = 6.12 \text{ mm } \textbf{Ans.}$$

Example 12.3. *A pinion having 30 teeth drives a gear having 80 teeth. The profile of the gears is involute with 20° pressure angle, 12 mm module and 10 mm addendum. Find the length of path of contact, arc of contact and the contact ratio.*


Worm.

Solution. Given : $t = 30$; $T = 80$; $\phi = 20°$; $m = 12$ mm ; Addendum = 10 mm

Length of path of contact

We know that pitch circle radius of pinion,

$$r = m.t / 2 = 12 \times 30 / 2 = 180 \text{ mm}$$

and pitch circle radius of gear,

$$R = m.T / 2 = 12 \times 80 / 2 = 480 \text{ mm}$$

∴ Radius of addendum circle of pinion,

$$r_A = r + \text{ Addendum} = 180 + 10 = 190 \text{ mm}$$

and radius of addendum circle of gear,

$$R_A = R + \text{Addendum} = 480 + 10 = 490 \text{ mm}$$

We know that length of the path of approach,

$$KP = \sqrt{(R_A)^2 - R^2 \cos^2 \phi} - R \sin \phi \qquad ...(\text{Refer Fig. 12.11})$$

$$= \sqrt{(490)^2 - (480)^2 \cos^2 20°} - 480 \sin 20° = 191.5 - 164.2 = 27.3 \text{ mm}$$

and length of the path of recess,

$$PL = \sqrt{(r_A)^2 - r^2 \cos^2 \phi} - r \sin \phi$$

$$= \sqrt{(190)^2 - (180)^2 \cos^2 20°} - 180 \sin 20° = 86.6 - 61.6 = 25 \text{ mm}$$

We know that length of path of contact,

$$KL = KP + PL = 27.3 + 25 = 52.3 \text{ mm } \textbf{Ans.}$$

Length of arc of contact

We know that length of arc of contact

$$= \frac{\text{Length of path of contact}}{\cos\phi} = \frac{52.3}{\cos 20°} = 55.66 \text{ mm } \textbf{Ans.}$$

Contact ratio

We know that circular pitch,

$$p_c = \pi.m = \pi \times 12 = 37.7 \text{ mm}$$

$$\therefore \quad \text{Contact ratio} = \frac{\text{Length of arc of contact}}{p_c} = \frac{55.66}{37.7} = 1.5 \text{ say } 2 \textbf{ Ans.}$$

Example 12.4. *Two involute gears of 20° pressure angle are in mesh. The number of teeth on pinion is 20 and the gear ratio is 2. If the pitch expressed in module is 5 mm and the pitch line speed is 1.2 m/s, assuming addendum as standard and equal to one module, find :*

1. The angle turned through by pinion when one pair of teeth is in mesh ; and

2. The maximum velocity of sliding.

Solution. Given : $\phi = 20°$; $t = 20$; $G = T/t = 2$; $m = 5$ mm ; $v = 1.2$ m/s ; addendum = 1 module = 5 mm

1. ***Angle turned through by pinion when one pair of teeth is in mesh***

We know that pitch circle radius of pinion,

$$r = m.t / 2 = 5 \times 20 / 2 = 50 \text{ mm}$$

and pitch circle radius of wheel,

$$R = m.T / 2 = m.G.t / 2 = 2 \times 20 \times 5 / 2 = 100 \text{ mm} \qquad \ldots (\because T = G.t)$$

$\therefore$ Radius of addendum circle of pinion,

$$r_A = r + \text{Addendum} = 50 + 5 = 55 \text{ mm}$$

and radius of addendum circle of wheel,

$$R_A = R + \text{Addendum} = 100 + 5 = 105 \text{ mm}$$

We know that length of the path of approach (*i.e.* the path of contact when engagement occurs),

$$KP = \sqrt{(R_A)^2 - R^2 \cos^2\phi} - R\sin\phi \qquad \ldots\text{(Refer Fig. 12.11)}$$

$$= \sqrt{(105)^2 - (100)^2 \cos^2 20°} - 100 \sin 20°$$

$$= 46.85 - 34.2 = 12.65 \text{ mm}$$

and the length of path of recess (*i.e.* the path of contact when disengagement occurs),

$$PL = \sqrt{(r_A)^2 - r^2 \cos^2\phi} - r\sin\phi$$

$$= \sqrt{(55)^2 - (50)^2 \cos^2 20°} - 50 \sin 20° = 28.6 - 17.1 = 11.5 \text{ mm}$$

$\therefore$ Length of the path of contact,

$$KL = KP + PL = 12.65 + 11.5 = 24.15 \text{ mm}$$

and length of the arc of contact

$$= \frac{\text{Length of path of contact}}{\cos\phi} = \frac{24.15}{\cos 20°} = 25.7 \text{ mm}$$

We know that angle turned through by pinion

$$= \frac{\text{Length of arc of contact} \times 360°}{\text{Circumference of pinion}} = \frac{25.7 \times 360°}{2\pi \times 50} = 29.45° \quad \textbf{Ans.}$$

2. ***Maximum velocity of sliding***

Let ω_1 = Angular speed of pinion, and

ω_2 = Angular speed of wheel.

We know that pitch line speed,

$$v = \omega_1 . r = \omega_2 . R$$

$$\therefore \quad \omega_1 = v/r = 120/5 = 24 \text{ rad/s}$$

and $\omega_2 = v/R = 120/10 = 12$ rad/s

$\therefore$ Maximum velocity of sliding,

$$v_S = (\omega_1 + \omega_2)\, KP \qquad ...(\because KP > PL)$$

$$= (24 + 12)\, 12.65 = 455.4 \text{ mm/s} \quad \textbf{Ans.}$$

Example 12.5. *A pair of gears, having 40 and 20 teeth respectively, are rotating in mesh, the speed of the smaller being 2000 r.p.m. Determine the velocity of sliding between the gear teeth faces at the point of engagement, at the pitch point, and at the point of disengagement if the smaller gear is the driver. Assume that the gear teeth are 20° involute form, addendum length is 5 mm and the module is 5 mm.*

Also find the angle through which the pinion turns while any pairs of teeth are in contact.

Solution. Given : $T = 40$; $t = 20$; $N_1 = 2000$ r.p.m. ; $\phi = 20°$; addendum = 5 mm ; $m = 5$ mm

We know that angular velocity of the smaller gear,

$$\omega_1 = \frac{2\pi N_1}{60} = \frac{2\pi \times 2000}{60} = 209.5 \text{ rad/s}$$

and angular velocity of the larger gear,

$$\omega_2 = \omega_1 \times \frac{t}{T} = 209.5 \times \frac{20}{40} = 104.75 \text{ rad/s} \qquad ...\left(\because \frac{\omega_2}{\omega_1} = \frac{t}{T}\right)$$

Pitch circle radius of the smaller gear,

$$r = m.t / 2 = 5 \times 20/2 = 50 \text{ mm}$$

and pitch circle radius of the larger gear,

$$R = m.t / 2 = 5 \times 40/2 = 100 \text{ mm}$$

$\therefore$ Radius of addendum circle of smaller gear,

$$r_A = r + \text{Addendum} = 50 + 5 = 55 \text{ mm}$$

and radius of addendum circle of larger gear,

$$R_A = R + \text{Addendum} = 100 + 5 = 105 \text{ mm}$$

The engagement and disengagement of the gear teeth is shown in Fig. 12.11. The point K is the point of engagement, P is the pitch point and L is the point of disengagement. MN is the common tangent at the points of contact.

We know that the distance of point of engagement K from the pitch point P or the length of the path of approach,

$$KP = \sqrt{(R_A)^2 - R^2 \cos^2 \phi} - R \sin \phi$$

$$= \sqrt{(105)^2 - (100)^2 \cos^2 20°} - 100 \sin 20°$$

$$= 46.85 - 34.2 = 12.65 \text{ mm}$$

and the distance of the pitch point P from the point of disengagement L or the length of the path of recess,

$$PL = \sqrt{(r_A)^2 - r^2 \cos^2 \phi} - r \sin \phi$$

$$= \sqrt{(55)^2 - (50)^2 \cos^2 20°} - 50 \sin 20° = 28.6 - 17.1 = 11.5 \text{ mm}$$

Velocity of sliding at the point of engagement

We know that velocity of sliding at the point of engagement K,

$$v_{SK} = (\omega_1 + \omega_2)\ KP = (209.5 + 104.75)\ 12.65 = 3975 \text{ mm/s} \quad \textbf{Ans.}$$

Velocity of sliding at the pitch point

Since the velocity of sliding is proportional to the distance of the contact point from the pitch point, therefore the velocity of sliding at the pitch point is zero. **Ans.**

Velocity of sliding at the point of disengagement

We know that velocity of sliding at the point of disengagement L,

$$v_{SL} = (\omega_1 + \omega_2)\ PL = (209.5 + 104.75)\ 11.5 = 3614 \text{ mm/s} \quad \textbf{Ans.}$$

Angle through which the pinion turns

We know that length of the path of contact,

$$KL = KP + PL = 12.65 + 11.5 = 24.15 \text{ mm}$$

and length of arc of contact $= \dfrac{KL}{\cos \phi} = \dfrac{24.15}{\cos 20°} = 25.7$ mm

Circumference of the smaller gear or pinion

$$= 2\,\pi\, r = 2\pi \times 50 = 314.2 \text{ mm}$$

∴ Angle through which the pinion turns

$$= \text{Length of arc of contact} \times \frac{360°}{\text{Circumference of pinion}}$$

$$= 25.7 \times \frac{360°}{314.2} = 29.45° \quad \textbf{Ans.}$$

Example 12.6. *The following data relate to a pair of 20° involute gears in mesh :*

Module = 6 mm, Number of teeth on pinion = 17, Number of teeth on gear = 49 ; Addenda on pinion and gear wheel = 1 module.

Find : **1.** *The number of pairs of teeth in contact ;* **2.** *The angle turned through by the pinion and the gear wheel when one pair of teeth is in contact, and* **3.** *The ratio of sliding to rolling motion when the tip of a tooth on the larger wheel* (*i*) *is just making contact,* (*ii*) *is just leaving contact with its mating tooth, and* (*iii*) *is at the pitch point.*

Solution. Given : $\phi = 20°$; $m = 6$ mm ; $t = 17$; $T = 49$; Addenda on pinion and gear wheel = 1 module = 6 mm

1. ***Number of pairs of teeth in contact***

We know that pitch circle radius of pinion,

$$r = m.t / 2 = 6 \times 17 / 2 = 51 \text{ mm}$$

and pitch circle radius of gear,

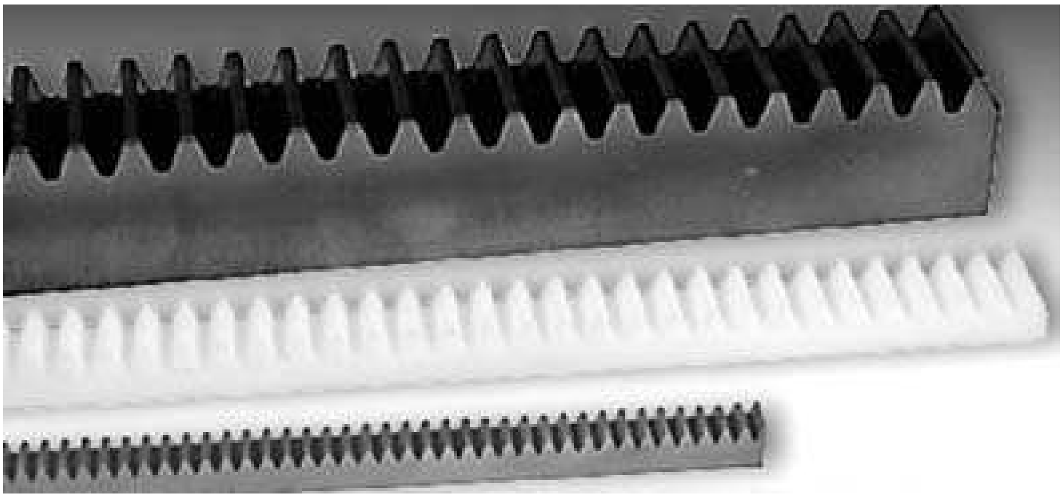
$$r = m.T / 2 = 6 \times 49 / 2 = 147 \text{ mm}$$

∴ Radius of addendum circle of pinion,

$$r_A = r + \text{Addendum} = 51 + 6 = 57 \text{ mm}$$

and radius of addendum circle of gear,

$$R_A = R + \text{Addendum} = 147 + 6 = 153 \text{ mm}$$



Racks

We know that the length of path of approach (*i.e.* the path of contact when engagement occurs),

$$KP = \sqrt{(R_A)^2 - R^2 \cos^2 \phi} - R \sin \phi \qquad \text{...(Refer Fig. 12.11)}$$

$$= \sqrt{(153)^2 - (147)^2 \cos^2 20°} - 147 \sin 20°$$

$$= 65.8 - 50.3 = 15.5 \text{ mm}$$

and length of path of recess (*i.e.* the path of contact when disengagement occurs),

$$PL = \sqrt{(r_A)^2 - r^2 \cos^2 \phi} - r \sin \phi$$

$$= \sqrt{(57)^2 - (51)^2 \cos^2 20°} - 51 \sin 20°$$

$$= 30.85 - 17.44 = 13.41 \text{ mm}$$

∴ Length of path of contact,

$$KL = KP + PL = 15.5 + 13.41 = 28.91 \text{ mm}$$

and length of arc of contact $= \dfrac{\text{Length of path of contact}}{\cos\phi} = \dfrac{28.91}{\cos 20°} = 30.8$ mm

We know that circular pitch,

$$p_c = \pi.m = \pi \times 6 = 18.852 \text{ mm}$$

∴ Number of pairs of teeth in contact (or contact ratio)

$$= \frac{\text{Length of arc of contact}}{\text{Circular pitch}} = \frac{30.8}{18.852} = 1.6 \text{ say } 2 \quad \textbf{Ans.}$$

2. ***Angle turned through by the pinion and gear wheel when one pair of teeth is in contact***

We know that angle turned through by the pinion

$$= \frac{\text{Length of arc of contact} \times 360°}{\text{Circumference of pinion}} = \frac{30.8 \times 360}{2\pi \times 51} = 34.6° \quad \textbf{Ans.}$$

and angle turned through by the gear wheel

$$= \frac{\text{Length of arc of contact} \times 360°}{\text{Circumference of gear}} = \frac{30.8 \times 360}{2\pi \times 147} = 12° \quad \textbf{Ans.}$$

3. ***Ratio of sliding to rolling motion***

Let ω_1 = Angular velocity of pinion, and

ω_2 = Angular velocity of gear wheel.

We know that $\omega_1 / \omega_2 = T/t$ or $\omega_2 = \omega_1 \times t/T = \omega_1 \times 17/49 = 0.347\ \omega_1$

and rolling velocity, $v_R = \omega_1.r = \omega_2.R = \omega_1 \times 51 = 51\ \omega_1$ mm/s

(*i*) At the instant when the tip of a tooth on the larger wheel is just making contact with its mating teeth (*i.e.* when the engagement commences), the sliding velocity

$$v_S = (\omega_1 + \omega_2)\ KP = (\omega_1 + 0.347\ \omega_1)\ 15.5 = 20.88\ \omega_1 \text{ mm/s}$$

∴ Ratio of sliding velocity to rolling velocity,

$$\frac{v_S}{v_R} = \frac{20.88\ \omega_1}{51 \omega_1} = 0.41 \quad \textbf{Ans.}$$

(*ii*) At the instant when the tip of a tooth on the larger wheel is just leaving contact with its mating teeth (*i.e.* when engagement terminates), the sliding velocity,

$$v_S = (\omega_1 + \omega_2)\ PL = (\omega_1 + 0.347 \omega_1)\ 13.41 = 18.1\ \omega_1 \text{ mm/s}$$

∴ Ratio of sliding velocity to rolling velocity

$$\frac{v_S}{v_R} = \frac{18.1\ \omega_1}{51\ \omega_1} = 0.355 \quad \textbf{Ans.}$$

(*iii*) Since at the pitch point, the sliding velocity is zero, therefore the ratio of sliding velocity to rolling velocity is zero. **Ans.**

Example 12.7. *A pinion having 18 teeth engages with an internal gear having 72 teeth. If the gears have involute profiled teeth with 20° pressure angle, module of 4 mm and the addenda on pinion and gear are 8.5 mm and 3.5 mm respectively, find the length of path of contact.*

Solution. Given : $t = 18$; $T = 72$; $\phi = 20°$; $m = 4$ mm ; Addendum on pinion = 8.5 mm ; Addendum on gear = 3.5 mm

Fig. 12.12 shows a pinion with centre O_1, in mesh with internal gear of centre O_2. It may be noted that the internal gears have the addendum circle and the tooth faces ***inside*** the pitch circle.

We know that the length of path of contact is the length of the common tangent to the two base circles cut by the addendum circles. From Fig. 12.12, we see that the addendum circles cut the common tangents at points K and L. Therefore the length of path of contact is KL which is equal to the sum of KP (*i.e.* path of approach) and PL (*i.e.* path of recess).

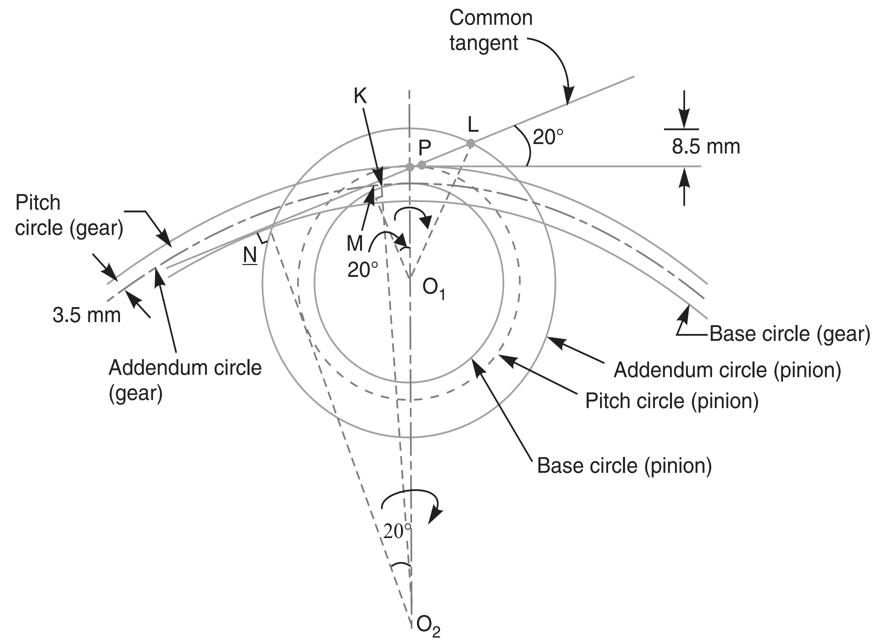


Fig. 12.12

We know that pitch circle radius of the pinion,

$$r = O_1P = m.t/2 = 4 \times 18/2 = 36 \text{ mm}$$

and pitch circle radius of the gear,

$$R = O_2P = m.T/2 = 4 \times 72/2 = 144 \text{ mm}$$

∴ Radius of addendum circle of the pinion,

$$r_A = O_1L = O_1P + \text{ Addendum on pinion} = 36 + 8.5 = 44.5 \text{ mm}$$

and radius of addendum circle of the gear,

$$R_A = O_2K = O_2P - \text{Addendum on wheel} = 144 - 3.5 = 140.5 \text{ mm}$$

From Fig. 12.12, radius of the base circle of the pinion,

$$O_1M = O_1P \cos\phi = r\cos\phi = 36 \cos 20° = 33.83 \text{ mm}$$

and radius of the base circle of the gear,

$$O_2N = O_2P\cos\phi = R\cos\phi = 144 \cos 20° = 135.32 \text{ mm}$$

We know that length of the path of approach,

$$KP = PN - KN = O_2P \sin 20° - \sqrt{(O_2K)^2 - (O_2N)^2}$$

$$= 144 \times 0.342 - \sqrt{(140.5)^2 - (135.32)^2} = 49.25 - 37.8 = 11.45 \text{ mm}$$

and length of the path of recess,

$$PL = ML - MP = \sqrt{(O_1L)^2 - (O_1M)^2} - O_1P \sin 20°$$

$$= \sqrt{(44.5)^2 - (33.83)^2} - 36 \times 0.342 = 28.9 - 12.3 = 16.6 \text{ mm}$$

∴ Length of the path of contact,

$$KL = KP + PL = 11.45 + 16.6 = 28.05 \text{ mm} \quad \textbf{Ans.}$$

12.19. Interference in Involute Gears

Fig. 12.13 shows a pinion with centre O_1, in mesh with wheel or gear with centre O_2. MN is the common tangent to the base circles and KL is the path of contact between the two mating teeth.

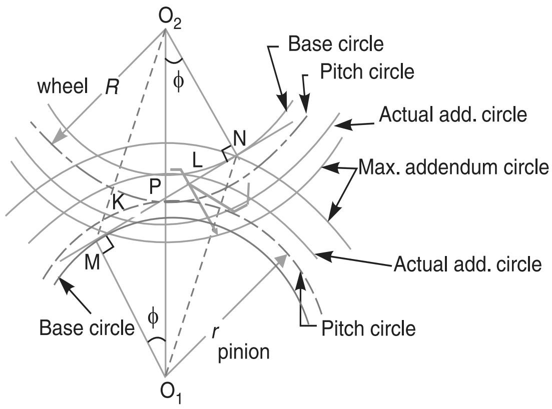


Fig. 12.13. Interference in involute gears.

A little consideration will show, that if the radius of the addendum circle of pinion is increased to O_1N, the point of contact L will move from L to N. When this radius is further increased, the point of contact L will be on the inside of base circle of wheel and not on the involute profile of tooth on wheel. The tip of tooth on the pinion will then undercut the tooth on the wheel at the root and remove part of the involute profile of tooth on the wheel. This effect is known as ***interference***, and occurs when the teeth are being cut. In brief, ***the phenomenon when the tip of tooth undercuts the root on its mating gear is known as interference***.

Similarly, if the radius of the addendum circle of the wheel increases beyond O_2M, then the tip of tooth on wheel will cause interference with the tooth on pinion. The points M and N are called ***interference points***. Obviously, interference may be avoided if the path of contact does not extend beyond interference points. The limiting value of the radius of the addendum circle of the pinion is *O_1N and of the wheel is O_2M.

From the above discussion, we conclude that the interference may only be avoided, if the point of contact between the two teeth is always on the involute profiles of both the teeth. In other

---

* From Fig. 12.13, we see that

$$O_1N = \sqrt{(O_1M)^2 + (MN)^2} = \sqrt{(r_b)^2 + [r + R) \sin \phi]^2}$$

where r_b = Radius of base circle of pinion = $O_1P \cos \phi = r \cos \phi$

and $$O_2M = \sqrt{(O_2N)^2 + (MN)^2} = \sqrt{(R_b)^2 + [r + R) \sin \phi]^2}$$

where R_b = Radius of base circle of wheel = $O_2P \cos \phi = R \cos \phi$

words, ***interference may only be prevented, if the addendum circles of the two mating gears cut the common tangent to the base circles between the points of tangency*.**

When interference is just avoided, the maximum length of path of contact is MN when the maximum addendum circles for pinion and wheel pass through the points of tangency N and M respectively as shown in Fig. 12.13. In such a case,

Maximum length of path of approach,

$$MP = r \sin \phi$$

and maximum length of path of recess,

$$PN = R \sin \phi$$

∴ Maximum length of path of contact,

$$MN = MP + PN = r \sin \phi + R \sin \phi = (r + R) \sin \phi$$

and maximum length of arc of contact

$$= \frac{(r + R) \sin \phi}{\cos \phi} = (r + R) \tan \phi$$

Note : In case the addenda on pinion and wheel is such that the path of approach and path of recess are half of their maximum possible values, then

Path of approach, $KP = \frac{1}{2} MP$

or $\sqrt{(R_A)^2 - R^2 \cos^2 \phi} - R \sin \phi = \dfrac{r \sin \phi}{2}$

and path of recess, $PL = \frac{1}{2} PN$

or $\sqrt{(r_A)^2 - r^2 \cos^2 \phi} - r \sin \phi = \dfrac{R \sin \phi}{2}$

∴ Length of the path of contact

$$= KP + PL = \tfrac{1}{2} MP + \tfrac{1}{2} PN = \frac{(r + R) \sin \phi}{2}$$

Example 12.8. *Two mating gears have 20 and 40 involute teeth of module 10 mm and 20° pressure angle. The addendum on each wheel is to be made of such a length that the line of contact on each side of the pitch point has half the maximum possible length. Determine the addendum height for each gear wheel, length of the path of contact, arc of contact and contact ratio.*

Solution. Given : $t = 20$; $T = 40$; $m = 10$ mm ; $\phi = 20°$

Addendum height for each gear wheel

We know that the pitch circle radius of the smaller gear wheel,

$$r = m.t / 2 = 10 \times 20 / 2 = 100 \text{ mm}$$

and pitch circle radius of the larger gear wheel,

$$R = m.T / 2 = 10 \times 40 / 2 = 200 \text{ mm}$$

Let R_A = Radius of addendum circle for the larger gear wheel, and

r_A = Radius of addendum circle for the smaller gear wheel.

Since the addendum on each wheel is to be made of such a length that the line of contact on each side of the pitch point (*i.e.* the path of approach and the path of recess) has half the maximum possible length, therefore

Path of approach, $KP = \frac{1}{2} MP$ ...(Refer Fig. 12.13)

or $\sqrt{(R_A)^2 - R^2 \cos^2 \phi} - R \sin \phi = \dfrac{r.\sin \phi}{2}$

or $\sqrt{(R_A)^2 - (200)^2 \cos^2 20°} - 200 \sin 20° = \dfrac{100 \times \sin 20°}{2} = 50 \sin 20°$

$$\sqrt{(R_A)^2 - 35\ 320} = 50 \sin 20° + 200 \sin 20° = 250 \times 0.342 = 85.5$$

$$(R_A)^2 - 35\ 320 = (85.5)^2 = 7310 \quad \text{...(Squaring both sides)}$$

$$(R_A)^2 = 7310 + 35\ 320 = 42\ 630 \quad \text{or} \quad R_A = 206.5 \text{ mm}$$

∴ Addendum height for larger gear wheel

$$= R_A - R = 206.5 - 200 = 6.5 \text{ mm } \textbf{Ans.}$$

Now path of recess, $PL = \frac{1}{2} PN$

or $\sqrt{(r_A)^2 - r^2 \cos^2 \phi} - r \sin \phi = \dfrac{R.\sin \phi}{2}$

or $\sqrt{(r_A)^2 - (100)^2 \cos^2 20°} - 100 \sin 20° = \dfrac{200 \sin 20°}{2} = 100 \sin 20°$

$$\sqrt{(r_A)^2 - (100)^2 \cos^2 20°} = 100 \sin 20° + 100 \sin 20° = 200 \times 0.342 = 68.4$$

$$(r_A)^2 - 8830 = (68.4)^2 = 4680 \quad \text{...(Squaring both sides)}$$

$$(r_A)^2 = 4680 + 8830 = 13\ 510 \quad \text{or} \quad r_A = 116.2 \text{ mm}$$

∴ Addendum height for smaller gear wheel

$$= r_A - r = 116.2 - 100 = 6.2 \text{ mm } \textbf{Ans.}$$

Length of the path of contact

We know that length of the path of contact

$$= KP + PL = \frac{1}{2} MP + \frac{1}{2} PN = \frac{(r + R) \sin \phi}{2}$$

$$= \frac{(100 + 200) \sin 20°}{2} = 51.3 \text{ mm } \textbf{Ans.}$$

Length of the arc of contact

We know that length of the arc of contact

$$= \frac{\text{Length of the path of contact}}{\cos \phi} = \frac{51.3}{\cos 20°} = 54.6 \text{ mm } \textbf{Ans.}$$

Contact ratio

We know that circular pitch,

$$P_c = \pi m = \pi \times 10 = 31.42 \text{ mm}$$

∴ $$\text{Contact ratio} = \frac{\text{Length of the path of contact}}{p_c} = \frac{54.6}{31.42} = 1.74 \text{ say } 2 \textbf{ Ans.}$$

12.20. Minimum Number of Teeth on the Pinion in Order to Avoid Interference

We have already discussed in the previous article that in order to avoid interference, the addendum circles for the two mating gears must cut the common tangent to the base circles between the points of tangency. The limiting condition reaches, when the addendum circles of pinion and wheel pass through points N and M (see Fig. 12.13) respectively.

Let t = Number of teeth on the pinion,,

T = Number of teeth on the wheel,

m = Module of the teeth,

r = Pitch circle radius of pinion = $m.t / 2$

G = Gear ratio = $T / t = R / r$

ϕ = Pressure angle or angle of obliquity.

From triangle O_1NP,

$$(O_1N)^2 = (O_1P)^2 + (PN)^2 - 2 \times O_1P \times PN \cos O_1PN$$

$$= r^2 + R^2 \sin^2\phi - 2r.R \sin\phi \cos(90° + \phi)$$

$$\ldots(\because PN = O_2P \sin\phi = R\sin\phi)$$

$$= r^2 + R^2 \sin^2\phi + 2r.R \sin^2\phi$$

$$= r^2\left[1 + \frac{R^2\sin^2\phi}{r^2} + \frac{2R\sin^2\phi}{r}\right] = r^2\left[1 + \frac{R}{r}\left(\frac{R}{r} + 2\right)\sin^2\phi\right]$$

$\therefore$ Limiting radius of the pinion addendum circle,

$$O_1N = r\sqrt{1 + \frac{R}{r}\left(\frac{R}{r} + 2\right)\sin^2\phi} = \frac{m.t}{2}\sqrt{1 + \frac{T}{t}\left[\frac{T}{t} + 2\right]\sin^2\phi}$$

Let $A_P.m$ = Addendum of the pinion, where A_P is a fraction by which the standard addendum of one module for the pinion should be multiplied in order to avoid interference.

We know that the addendum of the pinion

$$= O_1N - O_1P$$

$$\therefore \quad A_P.m = \frac{m.t}{2}\sqrt{1 + \frac{T}{t}\left(\frac{T}{t} + 2\right)\sin^2\phi} - \frac{m.t}{2} \qquad \ldots(\because O_1P = r = m.t/2)$$

$$= \frac{m.t}{2}\left[\sqrt{1 + \frac{T}{t}\left(\frac{T}{t} + 2\right)\sin^2\phi} - 1\right]$$

or

$$A_P = \frac{t}{2}\left[\sqrt{1 + \frac{T}{t}\left(\frac{T}{t} + 2\right)\sin^2\phi} - 1\right]$$

$$\therefore \quad t = \frac{2A_P}{\sqrt{1 + \frac{T}{t}\left(\frac{T}{t} + 2\right)\sin^2\phi} - 1} = \frac{2A_P}{\sqrt{1 + G(G + 2)\sin^2\phi} - 1}$$

This equation gives the minimum number of teeth required on the pinion in order to avoid interference.

Notes : 1. If the pinion and wheel have equal teeth, then $G = 1$. Therefore the above equation reduces to

$$t = \frac{2A_p}{\sqrt{1 + 3\sin^2\phi} - 1}$$

2. The minimum number of teeth on the pinion which will mesh with any gear (also rack) without interference are given in the following table :

Table 12.2. Minimum number of teeth on the pinion

S. No.	*System of gear teeth*	*Minimum number of teeth on the pinion*
1.	$14\frac{1}{2}°$ Composite	12
2.	$14\frac{1}{2}°$ Full depth involute	32
3.	20° Full depth involute	18
4.	20° Stub involute	14

12.21. Minimum Number of Teeth on the Wheel in Order to Avoid Interference

Let T = Minimum number of teeth required on the wheel in order to avoid interference,

and $A_W.m$ = Addendum of the wheel, where A_W is a fraction by which the standard addendum for the wheel should be multiplied.

Using the same notations as in Art. 12.20, we have from triangle O_2MP

$$(O_2M)^2 = (O_2P)^2 + (PM)^2 - 2 \times O_2P \times PM \cos O_2PM$$

$$= R^2 + r^2 \sin^2\phi - 2\, R.r \sin\phi \cos(90° + \phi)$$

$$...(\because PM = O_1P \sin\phi = r)$$

$$= R^2 + r^2 \sin^2\phi + 2R.r\sin^2\phi$$

$$= R^2\left[1 + \frac{r^2\sin^2\phi}{R^2} + \frac{2r\sin^2\phi}{R}\right] = R^2\left[1 + \frac{r}{R}\left(\frac{r}{R} + 2\right)\sin^2\phi\right]$$

∴ Limiting radius of wheel addendum circle,

$$O_2M = R\sqrt{1 + \frac{r}{R}\left(\frac{r}{R} + 2\right)\sin^2\phi} = \frac{m.T}{2}\sqrt{1 + \frac{t}{T}\left(\frac{t}{T} + 2\right)\sin^2\phi}$$

We know that the addendum of the wheel

$$= O_2M - O_2P$$

$$\therefore \quad A_W m = \frac{m.T}{2}\sqrt{1 + \frac{t}{T}\left(\frac{t}{T} + 2\right)\sin^2\phi} - \frac{m.T}{2} \qquad ...(\because O_2P = R = m.T/2)$$

$$= \frac{m.T}{2}\left[\sqrt{1 + \frac{t}{T}\left(\frac{t}{T} + 2\right)\sin^2\phi} - 1\right]$$

or

$$A_W = \frac{T}{2}\left[\sqrt{1 + \frac{t}{T}\left(\frac{t}{T} + 2\right)\sin^2\phi} - 1\right]$$

$$\therefore \qquad T = \frac{2\,A_W}{\sqrt{1 + \frac{t}{T}\left(\frac{t}{T} + 2\right)\sin^2\phi} - 1} = \frac{2\,A_W}{\sqrt{1 + \frac{1}{G}\left(\frac{1}{G} + 2\right)\sin^2\phi} - 1}$$

Notes : 1. From the above equation, we may also obtain the minimum number of teeth on pinion.

Multiplying both sides by $\frac{t}{T}$,

$$T \times \frac{t}{T} = \frac{2\,A_W \times \frac{t}{T}}{\sqrt{1 + \frac{1}{G}\left(\frac{1}{G} + 2\right)\sin^2\phi} - 1}$$

$$t = \frac{2\,A_W}{G\left[\sqrt{1 + \frac{1}{G}\left(\frac{1}{G} + 2\right)\sin^2\phi} - 1\right]}$$

2. If wheel and pinion have equal teeth, then $G = 1$, and

$$T = \frac{2\,A_W}{\sqrt{1 + 3\sin^2\phi} - 1}$$

Example 12.9. *Determine the minimum number of teeth required on a pinion, in order to avoid interference which is to gear with,*
1. a wheel to give a gear ratio of 3 to 1 ; and 2. an equal wheel.

The pressure angle is 20° and a standard addendum of 1 module for the wheel may be assumed.

Solution. Given : $G = T / t = 3$; $\phi = 20°$; $A_W = 1$ module

1. ***Minimum number of teeth for a gear ratio of 3 : 1***

We know that minimum number of teeth required on a pinion,

$$t = \frac{2 \times A_W}{G\left[\sqrt{1 + \frac{1}{G}\left(\frac{1}{G} + 2\right)\sin^2\phi} - 1\right]}$$

$$= \frac{2 \times 1}{3\left[\sqrt{1 + \frac{1}{3}\left(\frac{1}{3} + 2\right)\sin^2 20°} - 1\right]} = \frac{2}{0.133} = 15.04 \text{ or } 16 \quad \textbf{Ans.}$$

2. ***Minimum number of teeth for equal wheel***

We know that minimum number of teeth for equal wheel,

$$t = \frac{2 \times A_W}{\sqrt{1 + 3\sin^2\phi} - 1} = \frac{2 \times 1}{\sqrt{1 + 3\sin^2 20°} - 1} = \frac{2}{0.162}$$

$$= 12.34 \text{ or } 13 \quad \textbf{Ans.}$$

Example 12.10. *A pair of spur gears with involute teeth is to give a gear ratio of 4 : 1. The arc of approach is not to be less than the circular pitch and smaller wheel is the driver. The angle of pressure is 14.5°. Find : 1. the least number of teeth that can be used on each wheel, and 2. the addendum of the wheel in terms of the circular pitch ?*

Solution. *Given* : $G = T/t = R/r = 4$; $\phi = 14.5°$

1. ***Least number of teeth on each wheel***

Let t = Least number of teeth on the smaller wheel *i.e.* pinion,

T = Least number of teeth on the larger wheel *i.e.* gear, and

r = Pitch circle radius of the smaller wheel *i.e.* pinion.

We know that the maximum length of the arc of approach

$$= \frac{\text{Maximum length of the path of approach}}{\cos\phi} = \frac{r\sin\phi}{\cos\phi} = r\tan\phi$$

and circular pitch, $$p_c = \pi m = \frac{2\pi r}{t} \qquad \ldots\left(\because m = \frac{2r}{t}\right)$$

Since the arc of approach is not to be less than the circular pitch, therefore

$$r\tan\phi = \frac{2\pi r}{t} \quad \text{or} \quad t = \frac{2\pi}{\tan\phi} = \frac{2\pi}{\tan 14.5°} = 24.3 \text{ say } 25 \textbf{ Ans.}$$

and $$T = G.t = 4 \times 25 = 100 \textbf{ Ans.} \qquad \ldots(\because G = T/t)$$

2. ***Addendum of the wheel***

We know that addendum of the wheel

$$= \frac{m.T}{2}\left[\sqrt{1 + \frac{t}{T}\left(\frac{t}{T} + 2\right)\sin^2\phi} - 1\right]$$

$$= \frac{m \times 100}{2}\left[\sqrt{1 + \frac{25}{100}\left(\frac{25}{100} + 2\right)\sin^2 14.5°} - 1\right]$$

$$= 50\,m \times 0.017 = 0.85\,m = 0.85 \times p_c/\pi = 0.27\,p_c \textbf{ Ans.}$$

$$\ldots(\because m = p_c/\pi)$$

Example 12.11. *A pair of involute spur gears with 16° pressure angle and pitch of module 6 mm is in mesh. The number of teeth on pinion is 16 and its rotational speed is 240 r.p.m. When the gear ratio is 1.75, find in order that the interference is just avoided ;* **1.** *the addenda on pinion and gear wheel ;* **2.** *the length of path of contact ; and* **3.** *the maximum velocity of sliding of teeth on either side of the pitch point.*

Solution. Given : $\phi = 16°$; $m = 6$ mm ; $t = 16$; $N_1 = 240$ r.p.m. or $\omega_1 = 2\pi \times 240/60 = 25.136$ rad/s ; $G = T/t = 1.75$ or $T = G.t = 1.75 \times 16 = 28$

1. ***Addenda on pinion and gear wheel***

We know that addendum on pinion

$$= \frac{m.t}{2}\left[\sqrt{1 + \frac{T}{t}\left(\frac{T}{t} + 2\right)\sin^2\phi} - 1\right]$$

$$= \frac{6 \times 16}{2}\left[\sqrt{1 + \frac{28}{16}\left(\frac{28}{16} + 2\right)\sin^2 16°} - 1\right]$$

$$= 48\,(1.224 - 1) = 10.76 \text{ mm } \textbf{Ans.}$$

and addendum on wheel $$= \frac{m.T}{2}\left[\sqrt{1 + \frac{t}{T}\left(\frac{t}{T} + 2\right)\sin^2\phi} - 1\right]$$

$$= \frac{6 \times 28}{2}\left[\sqrt{1 + \frac{16}{28}\left(\frac{16}{28} + 2\right)\sin^2 16°} - 1\right]$$

$$= 84\ (1.054 - 1) = 4.56 \text{ mm} \quad \textbf{Ans.}$$

2. ***Length of path of contact***

We know that the pitch circle radius of wheel,

$$R = m.T/2 = 6 \times 28/2 = 84 \text{ mm}$$

and pitch circle radius of pinion,

$$r = m.t/2 = 6 \times 16/2 = 48 \text{ mm}$$

∴ Addendum circle radius of wheel,

$$R_A = R + \text{Addendum of wheel} = 84 + 10.76 = 94.76 \text{ mm}$$

and addendum circle radius of pinion,

$$r_A = r + \text{Addendum of pinion} = 48 + 4.56 = 52.56 \text{ mm}$$

We know that the length of path of approach,

$$KP = \sqrt{(R_A)^2 - R^2\cos^2\phi} - R\sin\phi \qquad \text{...(Refer Fig. 12.11)}$$

$$= \sqrt{(94.76)^2 - (84)^2\cos^2 16°} - 84\sin 16°$$

$$= 49.6 - 23.15 = 26.45 \text{ mm}$$

and the length of the path of recess,

$$PL = \sqrt{(r_A)^2 - r^2\cos^2\phi} - r\sin\phi$$

$$= \sqrt{(52.56)^2 - (48)^2\cos^2 16°} - 48\sin 16°$$

$$= 25.17 - 13.23 = 11.94 \text{ mm}$$

∴ Length of the path of contact,

$$KL = KP + PL = 26.45 + 11.94 = 38.39 \text{ mm} \quad \textbf{Ans.}$$

3. ***Maximum velocity of sliding of teeth on either side of pitch point***

Let ω_2 = Angular speed of gear wheel.

We know that $\dfrac{\omega_1}{\omega_2} = \dfrac{T}{t} = 1.75$ or $\omega_2 = \dfrac{\omega_1}{1.75} = \dfrac{25.136}{1.75} = 14.28$ rad/s

∴ Maximum velocity of sliding of teeth on the left side of pitch point *i.e.* at point K

$$= (\omega_1 + \omega_2)\ KP = (25.136 + 14.28)\ 26.45 = 1043 \text{ mm/s} \quad \textbf{Ans.}$$

and maximum velocity of sliding of teeth on the right side of pitch point *i.e.* at point L

$$= (\omega_1 + \omega_2)\ PL = (25.136 + 14.28)\ 11.94 = 471 \text{ mm/s} \quad \textbf{Ans.}$$

Example 12.12. *A pair of 20° full depth involute spur gears having 30 and 50 teeth respectively of module 4 mm are in mesh. The smaller gear rotates at 1000 r.p.m. Determine :* **1.** *sliding velocities at engagement and at disengagement of pair of a teeth, and* **2.** *contact ratio.*

Solution. Given: $\phi = 20°$; $t = 30$; $T = 50$; $m = 4$; $N_1 = 1000$ r.p.m. or $\omega_1 = 2\pi \times 1000/60 = 104.7$ rad/s

1. ***Sliding velocities at engagement and at disengagement of pair of a teeth***

First of all, let us find the radius of addendum circles of the smaller gear and the larger gear. We know that

Addendum of the smaller gear,

$$= \frac{m.t}{2}\left[\sqrt{1 + \frac{T}{t}\left(\frac{T}{t} + 2\right)\sin^2\phi} - 1\right]$$

$$= \frac{4 \times 30}{2}\left[\sqrt{1 + \frac{50}{30}\left(\frac{50}{30} + 2\right)\sin^2 20°} - 1\right]$$

$$= 60(1.31 - 1) = 18.6 \text{ mm}$$

and addendum of the larger gear,

$$= \frac{m.T}{2}\left[\sqrt{1 + \frac{t}{T}\left(\frac{t}{T} + 2\right)\sin^2\phi} - 1\right]$$

$$= \frac{4 \times 50}{2}\left[\sqrt{1 + \frac{30}{50}\left(\frac{30}{50} + 2\right)\sin^2 20°} - 1\right]$$

$$= 100(1.09 - 1) = 9 \text{ mm}$$

Pitch circle radius of the smaller gear,

$$r = m.t/2 = 4 \times 30/2 = 60 \text{ mm}$$

∴ Radius of addendum circle of the smaller gear,

$$r_A = r + \text{Addendum of the smaller gear} = 60 + 18.6 = 78.6 \text{ mm}$$

Pitch circle radius of the larger gear,

$$R = m.T/2 = 4 \times 50/2 = 100 \text{ mm}$$

∴ Radius of addendum circle of the larger gear,

$$R_A = R + \text{Addendum of the larger gear} = 100 + 9 = 109 \text{ mm}$$

We know that the path of approach (*i.e.* path of contact when engagement occurs),

$$KP = \sqrt{(R_A)^2 - R^2\cos^2\phi} - R\sin\phi \qquad \text{...(Refer Fig. 12.11)}$$

$$= \sqrt{(109)^2 - (100)^2\cos^2 20°} - 100\sin 20° = 55.2 - 34.2 = 21 \text{ mm}$$

and the path of recess (*i.e.* path of contact when disengagement occurs),

$$PL = \sqrt{(r_A)^2 - r^2\cos^2\phi} - r\sin\phi$$

$$= \sqrt{(78.6)^2 - (60)^2\cos^2 20°} - 60\sin 20° = 54.76 - 20.52 = 34.24 \text{ mm}$$

Let ω_2 = Angular speed of the larger gear in rad/s.

We know that $\dfrac{\omega_1}{\omega_2} = \dfrac{T}{t}$ or $\omega_2 = \dfrac{\omega_1 \times t}{T} = \dfrac{10.47 \times 30}{50} = 62.82$ rad/s

∴ Sliding velocity at engagement of a pair of teeth

$$= (\omega_1 + \omega_2)KP = (104.7 + 62.82)21 = 3518 \text{ mm/s}$$

$$= 3.518 \text{ m/s}$$ **Ans.**

and sliding velocity at disengagement of a pair of teeth

$$= (\omega_1 + \omega_2)\,PL = (104.7 + 62.82)\,34.24 = 5736 \text{ mm/s}$$

$$= 5.736 \text{ m/s} \quad \textbf{Ans.}$$

2. ***Contact ratio***

We know that the length of the arc of contact

$$= \frac{\text{Length of the path of contact}}{\cos\phi} = \frac{KP + PL}{\cos\phi} = \frac{21 + 34.24}{\cos 20°}$$

$$= 58.78 \text{ mm}$$

and Circular pitch $= \pi \times m = 3.142 \times 4 = 12.568$ mm

$$\therefore \quad \text{Contact ratio} = \frac{\text{Length of arc of contact}}{\text{Circular pitch}} = \frac{58.78}{12.568} = 4.67 \text{ say } 5 \quad \textbf{Ans.}$$

Example 12.13. *Two gear wheels mesh externally and are to give a velocity ratio of 3 to 1. The teeth are of involute form ; module = 6 mm, addendum = one module, pressure angle = 20°. The pinion rotates at 90 r.p.m. Determine : **1.** The number of teeth on the pinion to avoid interference on it and the corresponding number of teeth on the wheel, **2.** The length of path and arc of contact, **3.** The number of pairs of teeth in contact, and **4.** The maximum velocity of sliding.*

Solution. Given : $G = T / t = 3$; $m = 6$ mm ; $A_P = A_W = 1$ module $= 6$ mm ; $\phi = 20°$; $N_1 = 90$ r.p.m. or $\omega_1 = 2\pi \times 90 / 60 = 9.43$ rad/s

1. ***Number of teeth on the pinion to avoid interference on it and the corresponding number of teeth on the wheel***

We know that number of teeth on the pinion to avoid interference,

$$t = \frac{2A_P}{\sqrt{1 + G(G+2)\sin^2\phi} - 1} = \frac{2 \times 6}{\sqrt{1 + 3(3+2)\sin^2 20°} - 1}$$

$$= 18.2 \text{ say } 19 \quad \textbf{Ans.}$$

and corresponding number of teeth on the wheel,

$$T = G.t = 3 \times 19 = 57 \quad \textbf{Ans.}$$

2. ***Length of path and arc of contact***

We know that pitch circle radius of pinion,

$$r = m.t / 2 = 6 \times 19/2 = 57 \text{ mm}$$

$\therefore$ Radius of addendum circle of pinion,

$$r_A = r + \text{Addendum on pinion } (A_P) = 57 + 6 = 63 \text{ mm}$$

and pitch circle radius of wheel,

$$R = m.T / 2 = 6 \times 57 / 2 = 171 \text{ mm}$$

$\therefore$ Radius of addendum circle of wheel,

$$R_A = R + \text{Addendum on wheel } (A_W) = 171 + 6 = 177 \text{ mm}$$

We know that the path of approach (*i.e.* path of contact when engagement occurs),

$$KP = \sqrt{(R_A)^2 - R^2\cos^2\phi} - R\sin\phi$$ ...(Refer Fig. 12.11)

$$= \sqrt{(177)^2 - (171)^2\cos^2 20°} - 171\sin 20° = 74.2 - 58.5 = 15.7 \text{ mm}$$

and the path of recess (*i.e.* path of contact when disengagement occurs),

$$PL = \sqrt{(r_A)^2 - r^2 \cos^2 \phi} - r \sin \phi$$

$$= \sqrt{(63)^2 - (57)^2 \cos^2 20°} - 57 \sin 20° = 33.17 - 19.5 = 13.67 \text{ mm}$$

∴ Length of path of contact,

$$KL = KP + PL = 15.7 + 13.67 = 29.37 \text{ mm} \quad \textbf{Ans.}$$

We know that length of arc of contact

$$= \frac{\text{Length of path of contact}}{\cos \phi} = \frac{29.37}{\cos 20°} = 31.25 \text{ mm} \quad \textbf{Ans.}$$

3. ***Number of pairs of teeth in contact***

We know that circular pitch,

$$p_c = \pi \times m = \pi \times 6 = 18.852 \text{ mm}$$

∴ Number of pairs of teeth in contact

$$= \frac{\text{Length of arc of contact}}{p_c} = \frac{31.25}{18.852} = 1.66 \text{ say } 2 \quad \textbf{Ans.}$$

4. ***Maximum velocity of sliding***

Let ω_2 = Angular speed of wheel in rad/s.

We know that $\dfrac{\omega_1}{\omega_2} = \dfrac{T}{t}$ or $\omega_2 = \omega_1 \times \dfrac{t}{T} = 9.43 \times \dfrac{19}{57} = 3.14$ rad/s

∴ Maximum velocity of sliding,

$$v_S = (\omega_1 + \omega_2)\, KP \qquad \text{...}(\because KP > PL)$$

$$= (9.43 + 3.14)\, 15.7 = 197.35 \text{ mm/s} \quad \textbf{Ans.}$$

OBJECTIVE TYPE QUESTIONS

1. The two parallel and coplanar shafts are connected by gears having teeth parallel to the axis of the shaft. This arrangement is called

   (*a*) spur gearing (*b*) helical gearing (*c*) bevel gearing (*d*) spiral gearing

2. The type of gears used to connect two non-parallel non-intersecting shafts are

   (*a*) spur gears (*b*) helical gears (*c*) spiral gears (*d*) none of these

3. An imaginary circle which by pure rolling action, gives the same motion as the actual gear, is called

   (*a*) addendum circle (*b*) dedendum circle (*c*) pitch circle (*d*) clearance circle

4. The size of a gear is usually specified by

   (*a*) pressure angle (*b*) circular pitch (*c*) diametral pitch (*d*) pitch circle diameter

5. The radial distance of a tooth from the pitch circle to the bottom of the tooth, is called

   (*a*) dedendum (*b*) addendum (*c*) clearance (*d*) working depth

6. The product of the diametral pitch and circular pitch is equal to

   (*a*) 1 (*b*) $1/\pi$ (*c*) π (*d*) 2π

7. The module is the reciprocal of

   (*a*) diametral pitch (*b*) circular pitch (*c*) pitch diameter (*d*) none of these

8. Which is the incorrect relationship of gears?

   (*a*) Circular pitch × Diametral pitch = π (*b*) Module = P.C.D/No.of teeth

   (*c*) Dedendum = 1.157 module (*d*) Addendum = 2.157 module

9. If the module of a gear be m, the number of teeth T and pitch circle diameter D, then

   (*a*) $m = D/T$ (*b*) $D = T/m$ (*c*) $m = D/2T$ (*d*) none of these

10. Mitre gears are used for

    (*a*) great speed reduction (*b*) equal speed

    (*c*) minimum axial thrust (*d*) minimum backlash

11. The condition of correct gearing is

    (*a*) pitch line velocities of teeth be same

    (*b*) radius of curvature of two profiles be same

    (*c*) common normal to the pitch surface cuts the line of centres at a fixed point

    (*d*) none of the above

12. Law of gearing is satisfied if

    (*a*) two surfaces slide smoothly

    (*b*) common normal at the point of contact passes through the pitch point on the line joining the centres of rotation

    (*c*) number of teeth = P.C.D. / module

    (*d*) addendum is greater than dedendum

13. Involute profile is preferred to cyloidal because
(*a*) the profile is easy to cut
(*b*) only one curve is required to cut
(*c*) the rack has straight line profile and hence can be cut accurately
(*d*) none of the above

14. The contact ratio for gears is
(*a*) zero (*b*) less than one (*c*) greater than one

15. The maximum length of arc of contact for two mating gears, in order to avoid interference, is
(*a*) $(r + R) \sin \phi$ (*b*) $(r + R) \cos \phi$ (*c*) $(r + R) \tan \phi$ (*d*) none of these
where r = Pitch circle radius of pinion,
R = Pitch circle radius of driver, and
ϕ = Pressure angle.

16. When the addenda on pinion and wheel is such that the path of approach and path of recess are half of their maximum possible values, then the length of the path of contact is given by
(*a*) $\dfrac{(r + R) \sin\phi}{2}$ (*b*) $\dfrac{(r + R) \cos\phi}{2}$ (*c*) $\dfrac{(r + R) \tan\phi}{2}$ (*d*) none of these

17. Interference can be avoided in involute gears with 20° pressure angle by
(*a*) cutting involute correctly
(*b*) using as small number of teeth as possible
(*c*) using more than 20 teeth
(*d*) using more than 8 teeth

18. The ratio of face width to transverse pitch of a helical gear with α as the helix angle is normally
(*a*) more than $1.15/\tan \alpha$ (*b*) more than $1.05/\tan \alpha$
(*c*) more than $1/\tan \alpha$ (*d*) none of these

19. The maximum efficiency for spiral gears is
(*a*) $\dfrac{\sin (\theta + \phi) + 1}{\cos (\theta - \phi) + 1}$ (*b*) $\dfrac{\cos (\theta - \phi) + 1}{\sin (\theta + \phi) + 1}$
(*c*) $\dfrac{\cos (\theta + \phi) + 1}{\cos (\theta - \phi) + 1}$ (*d*) $\dfrac{\cos (\theta - \phi) + 1}{\cos (\theta + \phi) + 1}$
where θ = Shaft angle, and ϕ = Friction angle.

20. For a speed ratio of 100, smallest gear box is obtained by using
(*a*) a pair of spur gears
(*b*) a pair of helical and a pair of spur gear compounded
(*c*) a pair of bevel and a pair of spur gear compounded
(*d*) a pair of helical and a pair of worm gear compounded

ANSWERS

1. (*a*)	**2.** (*c*)	**3.** (*c*)	**4.** (*d*)	**5.** (*a*)
6. (*c*)	**7.** (*a*)	**8.** (*d*)	**9.** (*a*)	**10.** (*b*)
11. (*c*)	**12.** (*b*)	**13.** (*b*)	**14.** (*c*)	**15.** (*c*)
16. (*a*)	**17.** (*c*)	**18.** (*a*)	**19.** (*c*)	**20.** (*d*)

Gear Trains

Features

13.1. Introduction

Sometimes, two or more gears are made to mesh with each other to transmit power from one shaft to another. Such a combination is called ***gear train*** or ***train of toothed wheels***. The nature of the train used depends upon the velocity ratio required and the relative position of the axes of shafts. A gear train may consist of spur, bevel or spiral gears.

13.2. Types of Gear Trains

Following are the different types of gear trains, depending upon the arrangement of wheels :

1. Simple gear train, **2.** Compound gear train, **3.** Reverted gear train, and **4.** Epicyclic gear train.

In the first three types of gear trains, the axes of the shafts over which the gears are mounted are fixed relative to each other. But in case of epicyclic gear trains, the axes of the shafts on which the gears are mounted may move relative to a fixed axis.

13.3. Simple Gear Train

When there is only one gear on each shaft, as shown in Fig. 13.1, it is known as ***simple gear train***. The gears are represented by their pitch circles.

When the distance between the two shafts is small, the two gears 1 and 2 are made to mesh with each other to transmit motion from one shaft to the other, as shown in Fig. 13.1 (*a*). Since the gear 1 drives the gear 2, therefore gear 1 is called the ***driver*** and the gear 2 is called the ***driven*** or ***follower***. It may be noted that the motion of the driven gear is opposite to the motion of driving gear.

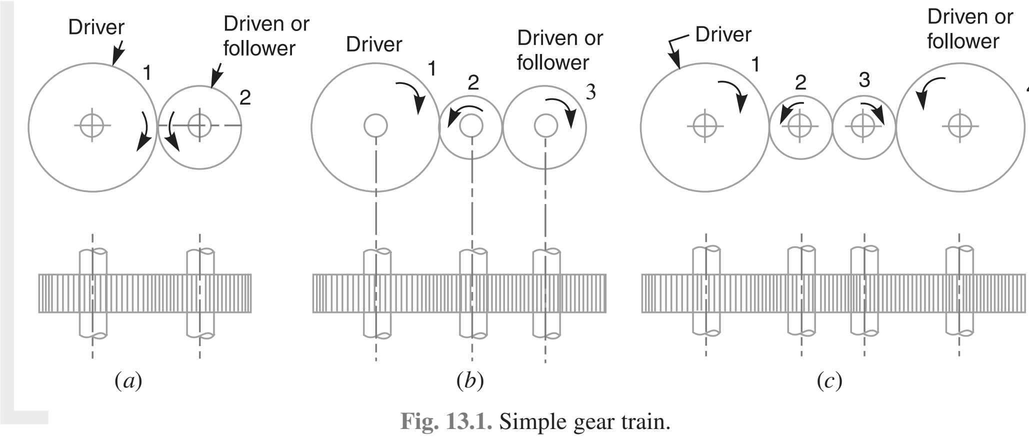


Fig. 13.1. Simple gear train.

Let N_1 = Speed of gear 1(or driver) in r.p.m.,

N_2 = Speed of gear 2 (or driven or follower) in r.p.m.,

T_1 = Number of teeth on gear 1, and

T_2 = Number of teeth on gear 2.

Since the speed ratio (or velocity ratio) of gear train is the ratio of the speed of the driver to the speed of the driven or follower and ratio of speeds of any pair of gears in mesh is the inverse of their number of teeth, therefore

$$\text{Speed ratio} = \frac{N_1}{N_2} = \frac{T_2}{T_1}$$

It may be noted that ratio of the speed of the driven or follower to the speed of the driver is known as ***train value*** of the gear train. Mathematically,

$$\text{Train value} = \frac{N_2}{N_1} = \frac{T_1}{T_2}$$

From above, we see that the train value is the reciprocal of speed ratio.

Sometimes, the distance between the two gears is large. The motion from one gear to another, in such a case, may be transmitted by either of the following two methods :

1. By providing the large sized gear, or **2.** By providing one or more intermediate gears.

A little consideration will show that the former method (*i.e.* providing large sized gears) is very inconvenient and uneconomical method ; whereas the latter method (*i.e.* providing one or more intermediate gear) is very convenient and economical.

It may be noted that when the number of intermediate gears are ***odd***, the motion of both the gears (*i.e.* driver and driven or follower) is ***like*** as shown in Fig. 13.1 (*b*).

But if the number of intermediate gears are ***even***, the motion of the driven or follower will be in the opposite direction of the driver as shown in Fig. 13.1 (*c*).

Now consider a simple train of gears with one intermediate gear as shown in Fig. 13.1 (*b*).

Let N_1 = Speed of driver in r.p.m.,

N_2 = Speed of intermediate gear in r.p.m.,

N_3 = Speed of driven or follower in r.p.m.,

T_1 = Number of teeth on driver,

T_2 = Number of teeth on intermediate gear, and

T_3 = Number of teeth on driven or follower.

Since the driving gear 1 is in mesh with the intermediate gear 2, therefore speed ratio for these two gears is

$$\frac{N_1}{N_2} = \frac{T_2}{T_1} \quad \text{...}(i)$$

Similarly, as the intermediate gear 2 is in mesh with the driven gear 3, therefore speed ratio for these two gears is

$$\frac{N_2}{N_3} = \frac{T_3}{T_2} \quad \text{...}(ii)$$

The speed ratio of the gear train as shown in Fig. 13.1 (*b*) is obtained by multiplying the equations (*i*) and (*ii*).

$$\therefore \quad \frac{N_1}{N_2} \cdot \frac{N_2}{N_3} = \frac{T_2}{T_1} \cdot \frac{T_3}{T_2} \quad \text{or} \quad \frac{N_1}{N_3} = \frac{T_3}{T_1}$$

i.e.

$$\text{Speed ratio} = \frac{\text{Speed of driver}}{\text{Speed of driven}} = \frac{\text{No. of teeth on driven}}{\text{No. of teeth on driver}}$$

and

$$\text{Train value} = \frac{\text{Speed of driven}}{\text{Speed of driver}} = \frac{\text{No. of teeth on driver}}{\text{No. of teeth on driven}}$$

Similarly, it can be proved that the above equation holds good even if there are any number of intermediate gears. From above, we see that the speed ratio and the train value, in a simple train of gears, is independent of the size and number of intermediate gears. These intermediate gears are called ***idle gears***, as they do not effect the speed ratio or train value of the system. The idle gears are used for the following two purposes :

1. To connect gears where a large centre distance is required, and

2. To obtain the desired direction of motion of the driven gear (*i.e.* clockwise or anticlockwise).

Gear trains inside a mechanical watch

13.4. Compound Gear Train

When there are more than one gear on a shaft, as shown in Fig. 13.2, it is called a ***compound train of gear***.

We have seen in Art. 13.3 that the idle gears, in a simple train of gears do not effect the speed ratio of the system. But these gears are useful in bridging over the space between the driver and the driven.

But whenever the distance between the driver and the driven or follower has to be bridged over by intermediate gears and at the same time a great (or much less) speed ratio is required, then the advantage of intermediate gears is intensified by providing compound gears on intermediate shafts. In this case, each intermediate shaft has two gears rigidly fixed to it so that they may have the same speed. One of these two gears meshes with the driver and the other with the driven or follower attached to the next shaft as shown in Fig.13.2.

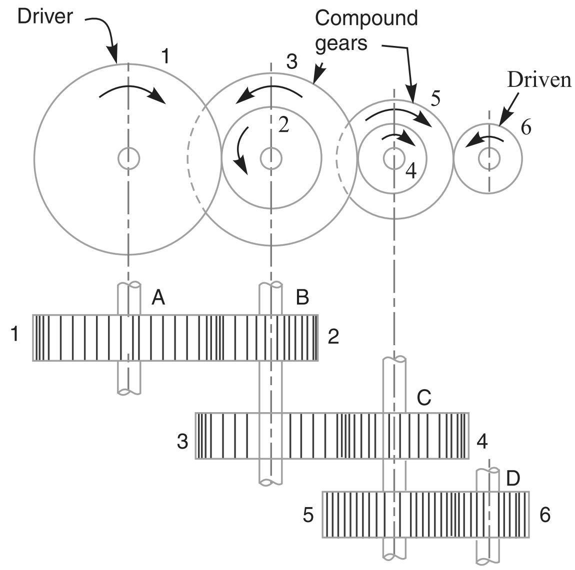


Fig. 13.2. Compound gear train.

In a compound train of gears, as shown in Fig. 13.2, the gear 1 is the driving gear mounted on shaft *A*, gears 2 and 3 are compound gears which are mounted on shaft *B*. The gears 4 and 5 are also compound gears which are mounted on shaft *C* and the gear 6 is the driven gear mounted on shaft *D*.

Let N_1 = Speed of driving gear 1,

T_1 = Number of teeth on driving gear 1,

$N_2, N_3 ..., N_6$ = Speed of respective gears in r.p.m., and

$T_2, T_3 ..., T_6$ = Number of teeth on respective gears.

Since gear 1 is in mesh with gear 2, therefore its speed ratio is

$$\frac{N_1}{N_2} = \frac{T_2}{T_1} \qquad ...(i)$$

Similarly, for gears 3 and 4, speed ratio is

$$\frac{N_3}{N_4} = \frac{T_4}{T_3} \qquad ...(ii)$$

and for gears 5 and 6, speed ratio is

$$\frac{N_5}{N_6} = \frac{T_6}{T_5} \qquad ...(iii)$$

The speed ratio of compound gear train is obtained by multiplying the equations (*i*), (*ii*) and (*iii*),

$$\therefore \qquad \frac{N_1}{N_2} \cdot \frac{N_3}{N_4} \cdot \frac{N_5}{N_6} = \frac{T_2}{T_1} \cdot \frac{T_4}{T_3} \cdot \frac{T_6}{T_5} \quad \text{or} \quad {}^*\frac{N_1}{N_6} = \frac{T_2 \cdot T_4 \cdot T_6}{T_1 \cdot T_3 \cdot T_5}$$

* Since gears 2 and 3 are mounted on one shaft *B*, therefore $N_2 = N_3$. Similarly gears 4 and 5 are mounted on shaft *C*, therefore $N_4 = N_5$.

i.e. $$\text{Speed ratio} = \frac{\text{Speed of the first driver}}{\text{Speed of the last driven or follower}}$$

$$= \frac{\text{Product of the number of teeth on the drivens}}{\text{Product of the number of teeth on the drivers}}$$

and $$\text{Train value} = \frac{\text{Speed of the last driven or follower}}{\text{Speed of the first driver}}$$

$$= \frac{\text{Product of the number of teeth on the drivers}}{\text{Product of the number of teeth on the drivens}}$$

The advantage of a compound train over a simple gear train is that a much larger speed reduction from the first shaft to the last shaft can be obtained with small gears. If a simple gear train is used to give a large speed reduction, the last gear has to be very large. Usually for a speed reduction in excess of 7 to 1, a simple train is not used and a compound train or worm gearing is employed.

Note: The gears which mesh must have the same circular pitch or module. Thus gears 1 and 2 must have the same module as they mesh together. Similarly gears 3 and 4, and gears 5 and 6 must have the same module.

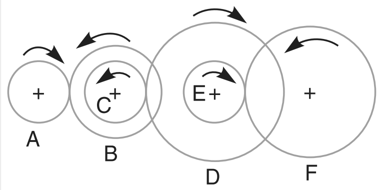


Fig. 13.3

Example 13.1. *The gearing of a machine tool is shown in Fig.* 13.3. *The motor shaft is connected to gear A and rotates at* 975 *r.p.m. The gear wheels B, C, D and E are fixed to parallel shafts rotating together. The final gear F is fixed on the output shaft. What is the speed of gear F ? The number of teeth on each gear are as given below* :

Gear	*A*	*B*	*C*	*D*	*E*	*F*
No. of teeth	*20*	*50*	*25*	*75*	*26*	*65*

Solution. Given : $N_A = 975$ r.p.m. ; $T_A = 20$; $T_B = 50$; $T_C = 25$; $T_D = 75$; $T_E = 26$; $T_F = 65$

From Fig. 13.3, we see that gears *A*, *C* and *E* are drivers while the gears *B*, *D* and *F* are driven or followers. Let the gear *A* rotates in clockwise direction. Since the gears *B* and *C* are mounted on the same shaft, therefore it is a compound gear and the direction or rotation of both these gears is same (*i.e.* anticlockwise). Similarly, the gears *D* and *E* are mounted on the same shaft, therefore it is also a compound gear and the direction of rotation of both these gears is same (*i.e.* clockwise). The gear *F* will rotate in anticlockwise direction.

Battery Car: Even though it is run by batteries, the power transmission, gears, clutches, brakes, etc. remain mechanical in nature.

Note : This picture is given as additional information and is not a direct example of the current chapter.

Let N_F = Speed of gear *F*, *i.e.* last driven or follower.

We know that

$$\frac{\text{Speed of the first driver}}{\text{Speed of the last driven}} = \frac{\text{Product of no. of teeth on drivens}}{\text{Product of no. of teeth on drivers}}$$

or $$\frac{N_A}{N_F} = \frac{T_B \cdot T_D \cdot T_F}{T_A \cdot T_C \cdot T_E} = \frac{50 \times 75 \times 65}{20 \times 25 \times 26} = 18.75$$

$\therefore$ $$N_F = \frac{N_A}{18.75} = \frac{975}{18.75} = 52 \text{ r. p. m.} \quad \textbf{Ans.}$$

13.5. Design of Spur Gears

Sometimes, the spur gears (*i.e*. driver and driven) are to be designed for the given velocity ratio and distance between the centres of their shafts.

Let x = Distance between the centres of two shafts,

N_1 = Speed of the driver,

T_1 = Number of teeth on the driver,

d_1 = Pitch circle diameter of the driver,

N_2, T_2 and d_2 = Corresponding values for the driven or follower, and

p_c = Circular pitch.

We know that the distance between the centres of two shafts,

$$x = \frac{d_1 + d_2}{2} \qquad \ldots(i)$$

and speed ratio or velocity ratio,

$$\frac{N_1}{N_2} = \frac{d_2}{d_1} = \frac{T_2}{T_1} \qquad \ldots(ii)$$

From the above equations, we can conveniently find out the values of d_1 and d_2 (or T_1 and T_2) and the circular pitch (p_c). The values of T_1 and T_2, as obtained above, may or may not be whole numbers. But in a gear since the number of its teeth is always a whole number, therefore a slight alterations must be made in the values of x, d_1 and d_2, so that the number of teeth in the two gears may be a complete number.

Example 13.2. *Two parallel shafts, about 600 mm apart are to be connected by spur gears. One shaft is to run at 360 r.p.m. and the other at 120 r.p.m. Design the gears, if the circular pitch is to be 25 mm.*

Solution. Given : x = 600 mm ; N_1 = 360 r.p.m. ; N_2 = 120 r.p.m. ; p_c = 25 mm

Let d_1 = Pitch circle diameter of the first gear, and

d_2 = Pitch circle diameter of the second gear.

We know that speed ratio,

$$\frac{N_1}{N_2} = \frac{d_2}{d_1} = \frac{360}{120} = 3 \quad \text{or} \quad d_2 = 3d_1 \qquad \ldots(i)$$

and centre distance between the shafts (x),

$$600 = \frac{1}{2}(d_1 + d_2) \quad \text{or} \quad d_1 + d_2 = 1200 \qquad \ldots(ii)$$

From equations *(i)* and *(ii)*, we find that

$$d_1 = 300 \text{ mm, and } d_2 = 900 \text{ mm}$$

$\therefore$ Number of teeth on the first gear,

$$T_1 = \frac{\pi d_2}{p_c} = \frac{\pi \times 300}{25} = 37.7$$

and number of teeth on the second gear,

$$T_2 = \frac{\pi d_2}{p_c} = \frac{\pi \cdot 900}{25} = 113.1$$

Since the number of teeth on both the gears are to be in complete numbers, therefore let us make the number of teeth on the first gear as 38. Therefore for a speed ratio of 3, the number of teeth on the second gear should be 38 × 3 = 114.

Now the exact pitch circle diameter of the first gear,

$$d_1' = \frac{T_1 \cdot p_c}{\pi} = \frac{38 \cdot 25}{\pi} = 302.36 \text{ mm}$$

and the exact pitch circle diameter of the second gear,

$$d_2' = \frac{T_2 \cdot p_c}{\pi} = \frac{114 \cdot 25}{\pi} = 907.1 \text{ mm}$$

∴ Exact distance between the two shafts,

$$x' = \frac{d_1' + d_2'}{2} = \frac{302.36 + 907.1}{2} = 604.73 \text{ mm}$$

Hence the number of teeth on the first and second gear must be 38 and 114 and their pitch circle diameters must be 302.36 mm and 907.1 mm respectively. The exact distance between the two shafts must be 604.73 mm. **Ans.**

13.6. Reverted Gear Train

When the axes of the first gear (*i.e.* first driver) and the last gear (*i.e.* last driven or follower) are co-axial, then the gear train is known as ***reverted gear train*** as shown in Fig. 13.4.

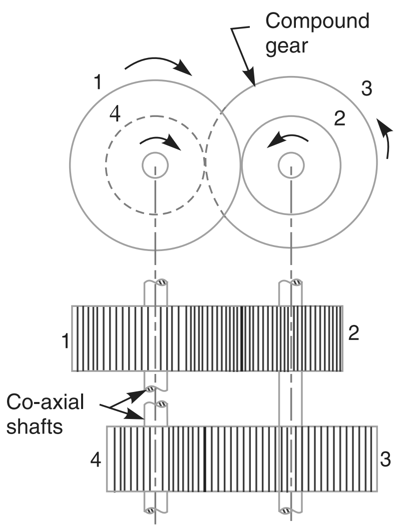


Fig. 13.4. Reverted gear train.

We see that gear 1 (*i.e.* first driver) drives the gear 2 (*i.e.* first driven or follower) in the opposite direction. Since the gears 2 and 3 are mounted on the same shaft, therefore they form a compound gear and the gear 3 will rotate in the same direction as that of gear 2. The gear 3 (which is now the second driver) drives the gear 4 (*i.e.* the last driven or follower) in the same direction as that of gear 1. Thus we see that in a reverted gear train, the motion of the first gear and the last gear is ***like***.

Let T_1 = Number of teeth on gear 1,

r_1 = Pitch circle radius of gear 1, and

N_1 = Speed of gear 1 in r.p.m.

Similarly,

T_2, T_3, T_4 = Number of teeth on respective gears,

r_2, r_3, r_4 = Pitch circle radii of respective gears, and

N_2, N_3, N_4 = Speed of respective gears in r.p.m.

Since the distance between the centres of the shafts of gears 1 and 2 as well as gears 3 and 4 is same, therefore

$$r_1 + r_2 = r_3 + r_4 \quad \text{...}(i)$$

Also, the circular pitch or module of all the gears is assumed to be same, therefore number of teeth on each gear is directly proportional to its circumference or radius.

$\therefore$ $$^*T_1 + T_2 = T_3 + T_4 \quad \text{...}(ii)$$

and $$\text{Speed ratio} = \frac{\text{Product of number of teeth on drivens}}{\text{Product of number of teeth on drivers}}$$

or $$\frac{N_1}{N_4} = \frac{T_2 \cdot T_4}{T_1 \cdot T_3} \quad \text{...}(iii)$$

From equations (*i*), (*ii*) and (*iii*), we can determine the number of teeth on each gear for the given centre distance, speed ratio and module only when the number of teeth on one gear is chosen arbitrarily.

The reverted gear trains are used in automotive transmissions, lathe back gears, industrial speed reducers, and in clocks (where the minute and hour hand shafts are co-axial).

Example 13.3. *The speed ratio of the reverted gear train, as shown in Fig. 13.5, is to be 12. The module pitch of gears A and B is 3.125 mm and of gears C and D is 2.5 mm. Calculate the suitable numbers of teeth for the gears. No gear is to have less than 24 teeth.*

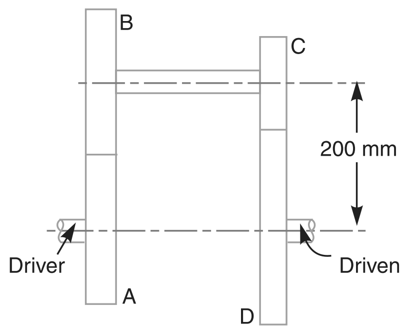


Fig. 13.5

Solution. Given : Speed ratio, $N_A/N_D = 12$; $m_A = m_B = 3.125$ mm ; $m_C = m_D = 2.5$ mm

Let N_A = Speed of gear A,

T_A = Number of teeth on gear A,

r_A = Pitch circle radius of gear A,

N_B, N_C, N_D = Speed of respective gears,

T_B, T_C, T_D = Number of teeth on respective gears, and

r_B, r_C, r_D = Pitch circle radii of respective gears.

---

* We know that circular pitch,

$$p_c = \frac{2\pi r}{T} = \pi m \quad \text{or} \quad r = \frac{m.T}{2}, \text{ where } m \text{ is the module.}$$

$\therefore$ $$r_1 = \frac{m.T_1}{2} \; ; \; r_2 = \frac{m.T_2}{2} \; ; \; r_3 = \frac{m.T_3}{2} \; ; \; r_4 = \frac{m.T_4}{2}$$

Now from equation (*i*),

$$\frac{m.T_1}{2} + \frac{m.T_2}{2} = \frac{m.T_3}{2} + \frac{m.T_4}{2}$$

$$T_1 + T_2 = T_3 + T_4$$

Since the speed ratio between the gears A and B and between the gears C and D are to be same, therefore

$$^{*}\frac{N_A}{N_B} = \frac{N_C}{N_D} = \sqrt{12} = 3.464$$

Also the speed ratio of any pair of gears in mesh is the inverse of their number of teeth, therefore

$$\frac{T_B}{T_A} = \frac{T_D}{T_C} = 3.464 \qquad \text{...}(i)$$

We know that the distance between the shafts

$$x = r_A + r_B = r_C + r_D = 200 \text{ mm}$$

or
$$\frac{m_A . T_A}{2} + \frac{m_B . T_B}{2} = \frac{m_C . T_C}{2} + \frac{m_D . T_D}{2} = 200 \qquad \text{...}\left(\because r = \frac{m.T}{2}\right)$$

$$3.125 (T_A + T_B) = 2.5 (T_C + T_D) = 400 \qquad \text{...}(\because m_A = m_B, \text{ and } m_C = m_D)$$

$$\therefore \quad T_A + T_B = 400 / 3.125 = 128 \qquad \text{...}(ii)$$

and
$$T_C + T_D = 400 / 2.5 = 160 \qquad \text{...}(iii)$$

From equation (*i*), $T_B = 3.464\ T_A$. Substituting this value of T_B in equation (*ii*),

$$T_A + 3.464\ T_A = 128 \quad \text{or} \quad T_A = 128 / 4.464 = 28.67 \text{ say } 28 \textbf{ Ans.}$$

and
$$T_B = 128 - 28 = 100 \textbf{ Ans.}$$

Again from equation (*i*), $T_D = 3.464\ T_C$. Substituting this value of T_D in equation (*iii*),

$$T_C + 3.464\ T_C = 160 \quad \text{or} \quad T_C = 160 / 4.464 = 35.84 \text{ say } 36 \textbf{ Ans.}$$

and
$$T_D = 160 - 36 = 124 \textbf{ Ans.}$$

Note : The speed ratio of the reverted gear train with the calculated values of number of teeth on each gear is

$$\frac{N_A}{N_D} = \frac{T_B \cdot T_D}{T_A \cdot T_C} = \frac{100 \cdot 124}{28 \cdot 36} = 12.3$$

13.7. Epicyclic Gear Train

We have already discussed that in an epicyclic gear train, the axes of the shafts, over which the gears are mounted, may move relative to a fixed axis. A simple epicyclic gear train is shown in Fig. 13.6, where a gear A and the arm C have a common axis at O_1 about which they can rotate. The gear B meshes with gear A and has its axis on the arm at O_2, about which the gear B can rotate. If the

---

* We know that speed ratio $= \dfrac{\text{Speed of first driver}}{\text{Speed of last driven}} = \dfrac{N_A}{N_D} = 12$

Also $\dfrac{N_A}{N_D} = \dfrac{N_A}{N_B} \cdot \dfrac{N_C}{N_D}$ ...($N_B = N_C$, being on the same shaft)

For $\dfrac{N_A}{N_B}$ and $\dfrac{N_C}{N_D}$ to be same, each speed ratio should be $\sqrt{12}$ so that

$$\frac{N_A}{N_D} = \frac{N_A}{N_B} \cdot \frac{N_C}{N_D} = \sqrt{12} \cdot \sqrt{12} = 12$$

arm is fixed, the gear train is simple and gear A can drive gear B or ***vice- versa***, but if gear A is fixed and the arm is rotated about the axis of gear A (*i.e.* O_1), then the gear B is forced to rotate ***upon*** and ***around*** gear A. Such a motion is called **epicyclic** and the gear trains arranged in such a manner that one or more of their members move upon and around another member are known as ***epicyclic gear trains*** (***epi.*** means upon and ***cyclic*** means around). The epicyclic gear trains may be ***simple*** or ***compound***.

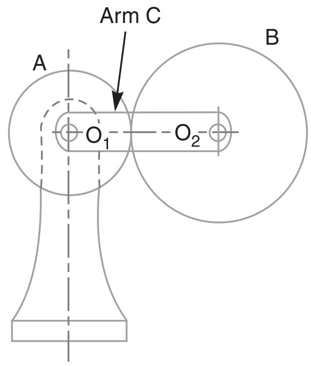


Fig. 13.6. Epicyclic gear train.

The epicyclic gear trains are useful for transmitting high velocity ratios with gears of moderate size in a comparatively lesser space. The epicyclic gear trains are used in the back gear of lathe, differential gears of the automobiles, hoists, pulley blocks, wrist watches etc.

13.8. Velocity Ratioz of Epicyclic Gear Train

The following two methods may be used for finding out the velocity ratio of an epicyclic gear train.

1. Tabular method, and **2.** Algebraic method.

These methods are discussed, in detail, as follows :

1. ***Tabular method***. Consider an epicyclic gear train as shown in Fig. 13.6.

Let T_A = Number of teeth on gear A, and

T_B = Number of teeth on gear B.

First of all, let us suppose that the arm is fixed. Therefore the axes of both the gears are also fixed relative to each other. When the gear A makes one revolution anticlockwise, the gear B will make *T_A / T_B revolutions, clockwise. Assuming the anticlockwise rotation as positive and clockwise as negative, we may say that when gear A makes + 1 revolution, then the gear B will make $(-T_A / T_B)$ revolutions. This statement of relative motion is entered in the first row of the table (see Table 13.1).


Inside view of a car engine.

Note : This picture is given as additional information and is not a direct example of the current chapter.

Secondly, if the gear A makes $+x$ revolutions, then the gear B will make $-x \times T_A / T_B$ revolutions. This statement is entered in the second row of the table. In other words, multiply the each motion (entered in the first row) by x.

Thirdly, each element of an epicyclic train is given $+y$ revolutions and entered in the third row. Finally, the motion of each element of the gear train is added up and entered in the fourth row.

* We know that $N_B / N_A = T_A / T_B$. Since N_A = 1 revolution, therefore $N_B = T_A / T_B$.

Table 13.1. Table of motions

Step No.	Conditions of motion	Revolutions of elements		
		Arm C	Gear A	Gear B
1.	Arm fixed-gear A rotates through + 1 revolution *i.e.* 1 rev. anticlockwise	0	+ 1	$-\dfrac{T_A}{T_B}$
2.	Arm fixed-gear A rotates through + x revolutions	0	$+x$	$-x \cdot \dfrac{T_A}{T_B}$
3.	Add + y revolutions to all elements	$+y$	$+y$	$+y$
4.	Total motion	$+y$	$x+y$	$y - x \cdot \dfrac{T_A}{T_B}$

A little consideration will show that when two conditions about the motion of rotation of any two elements are known, then the unknown speed of the third element may be obtained by substituting the given data in the third column of the fourth row.

2. *Algebraic method.* In this method, the motion of each element of the epicyclic train relative to the arm is set down in the form of equations. The number of equations depends upon the number of elements in the gear train. But the two conditions are, usually, supplied in any epicyclic train *viz.* some element is fixed and the other has specified motion. These two conditions are sufficient to solve all the equations ; and hence to determine the motion of any element in the epicyclic gear train.

Let the arm C be fixed in an epicyclic gear train as shown in Fig. 13.6. Therefore speed of the gear A relative to the arm C

$$= N_A - N_C$$

and speed of the gear B relative to the arm C,

$$= N_B - N_C$$

Since the gears A and B are meshing directly, therefore they will revolve in ***opposite*** directions.

$$\therefore \qquad \frac{N_B - N_C}{N_A - N_C} = -\frac{T_A}{T_B}$$

Since the arm C is fixed, therefore its speed, $N_C = 0$.

$$\therefore \qquad \frac{N_B}{N_A} = -\frac{T_A}{T_B}$$

If the gear A is fixed, then $N_A = 0$.

$$\frac{N_B - N_C}{0 - N_C} = -\frac{T_A}{T_B} \qquad \text{or} \qquad \frac{N_B}{N_C} = 1 + \frac{T_A}{T_B}$$

Note : The tabular method is easier and hence mostly used in solving problems on epicyclic gear train.

Example 13.4. *In an epicyclic gear train, an arm carries two gears A and B having 36 and 45 teeth respectively. If the arm rotates at 150 r.p.m. in the anticlockwise direction about the centre of the gear A which is fixed, determine the speed of gear B. If the gear A instead of being fixed, makes 300 r.p.m. in the clockwise direction, what will be the speed of gear B ?*

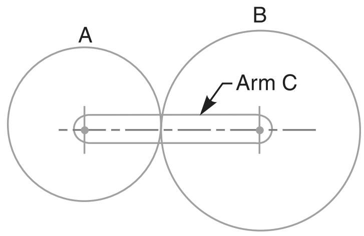


Fig. 13.7

Solution. Given : $T_A = 36$; $T_B = 45$; $N_C = 150$ r.p.m. (anticlockwise)

The gear train is shown in Fig. 13.7.

We shall solve this example, first by tabular method and then by algebraic method.

1. ***Tabular method***

First of all prepare the table of motions as given below :

Table 13.2. Table of motions.

Step No.	Conditions of motion	Revolutions of elements		
		Arm C	Gear A	Gear B
1.	Arm fixed-gear A rotates through + 1 revolution (*i.e.* 1 rev. anticlockwise)	0	+ 1	$-\frac{T_A}{T_B}$
2.	Arm fixed-gear A rotates through + x revolutions	0	$+x$	$-x \cdot \frac{T_A}{T_B}$
3.	Add + y revolutions to all elements	$+y$	$+y$	$+y$
4.	Total motion	$+y$	$x+y$	$y - x \cdot \frac{T_A}{T_B}$

Speed of gear B when gear A is fixed

Since the speed of arm is 150 r.p.m. anticlockwise, therefore from the fourth row of the table,

$$y = +150 \text{ r.p.m.}$$

Also the gear A is fixed, therefore

$$x + y = 0 \quad \text{or} \quad x = -y = -150 \text{ r.p.m.}$$

$\therefore$ Speed of gear B, $N_B = y - x \cdot \frac{T_A}{T_B} = 150 + 150 \cdot \frac{36}{45} = +270$ r.p.m.

$$= 270 \text{ r.p.m. (anticlockwise)} \quad \textbf{Ans.}$$

Speed of gear B when gear A makes 300 r.p.m. clockwise

Since the gear A makes 300 r.p.m.clockwise, therefore from the fourth row of the table,

$$x + y = -300 \quad \text{or} \quad x = -300 - y = -300 - 150 = -450 \text{ r.p.m.}$$

$\therefore$ Speed of gear B,

$$N_B = y - x \cdot \frac{T_A}{T_B} = 150 + 450 \cdot \frac{36}{45} = +510 \text{ r.p.m.}$$

$$= 510 \text{ r.p.m. (anticlockwise)} \quad \textbf{Ans.}$$

2. ***Algebraic method***

Let N_A = Speed of gear A.

N_B = Speed of gear B, and

N_C = Speed of arm C.

Assuming the arm C to be fixed, speed of gear A relative to arm C

$$= N_A - N_C$$

and speed of gear B relative to arm $C = N_B - N_C$

Since the gears A and B revolve in ***opposite*** directions, therefore

$$\frac{N_B - N_C}{N_A - N_C} = -\frac{T_A}{T_B} \qquad ...(i)$$

Speed of gear B when gear A is fixed

When gear A is fixed, the arm rotates at 150 r.p.m. in the anticlockwise direction, *i.e.*

$$N_A = 0, \quad \text{and} \quad N_C = +\,150 \text{ r.p.m.}$$

$$\therefore \qquad \frac{N_B - 150}{0 - 150} = -\frac{36}{45} = -\,0.8 \qquad ...[\text{From equation } (i)]$$

or $$N_B = -\,150 \times -\,0.8 + 150 = 120 + 150 = 270 \text{ r.p.m. } \textbf{Ans.}$$

Speed of gear B when gear A makes 300 r.p.m. clockwise

Since the gear A makes 300 r.p.m. clockwise, therefore

$$N_A = -\,300 \text{ r.p.m.}$$

$$\therefore \qquad \frac{N_B - 150}{-300 - 150} = -\frac{36}{45} = -\,0.8$$

or $$N_B = -\,450 \times -\,0.8 + 150 = 360 + 150 = 510 \text{ r.p.m. } \textbf{Ans.}$$

Example 13.5. *In a reverted epicyclic gear train, the arm A carries two gears B and C and a compound gear D - E. The gear B meshes with gear E and the gear C meshes with gear D. The number of teeth on gears B, C and D are 75, 30 and 90 respectively. Find the speed and direction of gear C when gear B is fixed and the arm A makes 100 r.p.m. clockwise.*

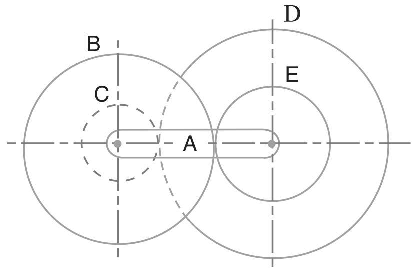


Fig. 13.8

Solution. Given : $T_B = 75$; $T_C = 30$; $T_D = 90$; $N_A = 100$ r.p.m. (clockwise)

The reverted epicyclic gear train is shown in Fig. 13.8. First of all, let us find the number of teeth on gear E (T_E). Let d_B, d_C, d_D and d_E be the pitch circle diameters of gears B, C, D and E respectively. From the geometry of the figure,

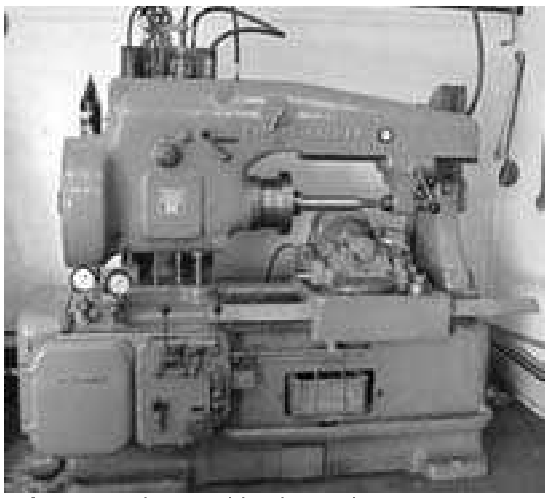
$$d_B + d_E = d_C + d_D$$

Since the number of teeth on each gear, for the same module, are proportional to their pitch circle diameters, therefore

$$T_B + T_E = T_C + T_D$$

$$\therefore \qquad T_E = T_C + T_D - T_B = 30 + 90 - 75 = 45$$

The table of motions is drawn as follows :



A gear-cutting machine is used to cut gears.

Note : This picture is given as additional information and is not a direct example of the current chapter.

Table 13.3. Table of motions.

Step No.	*Conditions of motion*	*Revolutions of elements*			
		Arm A	*Compound gear D-E*	*Gear B*	*Gear C*
1.	Arm fixed-compound gear *D-E* rotated through + 1 revolution (*i.e.* 1 rev. anticlockwise)	0	+ 1	$-\frac{T_E}{T_B}$	$-\frac{T_D}{T_C}$
2.	Arm fixed-compound gear *D-E* rotated through + *x* revolutions	0	+ *x*	$-x \cdot \frac{T_E}{T_B}$	$-x \cdot \frac{T_D}{T_C}$
3.	Add + *y* revolutions to all elements	+ *y*	+ *y*	+ *y*	+ *y*
4.	Total motion	+ *y*	*x* + *y*	$y - x \cdot \frac{T_E}{T_B}$	$y - x \cdot \frac{T_D}{T_C}$

Since the gear *B* is fixed, therefore from the fourth row of the table,

$$y - x \cdot \frac{T_E}{T_B} = 0 \quad \text{or} \quad y - x \cdot \frac{45}{75} = 0$$

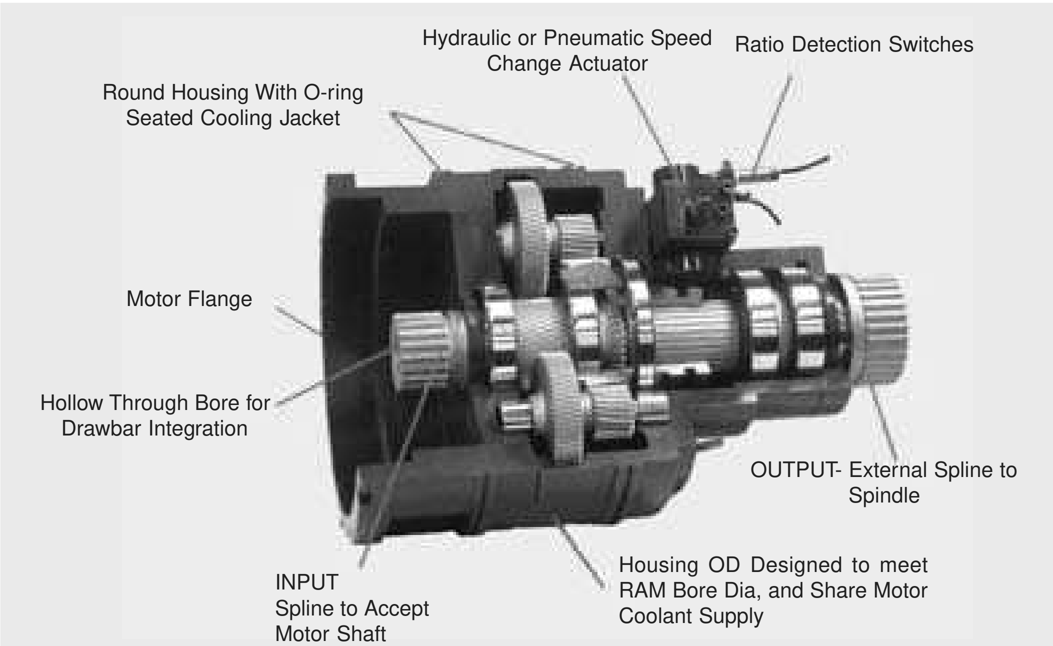
$$\therefore \qquad y - 0.6 = 0 \qquad ...(i)$$

Also the arm *A* makes 100 r.p.m. clockwise, therefore

$$y = -100 \qquad ...(ii)$$

Substituting $y = -100$ in equation (*i*), we get

$$-100 - 0.6\,x = 0 \quad \text{or} \quad x = -100 / 0.6 = -166.67$$



Model of sun and planet gears.

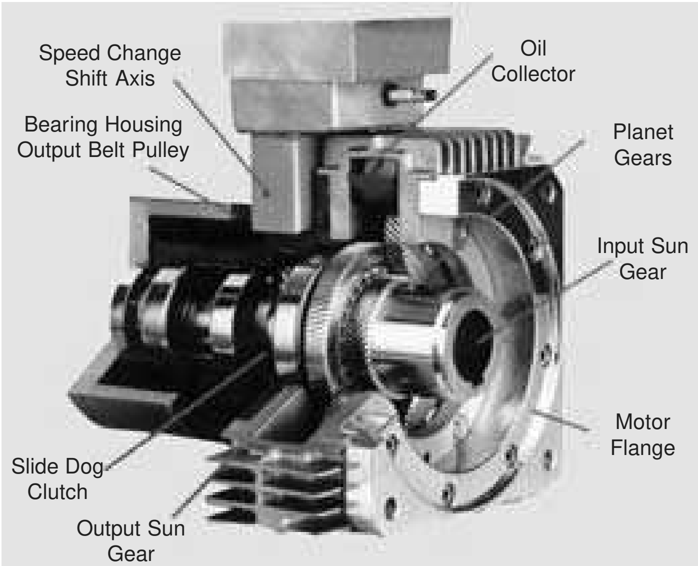
From the fourth row of the table, speed of gear C,

$$N_C = y - x \cdot \frac{T_D}{T_C} = -100 + 166.67 \cdot \frac{90}{30} = +400 \text{ r.p.m.}$$

$$= 400 \text{ r.p.m. (anticlockwise) } \textbf{Ans.}$$

13.9. Compound Epicyclic Gear Train—Sun and Planet Gear

A compound epicyclic gear train is shown in Fig. 13.9. It consists of two co-axial shafts S_1 and S_2, an annulus gear A which is fixed, the compound gear (or planet gear) B-C, the sun gear D and the arm H. The annulus gear has internal teeth and the compound gear is carried by the arm and revolves freely on a pin of the arm H. The sun gear is co-axial with the annulus gear and the arm but independent of them.



Sun and Planet gears.

The annulus gear A meshes with the gear B and the sun gear D meshes with the gear C. It may be noted that when the annulus gear is fixed, the sun gear provides the drive and when the sun gear is fixed, the annulus gear provides the drive. In both cases, the arm acts as a follower.

Note : The gear at the centre is called the ***sun gear*** and the gears whose axes move are called ***planet gears***.

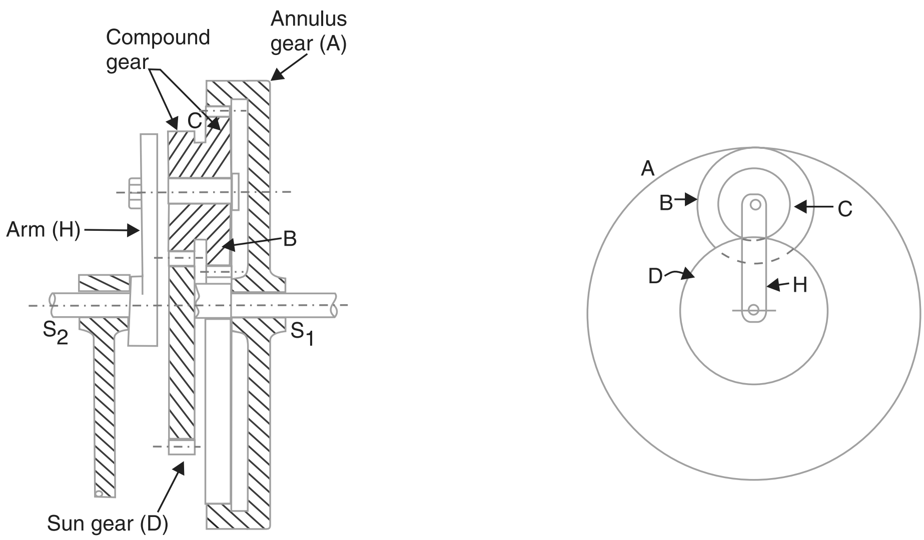


Fig. 13.9. Compound epicyclic gear train.

Let T_A, T_B, T_C, and T_D be the teeth and N_A, N_B, N_C and N_D be the speeds for the gears A, B, C and D respectively. A little consideration will show that when the arm is fixed and the sun gear D is turned anticlockwise, then the compound gear B-C and the annulus gear A will rotate in the clockwise direction.

The motion of rotations of the various elements are shown in the table below.

Table 13.4. Table of motions.

Step No.	Conditions of motion	*Revolutions of elements*			
		Arm	Gear D	Compound gear B-C	Gear A
1.	Arm fixed-gear D rotates through + 1 revolution	0	+ 1	$-\dfrac{T_D}{T_C}$	$-\dfrac{T_D}{T_C} \cdot \dfrac{T_B}{T_A}$
2.	Arm fixed-gear D rotates through + x revolutions	0	$+x$	$-x \cdot \dfrac{T_D}{T_C}$	$-x \cdot \dfrac{T_D}{T_C} \cdot \dfrac{T_B}{T_A}$
3.	Add + y revolutions to all elements	$+y$	$+y$	$+y$	$+y$
4.	Total motion	$+y$	$x+y$	$y - x \cdot \dfrac{T_D}{T_C}$	$y - x \cdot \dfrac{T_D}{T_C} \cdot \dfrac{T_B}{T_A}$

Note : If the annulus gear A is rotated through one revolution anticlockwise with the arm fixed, then the compound gear rotates through T_A / T_B revolutions in the same sense and the sun gear D rotates through $T_A / T_B \times T_C / T_D$ revolutions in clockwise direction.

Example 13.6. *An epicyclic gear consists of three gears A, B and C as shown in Fig. 13.10. The gear A has 72 internal teeth and gear C has 32 external teeth. The gear B meshes with both A and C and is carried on an arm EF which rotates about the centre of A at 18 r.p.m.. If the gear A is fixed, determine the speed of gears B and C.*

Solution. Given : $T_A = 72$; $T_C = 32$; Speed of arm EF = 18 r.p.m.

Considering the relative motion of rotation as shown in Table 13.5.

Table 13.5. Table of motions.

Step No.	Conditions of motion	*Revolutions of elements*			
		Arm EF	Gear C	Gear B	Gear A
1.	Arm fixed-gear C rotates through + 1 revolution (*i.e.* 1 rev. anticlockwise)	0	+ 1	$-\dfrac{T_C}{T_B}$	$-\dfrac{T_C}{T_B} \cdot \dfrac{T_B}{T_A} = -\dfrac{T_C}{T_A}$
2.	Arm fixed-gear C rotates through + x revolutions	0	$+x$	$-x \cdot \dfrac{T_C}{T_B}$	$-x \cdot \dfrac{T_C}{T_A}$
3.	Add + y revolutions to all elements	$+y$	$+y$	$+y$	$+y$
4.	Total motion	$+y$	$x+y$	$y - x \cdot \dfrac{T_C}{T_B}$	$y - x \cdot \dfrac{T_C}{T_A}$

Speed of gear C

We know that the speed of the arm is 18 r.p.m. therefore,

$$y = 18 \text{ r.p.m.}$$

and the gear A is fixed, therefore

$$y - x \cdot \frac{T_C}{T_A} = 0 \quad \text{or} \quad 18 - x \cdot \frac{32}{72} = 0$$

$\therefore$ $x = 18 \times 72 / 32 = 40.5$

$\therefore$ Speed of gear C $= x + y = 40.5 + 18$

$= + 58.5$ r.p.m.

$= 58.5$ r.p.m. in the direction of arm. **Ans.**

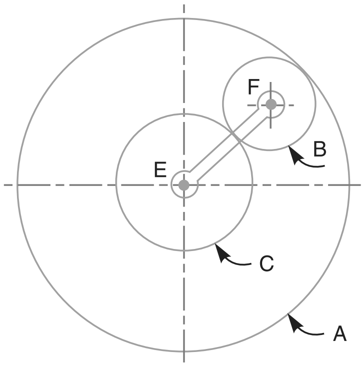


Fig. 13.10

Speed of gear B

Let d_A, d_B and d_C be the pitch circle diameters of gears A, B and C respectively. Therefore, from the geometry of Fig. 13.10,

$$d_B + \frac{d_C}{2} = \frac{d_A}{2} \quad \text{or} \quad 2\,d_B + d_C = d_A$$

Since the number of teeth are proportional to their pitch circle diameters, therefore

$$2\,T_B + T_C = T_A \quad \text{or} \quad 2\,T_B + 32 = 72 \quad \text{or} \quad T_B = 20$$

$\therefore$ Speed of gear B $= y - x \cdot \dfrac{T_C}{T_B} = 18 - 40.5 \cdot \dfrac{32}{20} = -46.8$ r.p.m.

$= 46.8$ r.p.m. in the opposite direction of arm. **Ans.**

Example 13.7. *An epicyclic train of gears is arranged as shown in Fig.13.11. How many revolutions does the arm, to which the pinions B and C are attached, make :*

1. when A makes one revolution clockwise and D makes half a revolution anticlockwise, and

2. when A makes one revolution clockwise and D is stationary ?

The number of teeth on the gears A and D are 40 and 90 respectively.

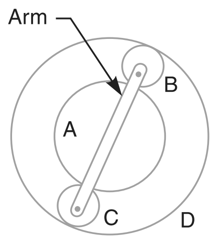


Fig. 13.11

Solution. Given : $T_A = 40$; $T_D = 90$

First of all, let us find the number of teeth on gears B and C (*i.e.* T_B and T_C). Let d_A, d_B, d_C and d_D be the pitch circle diameters of gears A, B, C and D respectively. Therefore from the geometry of the figure,

$$d_A + d_B + d_C = d_D \quad \text{or} \quad d_A + 2\,d_B = d_D \qquad ...(\because\ d_B = d_C)$$

Since the number of teeth are proportional to their pitch circle diameters, therefore,

$$T_A + 2\,T_B = T_D \quad \text{or} \quad 40 + 2\,T_B = 90$$

$$\therefore \qquad T_B = 25, \quad \text{and} \quad T_C = 25 \qquad ...(\because\ T_B = T_C)$$

The table of motions is given below :

Table 13.6. Table of motions.

Step No.	*Conditions of motion*	*Revolutions of elements*			
		Arm	*Gear A*	*Compound gear B-C*	*Gear D*
1.	Arm fixed, gear *A* rotates through – 1 revolution (*i.e.* 1 rev. clockwise)	0	– 1	$+\frac{T_A}{T_B}$	$+\frac{T_A}{T_B} \cdot \frac{T_B}{T_D} = +\frac{T_A}{T_D}$
2.	Arm fixed, gear *A* rotates through – *x* revolutions	0	$-x$	$+x \cdot \frac{T_A}{T_B}$	$+x \cdot \frac{T_A}{T_D}$
3.	Add – *y* revolutions to all elements	$-y$	$-y$	$-y$	$-y$
4.	Total motion	$-y$	$-x-y$	$x \cdot \frac{T_A}{T_B} - y$	$x \cdot \frac{T_A}{T_D} - y$

1. ***Speed of arm when A makes* 1 *revolution clockwise and D makes half revolution anticlockwise***

Since the gear *A* makes 1 revolution clockwise, therefore from the fourth row of the table,

$$-x-y=-1 \quad \text{or} \quad x+y=1 \qquad ...(i)$$

Also, the gear *D* makes half revolution anticlockwise, therefore

$$x \cdot \frac{T_A}{T_D} - y = \frac{1}{2} \quad \text{or} \quad x \cdot \frac{40}{90} - y = \frac{1}{2}$$

$$\therefore \quad 40\,x - 90\,y = 45 \quad \text{or} \quad x - 2.25\,y = 1.125 \qquad ...(ii)$$

From equations (*i*) and (*ii*), $x = 1.04$ and $y = -0.04$

$$\therefore \quad \text{Speed of arm} = -y = -(-0.04) = +0.04$$

= 0.04 revolution anticlockwise **Ans.**

2. ***Speed of arm when A makes* 1 *revolution clockwise and D is stationary***

Since the gear *A* makes 1 revolution clockwise, therefore from the fourth row of the table,

$$-x-y=-1 \quad \text{or} \quad x+y=1 \qquad ...(iii)$$

Also the gear *D* is stationary, therefore

$$x \cdot \frac{T_A}{T_D} - y = 0 \quad \text{or} \quad x \cdot \frac{40}{90} - y = 0$$

$$\therefore \quad 40\,x - 90\,y = 0 \quad \text{or} \quad x - 2.25\,y = 0 \qquad ...(iv)$$

From equations (*iii*) and (*iv*),

$$x = 0.692 \quad \text{and} \quad y = 0.308$$

$\therefore$ Speed of arm = $-y = -0.308 = 0.308$ revolution clockwise **Ans.**

Example 13.8. *In an epicyclic gear train, the internal wheels A and B and compound wheels C and D rotate independently about axis O. The wheels E and F rotate on pins fixed to the arm G. E gears with A and C and F gears with B and D. All the wheels have the same module and the number of teeth are : $T_C = 28$; $T_D = 26$; $T_E = T_F = 18$.*

1. Sketch the arrangement ; 2. Find the number of teeth on A and B ; 3. If the arm G makes 100 r.p.m. clockwise and A is fixed, find the speed of B ; and 4. If the arm G makes 100 r.p.m. clockwise and wheel A makes 10 r.p.m. counter clockwise ; find the speed of wheel B.

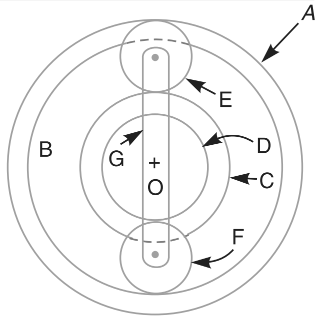


Fig. 13.12

Solution. Given : $T_C = 28$; $T_D = 26$; $T_E = T_F = 18$

1. *Sketch the arrangement*

The arrangement is shown in Fig. 13.12.

2. *Number of teeth on wheels A and B*

Let T_A = Number of teeth on wheel A, and

T_B = Number of teeth on wheel B.

If d_A, d_B, d_C, d_D, d_E and d_F are the pitch circle diameters of wheels A, B, C, D, E and F respectively, then from the geometry of Fig. 13.12,

$$d_A = d_C + 2\,d_E$$

and $$d_B = d_D + 2\,d_F$$

Since the number of teeth are proportional to their pitch circle diameters, for the same module, therefore

$$T_A = T_C + 2\,T_E = 28 + 2 \times 18 = 64 \quad \textbf{Ans.}$$

and $$T_B = T_D + 2\,T_F = 26 + 2 \times 18 = 62 \quad \textbf{Ans.}$$

3. *Speed of wheel B when arm G makes 100 r.p.m. clockwise and wheel A is fixed*

First of all, the table of motions is drawn as given below :

Table 13.7. Table of motions.

Step No.	*Conditions of motion*	*Revolutions of elements*					
		Arm G	*Wheel A*	*Wheel E*	*Compound wheel C-D*	*Wheel F*	*Wheel B*
1.	Arm fixed- wheel A rotates through + 1 revolution (*i.e.* 1 rev. anticlockwise)	0	+ 1	$+\frac{T_A}{T_E}$	$-\frac{T_A}{T_E} \cdot \frac{T_E}{T_C} = -\frac{T_A}{T_C}$	$+\frac{T_A}{T_C} \cdot \frac{T_D}{T_F}$	$+\frac{T_A}{T_C} \cdot \frac{T_D}{T_F} \cdot \frac{T_F}{T_B} = +\frac{T_A}{T_C} \cdot \frac{T_D}{T_B}$
2.	Arm fixed-wheel A rotates through + x revolutions	0	$+x$	$+x \cdot \frac{T_A}{T_E}$	$-x \cdot \frac{T_A}{T_C}$	$+x \cdot \frac{T_A}{T_C} \cdot \frac{T_D}{T_F}$	$+x \cdot \frac{T_A}{T_C} \cdot \frac{T_D}{T_B}$
3.	Add + y revolutions to all elements	$+y$	$+y$	$+y$	$+y$	$+y$	$+y$
4.	Total motion	$+y$	$x+y$	$y+x \cdot \frac{T_A}{T_E}$	$y-x \cdot \frac{T_A}{T_C}$	$y+x \cdot \frac{T_A}{T_C} \cdot \frac{T_D}{T_F}$	$y+x \cdot \frac{T_A}{T_C} \cdot \frac{T_D}{T_B}$

Since the arm G makes 100 r.p.m. clockwise, therefore from the fourth row of the table,

$$y = -100 \quad \ldots(i)$$

Also, the wheel A is fixed, therefore from the fourth row of the table,

$$x + y = 0 \quad \text{or} \quad x = -y = 100 \quad \ldots(ii)$$

$$\therefore \text{ Speed of wheel } B = y + x \cdot \frac{T_A}{T_C} \times \frac{T_D}{T_B} = -100 + 100 \times \frac{64}{28} \times \frac{26}{62} = -100 + 95.8 \text{ r.p.m.}$$

$$= -4.2 \text{ r.p.m.} = 4.2 \text{ r.p.m. clockwise} \quad \textbf{Ans.}$$

4. ***Speed of wheel B when arm G makes 100 r.p.m. clockwise and wheel A makes 10 r.p.m. counter clockwise***

Since the arm G makes 100 r.p.m. clockwise, therefore from the fourth row of the table

$$y = -100 \quad \ldots(iii)$$

Also the wheel A makes 10 r.p.m. counter clockwise, therefore from the fourth row of the table,

$$x + y = 10 \quad \text{or} \quad x = 10 - y = 10 + 100 = 110 \quad \ldots(iv)$$

$$\therefore \text{ Speed of wheel } B = y + x \cdot \frac{T_A}{T_C} \times \frac{T_D}{T_B} = -100 + 110 \times \frac{64}{28} \times \frac{26}{62} = -100 + 105.4 \text{ r.p.m.}$$

$$= +5.4 \text{ r.p.m.} = 5.4 \text{ r.p.m. counter clockwise} \quad \textbf{Ans.}$$

Example 13.9. *In an epicyclic gear of the 'sun and planet' type shown in Fig. 13.13, the pitch circle diameter of the internally toothed ring is to be 224 mm and the module 4 mm. When the ring D is stationary, the spider A, which carries three planet wheels C of equal size, is to make one revolution in the same sense as the sunwheel B for every five revolutions of the driving spindle carrying the sunwheel B. Determine suitable numbers of teeth for all the wheels.*

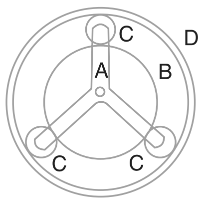


Fig. 13.13

Solution. Given : $d_D = 224$ mm ; $m = 4$ mm ; $N_A = N_B / 5$

Let T_B, T_C and T_D be the number of teeth on the sun wheel B, planet wheels C and the internally toothed ring D. The table of motions is given below :

Table 13.8. Table of motions.

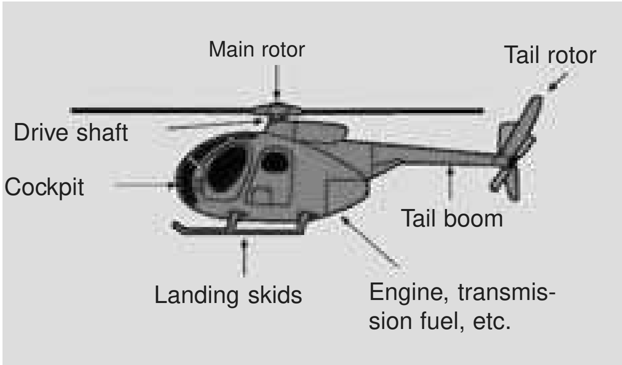
		Revolutions of elements			
Step No.	*Conditions of motion*	*Spider A*	*Sun wheel B*	*Planet wheel C*	*Internal gear D*
1.	Spider A fixed, sun wheel B rotates through + 1 revolution (*i.e.* 1 rev. anticlockwise)	0	+ 1	$-\frac{T_B}{T_C}$	$-\frac{T_B}{T_C} \times \frac{T_C}{T_D} = -\frac{T_B}{T_D}$
2.	Spider A fixed, sun wheel B rotates through + x revolutions	0	$+x$	$-x \cdot \frac{T_B}{T_C}$	$-x \cdot \frac{T_B}{T_D}$
3.	Add + y revolutions to all elements	$+y$	$+y$	$+y$	$+y$
4.	Total motion	$+y$	$x + y$	$y - x \cdot \frac{T_B}{T_C}$	$y - x \cdot \frac{T_B}{T_D}$

We know that when the sun wheel B makes + 5 revolutions, the spider A makes + 1 revolution. Therefore from the fourth row of the table,

$$y = +1 \; ; \text{ and } x + y = +5$$

$$\therefore \quad x = 5 - y = 5 - 1 = 4$$

Since the internally toothed ring D is stationary, therefore from the fourth row of the table,



Power transmission in a helicopter is essentially through gear trains.

Note : This picture is given as additional information and is not a direct example of the current chapter.

$$y - x \cdot \frac{T_B}{T_D} = 0$$

or $$1 - 4 \cdot \frac{T_B}{T_D} = 0$$

$$\therefore \quad \frac{T_B}{T_D} = \frac{1}{4} \quad \text{or} \quad T_D = 4\,T_B \qquad ...(i)$$

We know that $T_D = d_D / \text{m} = 224 / 4 = 56$ **Ans.**

$\therefore$ $T_B = T_D / 4 = 56 / 4 = 14$ **Ans.** ...[From equation (*i*)]

Let d_B, d_C and d_D be the pitch circle diameters of sun wheel B, planet wheels C and internally toothed ring D respectively. Assuming the pitch of all the gears to be same, therefore from the geometry of Fig. 13.13,

$$d_B + 2\,d_C = d_D$$

Since the number of teeth are proportional to their pitch circle diameters, therefore

$$T_B + 2\,T_C = T_D \quad \text{or} \quad 14 + 2\,T_C = 56$$

$$\therefore \quad T_C = 21 \text{ **Ans.**}$$

Example 13.10. *Two shafts A and B are co-axial. A gear C (50 teeth) is rigidly mounted on shaft A. A compound gear D-E gears with C and an internal gear G. D has 20 teeth and gears with C and E has 35 teeth and gears with an internal gear G. The gear G is fixed and is concentric with the shaft axis. The compound gear D-E is mounted on a pin which projects from an arm keyed to the shaft B. Sketch the arrangement and find the number of teeth on internal gear G assuming that all gears have the same module. If the shaft A rotates at 110 r.p.m., find the speed of shaft B.*

Solution. Given : $T_C = 50$; $T_D = 20$; $T_E = 35$; $N_A = 110$ r.p.m.

The arrangement is shown in Fig. 13.14.

Number of teeth on internal gear G

Let d_C, d_D, d_E and d_G be the pitch circle diameters of gears C, D, E and G respectively. From the geometry of the figure,

$$\frac{d_G}{2} = \frac{d_C}{2} + \frac{d_D}{2} + \frac{d_E}{2}$$

or $$d_G = d_C + d_D + d_E$$

Let T_C, T_D, T_E and T_G be the number of teeth on gears *C*, *D*, *E* and *G* respectively. Since all the gears have the same module, therefore number of teeth are proportional to their pitch circle diameters.

$$\therefore \qquad T_G = T_C + T_D + T_E = 50 + 20 + 35 = 105 \text{ Ans.}$$

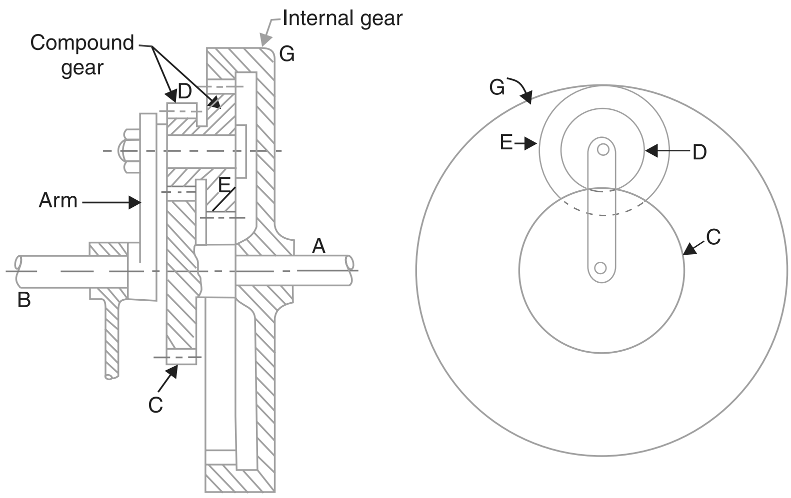


Fig. 13.14

Speed of shaft B

The table of motions is given below :

Table 13.9. Table of motions.

Step No.	*Conditions of motion*	*Revolutions of elements*			
		Arm	*Gear C (or shaft A)*	*Compound gear D-E*	*Gear G*
1.	Arm fixed - gear *C* rotates through + 1 revolution	0	+ 1	$-\frac{T_C}{T_D}$	$-\frac{T_C}{T_D} \cdot \frac{T_E}{T_G}$
2.	Arm fixed - gear *C* rotates through + *x* revolutions	0	+ *x*	$-x \cdot \frac{T_C}{T_D}$	$-x \cdot \frac{T_C}{T_D} \cdot \frac{T_E}{T_G}$
3.	Add + *y* revolutions to all elements	+ *y*	+ *y*	+ *y*	+ *y*
4.	Total motion	+ *y*	*x* + *y*	$y - x \cdot \frac{T_C}{T_D}$	$y - x \cdot \frac{T_C}{T_D} \cdot \frac{T_E}{T_G}$

Since the gear *G* is fixed, therefore from the fourth row of the table,

$$y - x \cdot \frac{T_C}{T_D} \cdot \frac{T_E}{T_G} = 0 \qquad \text{or} \qquad y - x \cdot \frac{50}{20} \cdot \frac{35}{105} = 0$$

$$\therefore \qquad y - \frac{5}{6}x = 0 \qquad \ldots(i)$$

Since the gear C is rigidly mounted on shaft A, therefore speed of gear C and shaft A is same. We know that speed of shaft A is 110 r.p.m., therefore from the fourth row of the table,

$$x + y = 100 \qquad \text{...}(ii)$$

From equations (*i*) and (*ii*), $x = 60$, and $y = 50$

$\therefore$ Speed of shaft B = Speed of arm = $+ y$ = 50 r.p.m. anticlockwise **Ans.**

Example 13.11. *Fig. 13.15 shows diagrammatically a compound epicyclic gear train. Wheels A, D and E are free to rotate independently on spindle O, while B and C are compound and rotate together on spindle P, on the end of arm OP. All the teeth on different wheels have the same module. A has 12 teeth, B has 30 teeth and C has 14 teeth cut externally. Find the number of teeth on wheels D and E which are cut internally.*

If the wheel A is driven clockwise at 1 r.p.s. while D is driven counter clockwise at 5 r.p.s., determine the magnitude and direction of the angular velocities of arm OP and wheel E.

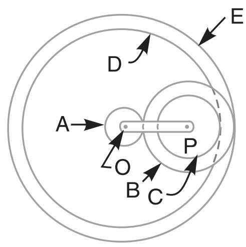


Fig. 13.15

Solution. Given : $T_A = 12$; $T_B = 30$; $T_C = 14$; $N_A = 1$ r.p.s. ; $N_D = 5$ r.p.s.

Number of teeth on wheels D and E

Let T_D and T_E be the number of teeth on wheels D and E respectively. Let d_A, d_B, d_C, d_D and d_E be the pitch circle diameters of wheels A, B, C, D and E respectively. From the geometry of the figure,

$$d_E = d_A + 2d_B \quad \text{and} \quad d_D = d_E - (d_B - d_C)$$

Since the number of teeth are proportional to their pitch circle diameters for the same module, therefore

$$T_E = T_A + 2T_B = 12 + 2 \times 30 = 72 \text{ **Ans.**}$$

and

$$T_D = T_E - (T_B - T_C) = 72 - (30 - 14) = 56 \text{ **Ans.**}$$

Magnitude and direction of angular velocities of arm OP and wheel E

The table of motions is drawn as follows :

Table 13.10. Table of motions.

Step No.	*Conditions of motion*	*Revolutions of elements*				
		Arm	*Wheel A*	*Compound wheel B-C*	*Wheel D*	*Wheel E*
1.	Arm fixed A rotated through – 1 revolution (*i.e.* 1 revolution clockwise)	0	-1	$+\frac{T_A}{T_B}$	$+\frac{T_A}{T_B} \cdot \frac{T_C}{T_D}$	$+\frac{T_A}{T_B} \cdot \frac{T_B}{T_E} = +\frac{T_A}{T_E}$
2.	Arm fixed-wheel A rotated through $-x$ revolutions	0	$-x$	$+x \cdot \frac{T_A}{T_B}$	$+x \cdot \frac{T_A}{T_B} \cdot \frac{T_C}{T_D}$	$+x \cdot \frac{T_A}{T_E}$
3.	Add $-y$ revolutions to all elements	$-y$	$-y$	$-y$	$-y$	$-y$
4.	Total motion	$-y$	$-x-y$	$x \cdot \frac{T_A}{T_B} - y$	$x \cdot \frac{T_A}{T_B} \cdot \frac{T_C}{T_D} - y$	$x \cdot \frac{T_A}{T_E} - y$

Since the wheel A makes 1 r.p.s. clockwise, therefore from the fourth row of the table,

$$-x - y = -1 \quad \text{or} \quad x + y = 1 \qquad ...(i)$$

Also, the wheel D makes 5 r.p.s. counter clockwise, therefore

$$x \cdot \frac{T_A}{T_B} \cdot \frac{T_C}{T_D} - y = 5 \quad \text{or} \quad x \cdot \frac{12}{30} \cdot \frac{14}{56} - y = 5$$

$$\therefore \qquad 0.1\, x - y = 5 \qquad ...(ii)$$

From equations (*i*) and (*ii*),

$$x = 5.45 \quad \text{and} \quad y = -4.45$$

$\therefore$ Angular velocity of arm OP

$$= -y = -(-4.45) = 4.45 \text{ r.p.s}$$

$$= 4.45 \times 2\,\pi = 27.964 \text{ rad/s (counter clockwise)} \textbf{ Ans.}$$

and angular velocity of wheel $E = x \cdot \dfrac{T_A}{T_E} - y = 5.45 \cdot \dfrac{12}{72} - (-4.45) = 5.36$ r.p.s.

$$= 5.36 \times 2\,\pi = 33.68 \text{ rad/s (counter clockwise)} \textbf{ Ans.}$$

Example 13.12. *An internal wheel B with 80 teeth is keyed to a shaft F. A fixed internal wheel C with 82 teeth is concentric with B. A compound wheel D-E gears with the two internal wheels; D has 28 teeth and gears with C while E gears with B. The compound wheels revolve freely on a pin which projects from a disc keyed to a shaft A co-axial with F. If the wheels have the same pitch and the shaft A makes 800 r.p.m., what is the speed of the shaft F ? Sketch the arrangement.*

Helicopter

Note : This picture is given as additional information and is not a direct example of the current chapter.

Solution. Given : $T_B = 80$; $T_C = 82$; $T_D = 28$; $N_A = 500$ r.p.m.

The arrangement is shown in Fig. 13.16.

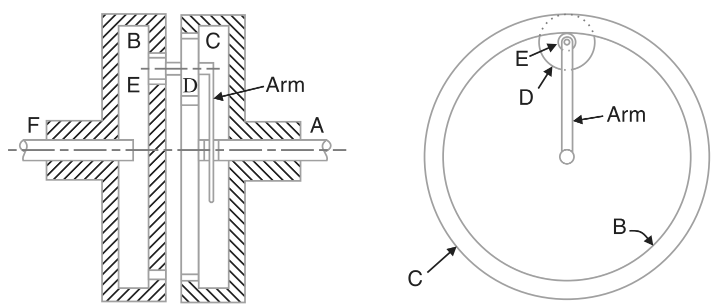


Fig. 13.16

First of all, let us find out the number of teeth on wheel E (T_E). Let d_B, d_C, d_D and d_E be the pitch circle diameter of wheels B, C, D and E respectively. From the geometry of the figure,

$$d_B = d_C - (d_D - d_E)$$

or $$d_E = d_B + d_D - d_C$$

Since the number of teeth are proportional to their pitch circle diameters for the same pitch, therefore

$$T_E = T_B + T_D - T_C = 80 + 28 - 82 = 26$$

The table of motions is given below :

Table 13.11. Table of motions.

Step No.	Conditions of motion	Revolutions of elements			
		Arm (or shaft A)	Wheel B (or shaft F)	Compound gear D-E	Wheel C
1.	Arm fixed - wheel *B* rotated through + 1 revolution (*i.e.* 1 revolution anticlockwise)	0	+ 1	$+\frac{T_B}{T_E}$	$+\frac{T_B}{T_E} \cdot \frac{T_D}{T_C}$
2.	Arm fixed - wheel *B* rotated through $+ x$ revolutions	0	$+ x$	$+ x \cdot \frac{T_B}{T_E}$	$+ x \cdot \frac{T_B}{T_E} \cdot \frac{T_D}{T_C}$
3.	Add $+ y$ revolutions to all elements	$+ y$	$+ y$	$+ y$	$+ y$
4.	Total motion	$+ y$	$x + y$	$y + x \cdot \frac{T_B}{T_E}$	$y + x \cdot \frac{T_B}{T_E} \cdot \frac{T_D}{T_C}$

Since the wheel *C* is fixed, therefore from the fourth row of the table,

$$y + x \cdot \frac{T_B}{T_E} \cdot \frac{T_D}{T_C} = 0 \qquad \text{or} \qquad y + x \cdot \frac{80}{26} \cdot \frac{28}{82} = 0$$

$$\therefore \qquad y + 1.05\, x = 0 \qquad \qquad ...(i)$$

Also, the shaft *A* (or the arm) makes 800 r.p.m., therefore from the fourth row of the table,

$$y = 800 \qquad \qquad ...(ii)$$

From equations (*i*) and (*ii*),

$$x = -762$$

$\therefore$ Speed of shaft F = Speed of wheel $B = x + y = -762 + 800 = +38$ r.p.m.

= 38 r.p.m. (anticlockwise) **Ans.**

DO YOU KNOW ?

1. What do you understand by 'gear train'? Discuss the various types of gear trains.
2. Explain briefly the differences between simple, compound, and epicyclic gear trains. What are the special advantages of epicyclic gear trains ?
3. Explain the procedure adopted for designing the spur wheels.
4. How the velocity ratio of epicyclic gear train is obtained by tabular method?
5. Explain with a neat sketch the 'sun and planet wheel'.
6. What are the various types of the torques in an epicyclic gear train ?

OBJECTIVE TYPE QUESTIONS

1. In a simple gear train, if the number of idle gears is odd, then the motion of driven gear will
   (*a*) be same as that of driving gear
   (*b*) be opposite as that of driving gear
   (*c*) depend upon the number of teeth on the driving gear
   (*d*) none of the above
2. The train value of a gear train is
   (*a*) equal to velocity ratio of a gear train (*b*) reciprocal of velocity ratio of a gear train
   (*c*) always greater than unity (*d*) always less than unity
3. When the axes of first and last gear are co-axial, then gear train is known as
   (*a*) simple gear train (*b*) compound gear train
   (*c*) reverted gear train (*d*) epicyclic gear train
4. In a clock mechanism, the gear train used to connect minute hand to hour hand, is
   (*a*) epicyclic gear train (*b*) reverted gear train
   (*c*) compound gear train (*d*) simple gear train
5. In a gear train, when the axes of the shafts, over which the gears are mounted, move relative to a fixed axis, is called
   (*a*) simple gear train (*b*) compound gear train
   (*c*) reverted gear train (*d*) epicyclic gear train
6. A differential gear in an automobile is a
   (*a*) simple gear train (*b*) epicyclic gear train
   (*c*) compound gear train (*d*) none of these
7. A differential gear in automobilies is used to
   (*a*) reduce speed (*b*) assist in changing speed
   (*c*) provide jerk-free movement of vehicle (*d*) help in turning

ANSWERS

1. (*a*) **2.** (*b*) **3.** (*c*) **4.** (*b*) **5.** (*d*)
6. (*b*) **7.** (*d*)

MODULE-III

Combined Static and Inertia Force Analysis: Inertia forces analysis, velocity and acceleration of slider crank mechanism by analytical method, engine force analysis -piston effort, force acting along the connecting rod, crank effort. dynamically equivalent system, compound pendulum, correction couple.

Inertia Forces in Reciprocating Parts

Features

1. *Introduction.*
2. *Resultant Effect of a System of Forces Acting on a Rigid Body.*
3. *D-Alembert's Principle.*
4. *Velocity and Acceleration of the Reciprocating Parts in Engines.*
5. *Klien's Construction.*
6. *Ritterhaus's Construction.*
7. *Bennett's Construction.*
8. *Approximate Analytical Method for Velocity and Acceleration of the Piston.*
9. *Angular Velocity and Acceleration of the Connecting Rod.*
10. *Forces on the Reciprocating Parts of an Engine Neglecting Weight of the Connecting Rod.*
11. *Equivalent Dynamical System.*
12. *Determination of Equivalent Dynamical System of Two Masses by Graphical Method.*
13. *Correction Couple to be Applied to Make the Two Mass Systems Dynamically Equivalent.*
14. *Inertia Forces in a Reciprocating Engine Considering the Weight of Connecting Rod.*
15. *Analytical Method for Inertia Torque.*

15.1. Introduction

The inertia force is an imaginary force, which when acts upon a rigid body, brings it in an equilibrium position. It is numerically equal to the accelerating force in magnitude, but ***opposite*** in direction. Mathematically,

$$\text{Inertia force} = -\text{Accelerating force} = -m.a$$

where m = Mass of the body, and

a = Linear acceleration of the centre of gravity of the body.

Similarly, the inertia torque is an imaginary torque, which when applied upon the rigid body, brings it in equilibrium position. It is equal to the accelerating couple in magnitude but ***opposite*** in direction.

15.2. Resultant Effect of a System of Forces Acting on a Rigid Body

Consider a rigid body acted upon by a system of forces. These forces may be reduced to a single resultant force

F whose line of action is at a distance *h* from the centre of gravity *G*. Now let us assume two equal and opposite forces (of magnitude *F*) acting through *G*, and parallel to the resultant force, without influencing the effect of the resultant force *F*, as shown in Fig. 15.1.

A little consideration will show that the body is now subjected to a couple (equal to $F \times h$) and a force, equal and parallel to the resultant force *F* passing through *G*. The force *F* through *G* causes linear acceleration of the c.g. and the moment of the couple ($F \times h$) causes angular acceleration of the body about an axis passing through *G* and perpendicular to the point in which the couple acts.

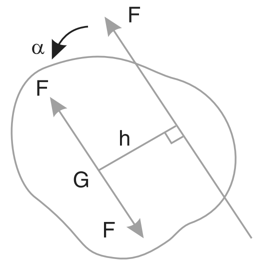


Fig. 15.1. Resultant effect of a system of forces acting on a rigid body.

Let α = Angular acceleration of the rigid body due to couple,

h = Perpendicular distance between the force and centre of gravity of the body,

m = Mass of the body,

k = Least radius of gyration about an axis through *G*, and

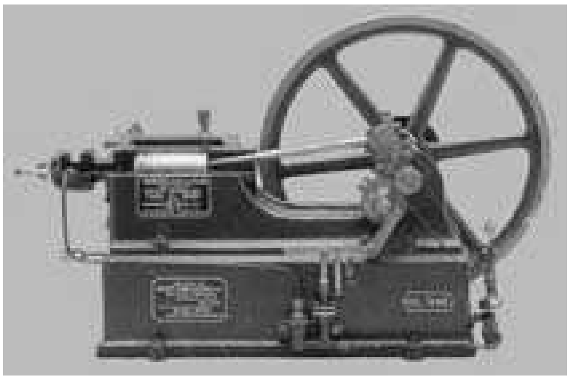
I = Moment of inertia of the body about an axis passing through its centre of gravity and perpendicular to the point in which the couple acts $= m.k^2$

We know that

Force, F = Mass × Acceleration = $m.a$ ...(*i*)

and $F.h = m.k^2.\alpha = I.\alpha$ ...($\because I = m.k^2$) ...(*ii*)

From equations (*i*) and (*ii*), we can find the values of *a* and α, if the values of *F*, *m*, *k*, and *h* are known.



The above picture shows the reciprocating parts of a 19th century oil engine.

15.3. D-Alembert's Principle

Consider a rigid body acted upon by a system of forces. The system may be reduced to a single resultant force acting on the body whose magnitude is given by the product of the mass of the body and the linear acceleration of the centre of mass of the body. According to Newton's second law of motion,

$$F = m.a \quad ...(i)$$

where F = Resultant force acting on the body,

m = Mass of the body, and

a = Linear acceleration of the centre of mass of the body.

The equation (*i*) may also be written as:

$$F - m.a = 0 \quad ...(ii)$$

A little consideration will show, that if the quantity $-m.a$ be treated as a force, equal, opposite and with the same line of action as the resultant force F, and include this force with the system of forces of which F is the resultant, then the complete system of forces will be in equilibrium. This principle is known as *D-Alembert's principle.* The equal and opposite force $-m.a$ is known as *reversed effective force* or the *inertia force* (briefly written as F_I). The equation (*ii*) may be written as

$$F + F_I = 0 \qquad ...(iii)$$

Thus, D-Alembert's principle states that *the resultant force acting on a body together with the reversed effective force (or inertia force), are in equilibrium.*

This principle is used to reduce a dynamic problem into an equivalent static problem.

15.8. Approximate Analytical Method for Velocity and Acceleration of the Piston

Consider the motion of a crank and connecting rod of a reciprocating steam engine as shown in Fig. 15.7. Let OC be the crank and PC the connecting rod. Let the crank rotates with angular velocity of ω rad/s and the crank turns through an angle θ from the inner dead centre (briefly written as I.D.C). Let x be the displacement of a reciprocating body P from I.D.C. after time t seconds, during which the crank has turned through an angle θ.

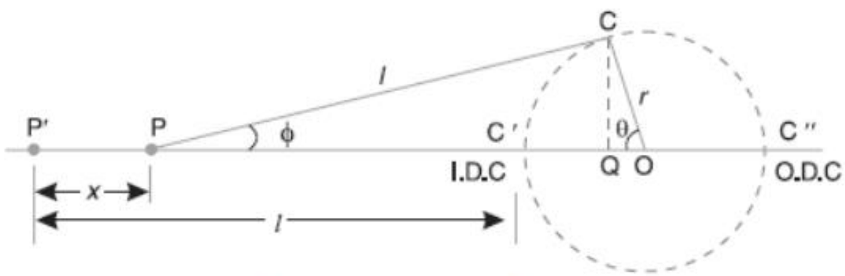


Fig. 15.7. Motion of a crank and connecting rod of a reciprocating steam engine.

Let l = Length of connecting rod between the centres,

r = Radius of crank or crank pin circle,

ϕ = Inclination of connecting rod to the line of stroke PO, and

n = Ratio of length of connecting rod to the radius of crank = l/r.

Velocity of the piston

From the geometry of Fig. 15.7,

$$x = P'P = OP' - OP = (P'C' + C'O) - (PQ + QO)$$

$$= (l + r) - (l \cos \phi + r \cos \theta) \qquad ...\begin{pmatrix} \because \; PQ = l \cos \phi, \\ \text{and } QO = r \cos \theta \end{pmatrix}$$

$$= r\,(1 - \cos\theta) + l\,(1 - \cos\phi) = r\left[(1 - \cos\theta) + \frac{l}{r}\,(1 - \cos\phi)\right]$$

$$= r\,[(1 - \cos\theta) + n\,(1 - \cos\phi)] \qquad ...(i)$$

From triangles CPQ and CQO,

$$CQ = l\sin\phi = r\sin\theta \text{ or } l/r = \sin\theta/\sin\phi$$

$$\therefore \qquad n = \sin\theta/\sin\phi \text{ or } \sin\phi = \sin\theta/n \qquad ...(ii)$$

We know that, $\cos\phi = \left(1 - \sin^2\phi\right)^{\frac{1}{2}} = \left(1 - \dfrac{\sin^2\theta}{n^2}\right)^{\frac{1}{2}}$

Expanding the above expression by binomial theorem, we get

$$\cos\phi = 1 - \frac{1}{2} \times \frac{\sin^2\theta}{n^2} + ... \qquad \text{...(Neglecting higher terms)}$$

or

$$1 - \cos\phi = \frac{\sin^2\theta}{2n^2} \qquad ...(iii)$$

Substituting the value of $(1 - \cos\phi)$ in equation *(i)*, we have

$$x = r\left[(1 - \cos\theta) + n \times \frac{\sin^2\theta}{2n^2}\right] = r\left[(1 - \cos\theta) + \frac{\sin^2\theta}{2n}\right] \qquad ..(iv)$$

Differentiating equation *(iv)* with respect to θ,

$$\frac{dx}{d\theta} = r\left[\sin\theta + \frac{1}{2n} \times 2\sin\theta.\cos\theta\right] = r\left(\sin\theta + \frac{\sin 2\theta}{2n}\right) \qquad ...(v)$$

$$(\because 2\sin\theta.\cos\theta = \sin 2\theta)$$

$\therefore$ Velocity of P with respect to O or velocity of the piston P,

$$v_{PO} = v_P = \frac{dx}{dt} = \frac{dx}{d\theta} \times \frac{d\theta}{dt} = \frac{dx}{d\theta} \times \omega$$

...($\because$ Ratio of change of angular velocity $= d\theta/dt = \omega$)

Substituting the value of $dx/d\theta$ from equation *(v)*, we have

$$v_{PO} = v_P = \omega r\left(\sin\theta + \frac{\sin 2\theta}{2n}\right) \qquad ...(vi)$$

Note: We know that by Klien's construction,

$$v_P = \omega \times OM$$

Comparing this equation with equation *(vi)*, we find that

$$OM = r\left(\sin\theta + \frac{\sin 2\theta}{2n}\right)$$

Acceleration of the piston

Since the acceleration is the rate of change of velocity, therefore acceleration of the piston P,

$$a_P = \frac{dv_P}{dt} = \frac{dv_P}{d\theta} \times \frac{d\theta}{dt} = \frac{dv_P}{d\theta} \times \omega$$

Differentiating equation (*vi*) with respect to θ,

$$\frac{dv_P}{d\theta} = \omega.r\left[\cos\theta + \frac{\cos 2\theta \times 2}{2n}\right] = \omega.r\left[\cos\theta + \frac{\cos 2\theta}{n}\right]$$

Substituting the value of $\frac{dv_P}{d\theta}$ in the above equation, we have

$$a_P = \omega.r\left[\cos\theta + \frac{\cos 2\theta}{n}\right] \times \omega = \omega^2.r\left[\cos\theta + \frac{\cos 2\theta}{n}\right] \quad \text{...(vii)}$$

Notes: **1.** When crank is at the inner dead centre (I.D.C.), then θ = 0°.

$$\therefore \quad a_P = \omega^2.r\left[\cos 0° + \frac{\cos 0°}{n}\right] = \omega^2.r\left(1 + \frac{1}{n}\right)$$

2. When the crank is at the outer dead centre (O.D.C.), then θ = 180°.

$$\therefore \quad a_P = \omega^2.r\left[\cos 180° + \frac{\cos 2 \times 180°}{n}\right] = \omega^2.r\left(-1 + \frac{1}{n}\right)$$

As the direction of motion is reversed at the outer dead centre therefore changing the sign of the above expression,

$$a_P = \omega^2.r\left[1 - \frac{1}{n}\right]$$

Above picture shows a diesel engine. Steam engine, petrol engine and diesel engine, all have reciprocating parts such as piston, piston rod, etc.

15.9. Angular Velocity and Acceleration of the Connecting Rod

Consider the motion of a connecting rod and a crank as shown in Fig. 15.7.From the geometry of the figure, we find that

$$CQ = l\sin\phi = r\sin\theta$$

$\therefore \qquad \sin\phi = \dfrac{r}{l} \times \sin\theta = \dfrac{\sin\theta}{n}$ $\qquad ...\left(\because n = \dfrac{l}{r}\right)$

Differentiating both sides with respect to time t,

$$\cos\phi \times \frac{d\phi}{dt} = \frac{\cos\theta}{n} \times \frac{d\theta}{dt} = \frac{\cos\theta}{n} \times \omega \qquad ...\left(\because \frac{d\theta}{dt} = \omega\right)$$

Since the angular velocity of the connecting rod PC is same as the angular velocity of point P with respect to C and is equal to $d\phi/dt$, therefore angular velocity of the connecting rod

$$\omega_{PC} = \frac{d\phi}{dt} = \frac{\cos\theta}{n} \times \frac{\omega}{\cos\phi} = \frac{\omega}{n} \times \frac{\cos\theta}{\cos\phi}$$

We know that, $\cos\phi = \left(1 - \sin^2\phi\right)^{\frac{1}{2}} = \left(1 - \dfrac{\sin^2\theta}{n^2}\right)^{\frac{1}{2}}$ $\qquad ...\left(\because \sin\phi = \dfrac{\sin\theta}{n}\right)$

$$\therefore \qquad \omega_{PC} = \frac{\omega}{n} \times \frac{\cos\theta}{\left(1 - \dfrac{\sin^2\theta}{n^2}\right)^{\frac{1}{2}}} = \frac{\omega}{n} \times \frac{\cos\theta}{\dfrac{1}{n}(n^2 - \sin^2\theta)^{1/2}}$$

$$= \frac{\omega\cos\theta}{(n^2 - \sin^2\theta)^{1/2}} \qquad ...(i)$$

Angular acceleration of the connecting rod PC,

$$\alpha_{PC} = \text{Angular acceleration of } P \text{ with respect to } C = \frac{d(\omega_{PC})}{dt}$$

We know that

$$\frac{d(\omega_{PC})}{dt} = \frac{d(\omega_{PC})}{d\theta} \times \frac{d\theta}{dt} = \frac{d(\omega_{PC})}{d\theta} \times \omega \qquad ...(ii)$$

$$...(\because d\theta/dt = \omega)$$

Now differentiating equation (*i*), we get

$$\frac{d(\omega_{PC})}{d\theta} = \frac{d}{d\theta}\left[\frac{\omega\cos\theta}{(n^2 - \sin^2\theta)^{1/2}}\right]$$

$$= \omega\left[\frac{(n^2 - \sin^2\theta)^{1/2}(-\sin\theta)] - [(\cos\theta) \times \frac{1}{2}(n^2 - \sin^2\theta)^{-1/2} \times -2\sin\theta\cos\theta}{n^2 - \sin^2\theta}\right]$$

$$= \omega\left[\frac{(n^2 - \sin^2\theta)^{1/2}(-\sin\theta) + (n^2 - \sin^2\theta)^{-1/2}\sin\theta\cos^2\theta}{n^2 - \sin^2\theta}\right]$$

$$= -\omega\sin\theta\left[\frac{(n^2 - \sin^2\theta)^{1/2} - (n^2 - \sin^2\theta)^{-1/2}\cos^2\theta}{n^2 - \sin^2\theta}\right]$$

$$= -\omega\sin\theta\left[\frac{(n^2 - \sin^2\theta) - \cos^2\theta}{(n^2 - \sin^2\theta)^{3/2}}\right] \qquad ...[\text{Dividing and multiplying by } (n^2 - \sin^2\theta)^{1/2}]$$

$$= \frac{-\omega \sin\theta}{(n^2 - \sin^2\theta)^{3/2}}\left[n^2 - (\sin^2\theta + \cos^2\theta)\right] = \frac{-\omega \sin\theta\,(n^2 - 1)}{(n^2 - \sin^2\theta)^{3/2}}$$

...($\because \sin^2\theta + \cos^2\theta = 1$)

$$\therefore\ \alpha_{PC} = \frac{d(\omega_{PC})}{d\theta} \times \omega = \frac{-\omega^2 \sin\theta\,(n^2 - 1)}{(n^2 - \sin^2\theta)^{3/2}} \quad \text{...[From equation } (ii)] \quad ...(iii)$$

The negative sign shows that the sense of the acceleration of the connecting rod is such that it tends to reduce the angle ϕ.

Notes: **1.** Since $\sin^2\theta$ is small as compared to n^2, therefore it may be neglected. Thus, equations (*i*) and (*iii*) are reduced to

$$\omega_{PC} = \frac{\omega\cos\theta}{n}, \quad \text{and} \quad \alpha_{PC} = \frac{-\omega^2 \sin\theta\,(n^2 - 1)}{n^3}$$

2. Also in equation (*iii*), unity is small as compared to n^2, hence the term unity may be neglected.

$$\therefore \qquad \alpha_{PC} = \frac{-\omega^2 \sin\theta}{n}$$

Example 15.3. *If the crank and the connecting rod are 300 mm and 1 m long respectively and the crank rotates at a constant speed of 200 r.p.m., determine:* **1.** *The crank angle at which the maximum velocity occurs, and* **2.** *Maximum velocity of the piston.*

Solution. Given : $r = 300$ mm = 0.3 m ; $l = 1$ m ; $N = 200$ r.p.m. or $\omega = 2\pi \times 200/60 = 20.95$ rad/s

1. ***Crank angle at which the maximum velocity occurs***

Let θ = Crank angle from the inner dead centre at which the maximum velocity occurs.

We know that ratio of length of connecting rod to crank radius,

$$n = l/r = 1/0.3 = 3.33$$

and velocity of the piston,

$$v_P = \omega.r\left(\sin\theta + \frac{\sin 2\theta}{2n}\right) \qquad ...(i)$$

For maximum velocity of the piston,

$$\frac{dv_P}{d\theta} = 0 \quad i.e. \quad \omega.r\left(\cos\theta + \frac{2\cos 2\theta}{2n}\right) = 0$$

or $n\cos\theta + 2\cos^2\theta - 1 = 0$ ...($\because \cos 2\theta = 2\cos^2\theta - 1$)

$$2\cos^2\theta + 3.33\cos\theta - 1 = 0$$

$$\therefore \qquad \cos\theta = \frac{-3.33 \pm \sqrt{(3.33)^2 + 4 \times 2 \times 1}}{2 \times 2} = 0.26 \qquad \text{...(Taking + ve sign)}$$

or $\theta = 75°$ **Ans.**

2. ***Maximum velocity of the piston***

Substituting the value of $\theta = 75°$ in equation (*i*), maximum velocity of the piston,

$$v_{P(max)} = \omega.r\left[\sin 75° + \frac{\sin 150°}{2n}\right] = 20.95 \times 0.3\left[0.966 + \frac{0.5}{3.33}\right] \text{ m/s}$$

$$= 6.54 \text{ m/s} \quad \textbf{Ans.}$$

Example 15.4. *The crank and connecting rod of a steam engine are 0.3 m and 1.5 m in length. The crank rotates at 180 r.p.m. clockwise. Determine the velocity and acceleration of the piston when the crank is at 40 degrees from the inner dead centre position. Also determine the position of the crank for zero acceleration of the piston.*

Solution. Given : $r = 0.3$; $l = 1.5$ m ; $N = 180$ r.p.m. or $\omega = \pi \times 180/60 = 18.85$ rad/s; $\theta = 40°$

Velocity of the piston

We know that ratio of lengths of the connecting rod and crank,

$$n = l/r = 1.5/0.3 = 5$$

∴ Velocity of the piston,

$$v_P = \omega.r\left(\sin\theta + \frac{\sin 2\theta}{2n}\right) = 18.85 \times 0.3\left(\sin 40° + \frac{\sin 80°}{2 \times 5}\right) \text{m/s}$$

$$= 4.19 \text{ m/s} \quad \textbf{Ans.}$$

Acceleration of the piston

We know that acceleration of piston,

$$a_P = \omega^2.r\left(\cos\theta + \frac{\cos 2\theta}{n}\right) = (18.85)^2 \times 0.3\left(\cos 40° + \frac{\cos 80°}{5}\right) \text{m/s}^2$$

$$= 85.35 \text{ m/s}^2 \quad \textbf{Ans.}$$

Position of the crank for zero acceleration of the piston

Let θ_1 = Position of the crank from the inner dead centre for zero acceleration of the piston.

We know that acceleration of piston,

$$a_P = \omega^2.r\left(\cos\theta_1 + \frac{\cos 2\theta_1}{n}\right)$$

or

$$0 = \frac{\omega^2.r}{n}(n\cos\theta_1 + \cos 2\theta_1) \qquad ...(\because a_P = 0)$$

$$\therefore \quad n\cos\theta_1 + \cos 2\theta_1 = 0$$

$$5\cos\theta_1 + 2\cos^2\theta_1 - 1 = 0 \quad \text{or} \quad 2\cos^2\theta_1 + 5\cos\theta_1 - 1 = 0$$

$$\therefore \quad \cos\theta_1 = \frac{-5 \pm \sqrt{5^2 + 4 \times 1 \times 2}}{2 \times 2} = 0.1862 \qquad ...(\text{Taking + ve sign})$$

or

$$\theta_1 = 79.27° \text{ or } 280.73° \quad \textbf{Ans.}$$

Example 15.5. *In a slider crank mechanism, the length of the crank and connecting rod are 150 mm and 600 mm respectively. The crank position is 60° from inner dead centre. The crank shaft speed is 450 r.p.m. (clockwise). Using analytical method, determine: 1. Velocity and acceleration of the slider, and 2. Angular velocity and angular acceleration of the connecting rod.*

Solution. Given : r = 150 mm = 0.15 m ; l = 600 mm = 0.6 m ; $\theta = 60°$; N = 400 r.p.m or $\omega = \pi \times 450/60 = 47.13$ rad/s

1. ***Velocity and acceleration of the slider***

We know that ratio of the length of connecting rod and crank,

$$n = l / r = 0.6 / 0.15 = 4$$

∴ Velocity of the slider,

$$v_P = \omega.r\left(\sin\theta + \frac{\sin 2\theta}{2n}\right) = 47.13 \times 0.15\left(\sin 60° + \frac{\sin 120°}{2\times 4}\right) \text{m/s}$$

$$= 6.9 \text{ m/s} \quad \textbf{Ans.}$$

and acceleration of the slider,

$$a_P = \omega^2.r\left(\cos\theta + \frac{\cos 2\theta}{n}\right) = (47.13)^2 \times 0.15\left(\cos 60° + \frac{\cos 120°}{4}\right) \text{m/s}^2$$

$$= 124.94 \text{ m/s}^2 \quad \textbf{Ans.}$$

2. ***Angular velocity and angular acceleration of the connecting rod***

We know that angular velocity of the connecting rod,

$$\omega_{PC} = \frac{\omega\cos\theta}{n} = \frac{47.13\times\cos 60°}{4} = 5.9 \text{ rad/s} \quad \textbf{Ans.}$$

and angular acceleration of the connecting rod,

$$\alpha_{PC} = \frac{\omega^2 \sin\theta}{n} = \frac{(47.13)^2 \times \sin 60°}{4} = 481 \text{ rad/s}^2 \quad \textbf{Ans.}$$

15.10. Forces on the Reciprocating Parts of an Engine, Neglecting the Weight of the Connecting Rod

The various forces acting on the reciprocating parts of a horizontal engine are shown in Fig. 15.8. The expressions for these forces, neglecting the weight of the connecting rod, may be derived as discussed below :

1. ***Piston effort***. It is the net force acting on the piston or crosshead pin, along the line of stroke. It is denoted by F_P in Fig. 15.8.

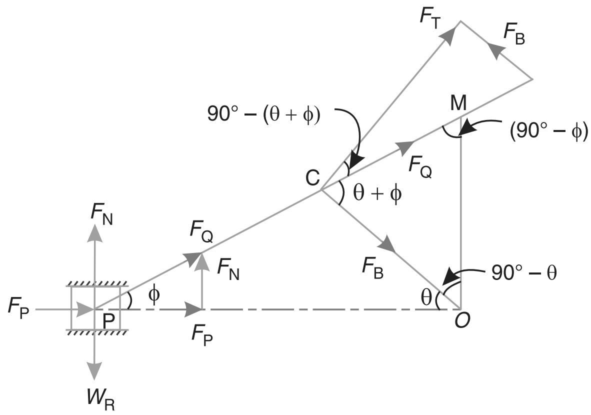


Fig. 15.8. Forces on the reciprocating parts of an engine.

Let m_R = Mass of the reciprocating parts, *e.g.* piston, crosshead pin or gudgeon pin etc., in kg, and

W_R = Weight of the reciprocating parts in newtons = $m_R.g$

We know that acceleration of the reciprocating parts,

$$a_R = a_P = \omega^2.r\left(\cos\theta + \frac{\cos 2\theta}{n}\right)$$

∴ *Accelerating force or inertia force of the reciprocating parts,

$$F_I = m_R \cdot a_R = m_R \cdot \omega^2 \cdot r \left(\cos \theta + \frac{\cos 2\theta}{n} \right)$$

It may be noted that in a horizontal engine, the reciprocating parts are accelerated from rest, during the latter half of the stroke (*i.e.* when the piston moves from inner dead centre to outer dead centre). It is, then, retarded during the latter half of the stroke (*i.e.* when the piston moves from outer dead centre to inner dead centre). The inertia force due to the acceleration of the reciprocating parts, opposes the force on the piston due to the difference of pressures in the cylinder on the two sides of the piston. On the other hand, the inertia force due to retardation of the reciprocating parts, helps the force on the piston. Therefore,

Connecting rod of a petrol engine.

Piston effort, F_P = Net load on the piston $\mp$ Inertia force

$= F_L \mp F_I$ ...(Neglecting frictional resistance)

$= F_L \mp F_I - R_F$ ...(Considering frictional resistance)

where R_F = Frictional resistance.

The –ve sign is used when the piston is accelerated, and +ve sign is used when the piston is retarded.

In a double acting reciprocating steam engine, net load on the piston,

$$F_L = p_1 A_1 - p_2 A_2 = p_1 A_1 - p_2 (A_1 - a)$$

where p_1, A_1 = Pressure and cross-sectional area on the back end side of the piston,

p_2, A_2 = Pressure and cross-sectional area on the crank end side of the piston,

a = Cross-sectional area of the piston rod.

Notes : **1.** If 'p' is the net pressure of steam or gas on the piston and D is diameter of the piston, then

Net load on the piston, F_L = Pressure × Area $= p \times \frac{\pi}{4} \times D^2$

2. In case of a vertical engine, the weight of the reciprocating parts assists the piston effort during the downward stroke (*i.e.* when the piston moves from top dead centre to bottom dead centre) and opposes during the upward stroke of the piston (*i.e.* when the piston moves from bottom dead centre to top dead centre).

∴ Piston effort, $F_P = F_L \mp F_I \pm W_R - R_F$

2. ***Force acting along the connecting rod.*** It is denoted by F_Q in Fig. 15.8. From the geometry of the figure, we find that

$$F_Q = \frac{F_P}{\cos \phi}$$

---

* The acceleration of the reciprocating parts by Klien's construction is,

$$a_P = \omega^2 \times NO$$

∴ $F_I = m_R \cdot \omega^2 \times NO$

We know that $\cos\phi = \sqrt{1 - \dfrac{\sin^2\theta}{n^2}}$

$$\therefore \quad F_Q = \frac{F_P}{\sqrt{1 - \dfrac{\sin^2\theta}{n^2}}}$$

3. ***Thrust on the sides of the cylinder walls or normal reaction on the guide bars.*** It is denoted by F_N in Fig. 15.8. From the figure, we find that

$$F_N = F_Q \sin\phi = \frac{F_P}{\cos\phi} \times \sin\phi = F_P \tan\phi \qquad \ldots\left[\because F_Q = \frac{F_P}{\cos\phi}\right]$$

4. ***Crank-pin effort and thrust on crank shaft bearings*.** The force acting on the connecting rod F_Q may be resolved into two components, one perpendicular to the crank and the other along the crank. The component of F_Q perpendicular to the crank is known as ***crank-pin effort*** and it is ***denoted by F_T*** in Fig. 15.8. The component of F_Q along the crank produces a thrust on the crank shaft bearings and it is denoted by F_B in Fig. 15.8.

Resolving F_Q perpendicular to the crank,

$$F_T = F_Q \sin(\theta + \phi) = \frac{F_P}{\cos\phi} \times \sin(\theta + \phi)$$

and resolving F_Q along the crank,

$$F_B = F_Q \cos(\theta + \phi) = \frac{F_P}{\cos\phi} \times \cos(\theta + \phi)$$

5. ***Crank effort*** or ***turning moment or torque on the crank shaft.*** The product of the crank-pin effort (F_T) and the crank pin radius (r) is known as ***crank effort*** or ***turning moment*** or ***torque on the crank shaft***. Mathematically,

$$\text{Crank effort,} \quad T = F_T \times r = \frac{F_P \sin(\theta + \phi)}{\cos\phi} \times r$$

$$= \frac{F_P (\sin\theta\cos\phi + \cos\theta\sin\phi)}{\cos\phi} \times r$$

$$= F_P\left(\sin\theta + \cos\theta \times \frac{\sin\phi}{\cos\phi}\right) \times r$$

$$= F_P (\sin\theta + \cos\theta\tan\phi) \times r \qquad \ldots(i)$$

We know that $l \sin\phi = r \sin\theta$

$$\sin\phi = \frac{r}{l}\sin\theta = \frac{\sin\theta}{n} \qquad \ldots\left(\because n = \frac{l}{r}\right)$$

and

$$\cos\phi = \sqrt{1 - \sin^2\phi} = \sqrt{1 - \frac{\sin^2\theta}{n^2}} = \frac{1}{n}\sqrt{n^2 - \sin^2\theta}$$

$$\therefore \quad \tan\phi = \frac{\sin\phi}{\cos\phi} = \frac{\sin\theta}{n} \times \frac{n}{\sqrt{n^2 - \sin^2\theta}} = \frac{\sin\theta}{\sqrt{n^2 - \sin^2\theta}}$$

Substituting the value of tan ϕ in equation *(i)*, we have crank effort,

$$T = F_P\left(\sin\theta + \frac{\cos\theta\sin\theta}{\sqrt{n^2 - \sin^2\theta}}\right)\times r$$

$$= F_P \times r\left(\sin\theta + \frac{\sin 2\theta}{2\sqrt{n^2 - \sin^2\theta}}\right) \quad \text{...}(ii)$$

...($\because$ 2 cos θ sin θ = sin 2θ)

Note: Since $\sin^2\theta$ is very small as compared to n^2 therefore neglecting $\sin^2\theta$, we have,

Crank effort, $$T = F_P \times r\left(\sin\theta + \frac{\sin 2\theta}{2n}\right) = F_P \times OM$$

We have seen in Art. 15.8, that

$$OM = r\left(\sin\theta + \frac{\sin 2\theta}{2n}\right)$$

Therefore, it is convenient to find OM instead of solving the large expression.

Example 15.6. *Find the inertia force for the following data of an I.C. engine.*

Bore = 175 mm, stroke = 200 mm, engine speed = 500 r.p.m., length of connecting rod = 400 mm, crank angle = 60° from T.D.C and mass of reciprocating parts = 180 kg.

Solution. Given : *D =175 mm ; L = 200 mm = 0.2 m or $r = L/2$ = 0.1 m ; N = 500 r.p.m. or $\omega = 2\pi \times 500/60$ =52.4 rad/s ; l = 400 mm = 0.4 m ; m_R = 180 kg

The inertia force may be calculated by graphical method or analytical method as discussed below:

1. ***Graphical method***

First of all, draw the Klien's acceleration diagram $OCQN$ to some suitable scale as shown in Fig. 15.9. By measurement,

$$ON = 38 \text{ mm} = 0.038 \text{ m}$$

$\therefore$ Acceleration of the reciprocating parts,

$$a_R = \omega^2 \times ON$$
$$= (52.4)^2 \times 0.038 = 104.34 \text{ m/s}$$

We know that inertia force,

$$F_I = m_R \times a_R = 180 \times 104.34 \text{ N}$$
$$= 18\ 780 \text{ N} = 18.78 \text{ kN} \quad \textbf{Ans.}$$

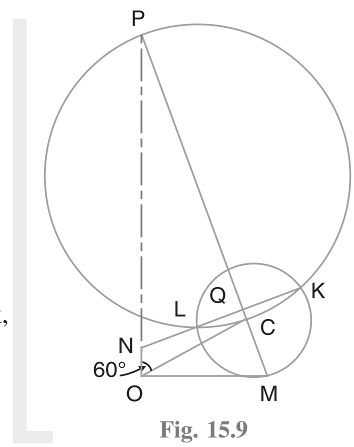


Fig. 15.9

2. ***Analytical method***

We know that ratio of lengths of connecting rod and crank,

$$n = l/r = 0.4/0.1 = 4$$

$\therefore$ Inertia force, $$F_I = m_R.\omega^2.r\left(\cos\theta + \frac{\cos 2\theta}{n}\right)$$

$$= 180 \times (52.4)^2 \times 0.1\left(\cos 60° + \frac{\cos 120°}{4}\right) = 18\ 530 \text{ N}$$

$$= 18.53 \text{ kN} \quad \textbf{Ans.}$$

* Superfluous data.

Example 15.7. *The crank-pin circle radius of a horizontal engine is 300 mm. The mass of the reciprocating parts is 250 kg. When the crank has travelled 60° from I.D.C., the difference between the driving and the back pressures is 0.35 N/mm². The connecting rod length between centres is 1.2 m and the cylinder bore is 0.5 m. If the engine runs at 250 r.p.m. and if the effect of piston rod diameter is neglected, calculate :* **1.** *pressure on slide bars,* **2.** *thrust in the connecting rod,* **3.** *tangential force on the crank-pin, and* **4.** *turning moment on the crank shaft.*

Solution. Given: $r = 300$ mm $= 0.3$ m ; $m_R = 250$ kg; $\theta = 60°$; $p_1 - p_2 = 0.35$ N/mm²; $l = 1.2$ m ; $D = 0.5$ m $= 500$ mm ; $N = 250$ r.p.m. or $\omega = 2\pi \times 250/60 = 26.2$ rad/s

First of all, let us find out the piston effort (F_P).

We know that net load on the piston,

$$F_L = (p_1 - p_2)\frac{\pi}{4} \times D^2 = 0.35 \times \frac{\pi}{4}(500)^2 = 68\,730 \text{ N}$$

...($\because$ Force = Pressure × Area)

Ratio of length of connecting rod and crank,

$$n = l / r = 1.2 / 0.3 = 4$$

and accelerating or inertia force on reciprocating parts,

$$F_I = m_R \,.\, \omega^2 r \left(\cos\theta + \frac{\cos 2\theta}{n}\right)$$

$$= 250\,(26.2)^2\; 0.3 \left(\cos 60° + \frac{\cos 120°}{4}\right) = 19\,306 \text{ N}$$

$\therefore$ Piston effort, $\quad F_P = F_L - F_I = 68\,730 - 19\,306 = 49\,424 \text{ N} = 49.424 \text{ kN}$

1. ***Pressure on slide bars***

Let $\quad \phi$ = Angle of inclination of the connecting rod to the line of stroke.

We know that, $\quad \sin\phi = \dfrac{\sin\theta}{n} = \dfrac{\sin 60°}{4} = \dfrac{0.866}{4} = 0.2165$

$\therefore \quad \phi = 12.5°$

We know that pressure on the slide bars,

$$F_N = F_P \tan\phi = 49.424 \times \tan 12.5° = 10.96 \text{ kN} \quad \textbf{Ans.}$$

2. ***Thrust in the connecting rod***

We know that thrust in the connecting rod,

$$F_Q = \frac{F_P}{\cos\phi} = \frac{49.424}{\cos 12.5°} = 50.62 \text{ kN} \;\textbf{Ans.}$$

3. ***Tangential force on the crank-pin***

We know that tangential force on the crank pin,

$$F_T = F_Q \sin(\theta + \phi) = 50.62 \sin(60° + 12.5°) = 48.28 \text{ kN} \;\textbf{Ans.}$$

4. ***Turning moment on the crank shaft***

We know that turning moment on the crank shaft,

$$T = F_T \times r = 48.28 \times 0.3 = 14.484 \text{ kN-m} \;\textbf{Ans.}$$

Example 15.8. *A vertical double acting steam engine has a cylinder 300 mm diameter and 450 mm stroke and runs at 200 r.p.m. The reciprocating parts has a mass of 225 kg and the piston rod is 50 mm diameter. The connecting rod is 1.2 m long. When the crank has turned through 125° from the top dead centre, the steam pressure above the piston is 30 kN/m² and below the piston is 1.5 kN/m². Calculate the effective turning moment on the crank shaft.*

Solution. Given : D = 300 mm = 0.3 m ; L = 450 mm or $r = L/2$ = 225 mm = 0.225 m ; N = 200 r.p.m. or $\omega = 2\pi \times 200/60 = 20.95$ rad/s ; m_R = 225 kg ; d = 50 mm = 0.05 m ; l = 1.2 m ; $\theta = 125°$; $p_1 = 30$ kN/m² $= 30 \times 10^3$ N/m² ; $p_2 = 1.5$ kN/m² $= 1.5 \times 10^3$ N/m²

We know that area of the piston,

$$A_1 = \frac{\pi}{4} \times D^2 = \frac{\pi}{4} \times (0.3)^2 = 0.0707 \text{ m}^2$$

and area of the piston rod,
$$a = \frac{\pi}{4} \times d^2 = \frac{\pi}{4} \times (0.05)^2 = 0.001\,96 \text{ m}^2$$

∴ Force on the piston due to steam pressure,

$$F_L = p_1.A_1 - p_2 (A_1 - a)$$
$$= 30 \times 10^3 \times 0.0707 - 1.5 \times 10^3 (0.0707 - 0.001\,96) \text{ N}$$
$$= 2121 - 103 = 2018 \text{ N}$$

Ratio of lengths of connecting rod and crank,

$$n = l / r = 1.2 / 0.225 = 5.33$$

and inertia force on the reciprocating parts,

$$F_I = m_R.\omega^2.r\left(\cos\theta + \frac{\cos 2\theta}{n}\right)$$
$$= 225\,(20.95)^2 \times 0.225\left(\cos 125° + \frac{\cos 250°}{5.33}\right) = -\,14\,172 \text{ N}$$

We know that for a vertical engine, net force on the piston or piston effort,

$$F_P = F_L - F_I + m_R.g$$
$$= 2018 - (-\,14\,172) + 225 \times 9.81 = 18\,397 \text{ N}$$

Let ϕ = Angle of inclination of the connecting rod to the line of stroke.

We know that,
$$\sin\phi = \frac{\sin\theta}{n} = \frac{\sin 125°}{5.33} = \frac{0.8191}{5.33} = 0.1537$$

∴ $\phi = 8.84°$

We know that effective turning moment on the crank shaft,

$$T = \frac{F_P \times \sin(\theta + \phi)}{\cos\phi} \times r = \frac{18\,397\,\sin(125° + 8.84°)}{\cos 8.84°} \times 0.225 \text{ N-m}$$
$$= 3021.6 \text{ N-m } \textbf{Ans.}$$

Example 15.9. *The crank and connecting rod of a petrol engine, running at 1800 r.p.m.are 50 mm and 200 mm respectively. The diameter of the piston is 80 mm and the mass of the reciprocating parts is 1 kg. At a point during the power stroke, the pressure on the piston is 0.7 N/mm², when it has moved 10 mm from the inner dead centre. Determine :* **1.** *Net load on the gudgeon pin,* **2.** *Thrust in the connecting rod,* **3.** *Reaction between the piston and cylinder, and* **4.** *The engine speed at which the above values become zero.*

Solution. Given : N = 1800 r.p.m. or $\omega = 2\pi \times 1800/60 = 188.52$ rad/s ; r = 50 mm = 0.05 m; l = 200 mm ; D = 80 mm ; m_R = 1 kg ; p = 0.7 N/mm² ; x = 10 mm

1. ***Net load on the gudgeon pin***

We know that load on the piston,

$$F_L = \frac{\pi}{4} D^2 \times p = \frac{\pi}{4} \times (80)^2 \times 0.7 = 3520 \text{ N}$$

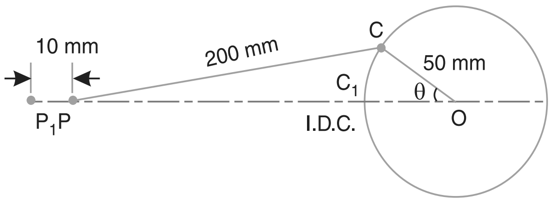


Fig. 15.10

When the piston has moved 10 mm from the inner dead centre, *i.e.* when P_1P = 10 mm, the crank rotates from OC_1 to OC through an angle θ as shown in Fig. 15.10.

By measurement, we find that *θ = 33°.

We know that ratio of lengths of connecting rod and crank,

$$n = l/r = 200/50 = 4$$

and inertia force on the reciprocating parts,

$$F_I = m_R . a_R = m_R . \omega^2 . r \left(\cos\theta + \frac{\cos 2\theta}{n} \right)$$

$$= 1 \times (188.52)^2 \times 0.05 \left(\cos 33° + \frac{\cos 66°}{4} \right) = 1671 \text{ N}$$

We know that net load on the gudgeon pin,

$$F_P = F_L - F_I = 3520 - 1671 = 1849 \text{ N} \quad \textbf{Ans.}$$

2. ***Thrust in the connecting rod***

Let ϕ = Angle of inclination of the connecting rod to the line of stroke.

We know that, $$\sin\phi = \frac{\sin\theta}{n} = \frac{\sin 33°}{4} = \frac{0.5446}{4} = 0.1361$$

∴ $\phi = 7.82°$

---

* The angle θ may also be obtained as follows:

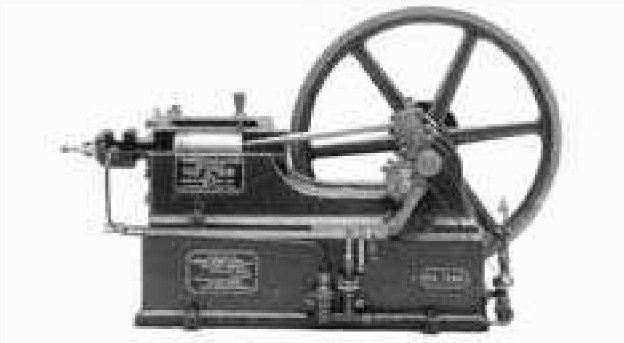
We know that $$x = r\left[(1-\cos\theta) + \frac{\sin^2\theta}{2n}\right] = r\left[(1-\cos\theta) + \frac{1-\cos^2\theta}{2n}\right]$$

$$10 = 50\left[(1-\cos\theta) + \frac{1-\cos^2\theta}{2\times 4}\right] = \frac{50}{8}\left[(8 - 8\cos\theta + 1 - \cos^2\theta\right]$$

$$= 50 - 50\cos\theta + 6.25 - 6.25\cos^2\theta$$

or $6.25\cos^2\theta + 50\cos\theta - 56.25 = 0$

Solving this quadratic equation, we get θ = 33.14°



Twin-cylinder aeroplane engine.

We know that thrust in the connecting rod,

$$F_Q = \frac{F_P}{\cos\phi} = \frac{1849}{\cos 7.82°} = 1866.3 \text{ N} \quad \textbf{Ans.}$$

3. ***Reaction between the piston and cylinder***

We know that reaction between the piston and cylinder,

$$F_N = F_P \tan\phi = 1849 \tan 7.82° = 254 \text{ N} \quad \textbf{Ans.}$$

4. ***Engine speed at which the above values will become zero***

A little consideration will show that the above values will become zero, if the inertia force on the reciprocating parts (F_I) is equal to the load on the piston (F_L). Let ω_1 be the speed in rad/s, at which $F_I = F_L$.

$$\therefore \quad m_R\,(\omega_1)^2\,r\left(\cos\theta + \frac{\cos 2\theta}{n}\right) = \frac{\pi}{4}\,D^2 \times p$$

$$1\,(\omega_1)^2 \times 0.05\left(\cos 33° + \frac{\cos 66°}{4}\right) = \frac{\pi}{4} \times (80)^2 \times 0.7 \quad \text{or} \quad 0.0\,47\,(\omega_1)^2 = 3520$$

$$\therefore \quad (\omega_1)^2 = 3520 / 0.047 = 74\,894 \text{ or } \omega_1 = 273.6 \text{ rad/s}$$

$\therefore$ Corresponding speed in r.p.m.,

$$N_1 = 273.6 \times 60 / 2\pi = 2612 \text{ r.p.m.} \quad \textbf{Ans.}$$

Example 15.10. *During a trial on steam engine, it is found that the acceleration of the piston is 36 m/s² when the crank has moved 30° from the inner dead centre position. The net effective steam pressure on the piston is 0.5 N/mm² and the frictional resistance is equivalent to a force of 600 N. The diameter of the piston is 300 mm and the mass of the reciprocating parts is 180 kg. If the length of the crank is 300 mm and the ratio of the connecting rod length to the crank length is 4.5, find:* **1.** *Reaction on the guide bars,* **2.** *Thrust on the crank shaft bearings, and* **3.** *Turning moment on the crank shaft.*

Solution. Given : $a_P = 36$ m/s^2 ; $\theta = 30°$; $p = 0.5$ N/mm^2 ; $R_F = 600$ N; $D = 300$ mm ; $m_R = 180$ kg ; $r = 300$ mm $= 0.3$ m ; $n = l / r = 4.5$

1. ***Reaction on the guide bars***

First of all, let us find the piston effort (F_P). We know that load on the piston,

$$F_L = p \times \frac{\pi}{4} \times D^2 = 0.5 \times \frac{\pi}{4} \times (300)^2 = 35\,350 \text{ N}$$

and inertia force due to reciprocating parts,

$$F_I = m_R \times a_P = 180 \times 36 = 6480 \text{ N}$$

∴ Piston effort, $F_P = F_L - F_I - R_F = 35\,350 - 6480 - 600 = 28\,270 \text{ N} = 28.27 \text{ kN}$

Let ϕ = Angle of inclination of the connecting rod to the line of stroke.

We know that $\sin\phi = \sin\theta/n = \sin 30°/4.5 = 0.1111$

∴ $\phi = 6.38°$

We know that reaction on the guide bars,

$$F_N = F_P \tan\phi = 28.27 \tan 6.38° = 3.16 \text{ kN}$$ **Ans.**

2. ***Thrust on the crank shaft bearing***

We know that thrust on the crank shaft bearings,

$$F_B = \frac{F_P \cos(\theta + \phi)}{\cos\phi} = \frac{28.27 \cos(30° + 6.38°)}{\cos 6.38°} = 22.9 \text{ kN}$$ **Ans.**

3. ***Turning moment on the crank shaft***

We know that turning moment on the crank shaft,

$$T = \frac{F_P \sin(\theta + \phi)}{\cos\phi} \times r = \frac{28.27 \sin(30° + 6.38°)}{\cos 6.38°} \times 0.3 \text{ kN-m}$$

$$= 5.06 \text{ kN-m}$$

Example 15.11. *A vertical petrol engine 100 mm diameter and 120 mm stroke has a connecting rod 250 mm long. The mass of the piston is 1.1 kg. The speed is 2000 r.p.m. On the expansion stroke with a crank 20° from top dead centre, the gas pressure is 700 kN/m². Determine:*

1. Net force on the piston, 2. Resultant load on the gudgeon pin, 3. Thrust on the cylinder walls, and 4. Speed above which, other things remaining same, the gudgeon pin load would be reversed in direction.

Solution. Given: D = 100 mm = 0.1 m ; L = 120 mm = 0.12 m or $r = L/2$ = 0.06 m ; l = 250 mm = 0.25 m ; m_R = 1.1 kg ; N = 2000 r.p.m. or $\omega = 2\pi \times 2000/60 = 209.5$ rad/s ; $\theta = 20°$; p = 700 kN/m²

1. ***Net force on the piston***

The configuration diagram of a vertical engine is shown in Fig. 15.11.

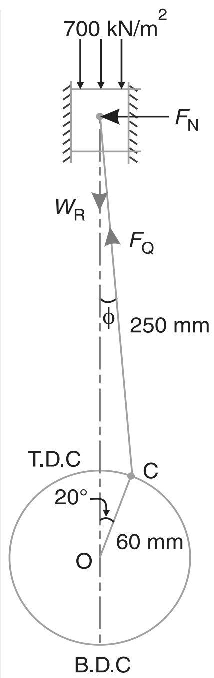


Fig. 15.11

We know that force due to gas pressure,

$$F_L = p \times \frac{\pi}{4} \times D^2 = 700 \times \frac{\pi}{4} \times (0.1)^2 = 5.5 \text{ kN}$$

$$= 5500 \text{ N}$$

and ratio of lengths of the connecting rod and crank,

$$n = l/r = 0.25/0.06 = 4.17$$

∴ Inertia force on the piston,

$$F_I = m_R \cdot \omega^2 \cdot r\left(\cos\theta + \frac{\cos 2\theta}{n}\right)$$

$$= 1.1 \times (209.5)^2 \times 0.06 \times \left(\cos 20° + \frac{\cos 40°}{4.17}\right)$$

$$= 3254 \text{ N}$$

We know that for a vertical engine, net force on the piston,

$$F_P = F_L - F_I + W_R = F_L - F_I + m_R . g$$
$$= 5500 - 3254 + 1.1 \times 9.81 = 2256.8 \text{ N } \textbf{Ans.}$$

2. *Resultant load on the gudgeon pin*

Let ϕ = Angle of inclination of the connecting rod to the line of stroke.

We know that,

$$\sin \phi = \sin \theta / n = \sin 20°/4.17 = 0.082$$

$\therefore$ $\phi = 4.7°$

We know that resultant load on the gudgeon pin,

$$F_Q = \frac{F_P}{\cos \phi} = \frac{2256.8}{\cos 4.7°} = 2265 \text{ N } \textbf{Ans.}$$

3. *Thrust on the cylinder walls*

We know that thrust on the cylinder walls,

$$F_N = F_P \tan \phi = 2256.8 \times \tan 4.7° = 185.5 \text{ N } \textbf{Ans.}$$

4. *Speed, above which, the gudgeon pin load would be reversed in direction*

Let N_1 = Required speed, in r.p.m.

The gudgeon pin load *i.e.* F_Q will be reversed in direction, if F_Q becomes negative. This is only possible when F_P is negative. Therefore, for F_P to be negative, F_I must be greater than $(F_L + W_R)$,

i.e. $$m_R (\omega_1)^2 r \left(\cos \theta + \frac{\cos 2\theta}{n}\right) > 5500 + 1.1 \times 9.81$$

$$1.1 \times (\omega_1)^2 \times 0.06 \left(\cos 20° + \frac{\cos 40°}{4.17}\right) > 5510.8$$

$$0.074 (\omega_1)^2 > 5510.8 \quad \text{or} \quad (\omega_1)^2 > 5510.8/0.074 \text{ or } 74\,470$$

or $$\omega_1 > 273 \text{ rad/s}$$

$\therefore$ Corresponding speed in r.p.m.,

$$N_1 > 273 \times 60/2\pi \quad \text{or} \quad 2606 \text{ r.p.m. } \textbf{Ans.}$$

15.11. Equivalent Dynamical System

In order to determine the motion of a rigid body, under the action of external forces, it is usually convenient to replace the rigid body by two masses placed at a fixed distance apart, in such a way that,

1. the sum of their masses is equal to the total mass of the body ;
2. the centre of gravity of the two masses coincides with that of the body ; and
3. the sum of mass moment of inertia of the masses about their centre of gravity is equal to the mass moment of inertia of the body.

When these three conditions are satisfied, then it is said to be an ***equivalent dynamical system.*** Consider a rigid body, having its centre of gravity at G, as shown in Fig. 15.14.

Let m = Mass of the body,

k_G = Radius of gyration about its centre of gravity G,

m_1 and m_2 = Two masses which form a dynamical equivalent system,

l_1 = Distance of mass m_1 from G,

l_2 = Distance of mass m_2 from G, and

L = Total distance between the masses m_1 and m_2.

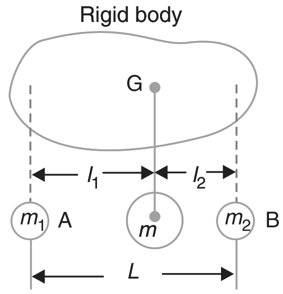


Fig. 15.14. Equivalent dynamical system.

Thus, for the two masses to be dynamically equivalent,

$$m_1 + m_2 = m \qquad \text{...}(i)$$

$$m_1.l_1 = m_2.l_2 \qquad \text{...}(ii)$$

and
$$m_1 (l_1)^2 + m_2 (l_2)^2 = m\,(k_G)^2 \qquad \text{...}(iii)$$

From equations *(i)* and *(ii)*,

$$m_1 = \frac{l_2.m}{l_1 + l_2} \qquad \text{...}(iv)$$

and
$$m_2 = \frac{l_1.m}{l_1 + l_2} \qquad \text{...}(v)$$

Substituting the value of m_1 and m_2 in equation *(iii)*, we have

$$\frac{l_2.m}{l_1 + l_2}(l_1)^2 + \frac{l_1.m}{l_1 + l_2}(l_2)^2 = m\,(k_G)^2 \quad \text{or} \quad \frac{l_1.l_2\,(l_1 + l_2)}{l_1 + l_2} = (k_G)^2$$

$$\therefore \qquad l_1.l_2 = (k_G)^2 \qquad \text{...}(vi)$$

This equation gives the essential condition of placing the two masses, so that the system becomes dynamical equivalent. The distance of one of the masses (*i.e.* either l_1 or l_2) is arbitrary chosen and the other distance is obtained from equation *(vi)*.

Note : When the radius of gyration k_G is not known, then the position of the second mass may be obtained by considering the body as a compound pendulum. We have already discussed, that the length of the simple pendulum which gives the same frequency as the rigid body (*i.e.* compound pendulum) is

$$L = \frac{(k_G)^2 + h^2}{h} = \frac{(k_G)^2 + (l_1)^2}{l_1} \qquad \text{..(Replacing } h \text{ by } l_1)$$

We also know that $l_1.l_2 = (k_G)^2$

$$\therefore \qquad L = \frac{l_1.l_2 + (l_1)^2}{l_1} = l_2 + l_1$$

This means that the second mass is situated at the ***centre of oscillation*** **or** ***percussion*** of the body, which is at a distance of $l_2 = (k_G)^2/l_1$.

15.12. Determination of Equivalent Dynamical System of Two Masses by Graphical Method

Consider a body of mass m, acting at G as shown in Fig. 15.15. This mass m, may be replaced by two masses m_1 and m_2 so that the system becomes dynamical equivalent. The position of mass m_1 may be fixed arbitrarily at A. Now draw perpendicular CG at G, equal in length of the radius of gyration of the body, k_G. Then join AC and draw CB perpendicular to AC intersecting AG produced in B. The point B now fixes the position of the second mass m_2.

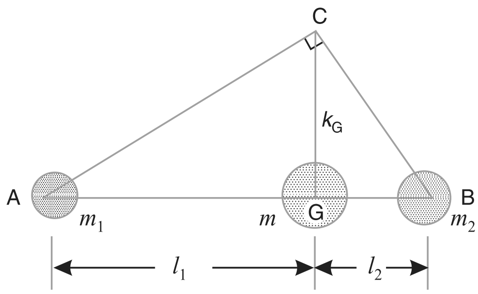


Fig. 15.15. Determination of equivalent dynamical system by graphical method.

A little consideration will show that the triangles ACG and BCG are similar. Therefore,

$$\frac{k_G}{l_1} = \frac{l_2}{k_G} \quad \text{or} \quad (k_G)^2 = l_1.l_2$$

...(Same as before)

Example 15.15. *The connecting rod of a gasoline engine is 300 mm long between its centres. It has a mass of 15 kg and mass moment of inertia of 7000 kg-mm^2. Its centre of gravity is at 200 mm from its small end centre. Determine the dynamical equivalent two-mass system of the connecting rod if one of the masses is located at the small end centre.*

Solution. Given : $l = 300$ mm ; $m = 15$ kg; $I = 7000$ kg-mm^2 ; $l_1 = 200$ mm

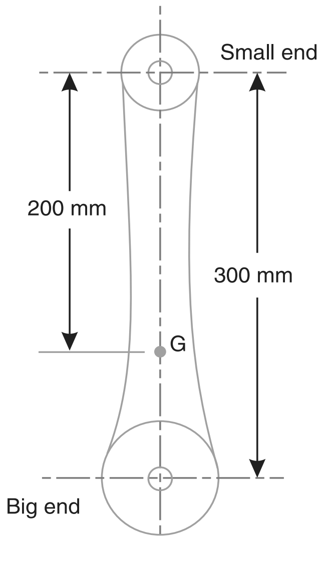


Fig. 15.16

The connecting rod is shown in Fig. 15.16.

Let k_G = Radius of gyration of the connecting rod about an axis passing through its centre of gravity G.

We know that mass moment of inertia (I),

$$7000 = m\,(k_G)^2 = 15\,(k_G)^2$$

$$\therefore \quad (k_G)^2 = 7000/15 = 466.7 \text{ mm}^2 \text{ or } k_G = 21.6 \text{ mm}$$

It is given that one of the masses is located at the small end centre. Let the other mass is placed at a distance l_2 from the centre of gravity G, as shown in Fig. 15.17.

We know that for a dynamical equivalent system,

$$l_1.l_2 = (k_G)^2$$

$$\therefore \quad l_2 = \frac{(k_G)^2}{l_1} = \frac{466.7}{200} = 2.33 \text{ mm}$$

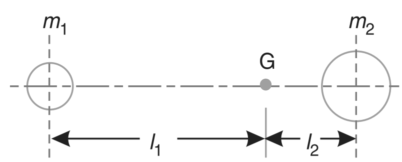


Fig. 15.17

Let m_1 = Mass placed at the small end centre, and

m_2 = Mass placed at a distance l_2 from the centre of gravity G.

We know that

$$m_1 = \frac{l_2.m}{l_1 + l_2} = \frac{2.33 \times 15}{200 + 2.33} = 0.17 \text{ kg } \textbf{Ans.}$$

and
$$m_2 = \frac{l_1.m}{l_1 + l_2} = \frac{200 \times 15}{200 + 2.33} = 14.83 \text{ kg } \textbf{Ans.}$$

Example 15.16. *A connecting rod is suspended from a point 25 mm above the centre of small end, and 650 mm above its centre of gravity, its mass being 37.5 kg. When permitted to oscillate, the time period is found to be 1.87 seconds. Find the dynamical equivalent system constituted of two masses, one of which is located at the small end centre.*

Solution. Given : $h = 650$ mm $= 0.65$ m ; $l_1 = 650 - 25 = 625$ mm $= 0.625$ m ; $m = 37.5$ kg ; $t_p = 1.87$ s

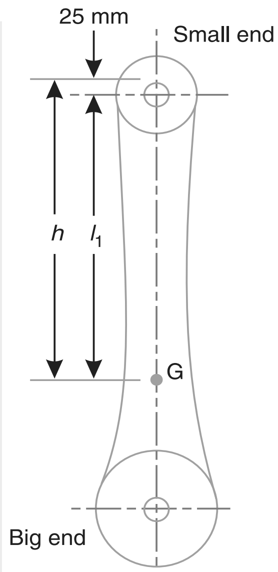


Fig. 15.18

First of all, let us find the radius of gyration (k_G) of the connecting rod (considering it is a compound pendulum), about an axis passing through its centre of gravity, G.

We know that for a compound pendulum, time period of oscillation (t_p),

$$1.87 = 2\pi\sqrt{\frac{(k_G)^2 + h^2}{g.h}} \quad \text{or} \quad \frac{1.87}{2\pi} = \sqrt{\frac{(k_G)^2 + (0.65)^2}{9.81 \times 0.65}}$$

Squaring both sides, we have

$$0.0885 = \frac{(k_G)^2 + 0.4225}{6.38}$$

$$(k_G)^2 = 0.0885 \times 6.38 - 0.4225 = 0.1425 \text{ m}^2$$

$$\therefore \quad k_G = 0.377 \text{ m}$$

It is given that one of the masses is located at the small end centre. Let the other mass is located at a distance l_2 from the centre of gravity G, as shown in Fig. 15.19. We know that, for a dynamically equivalent system,

$$l_1.l_2 = (k_G)^2$$

$$\therefore \quad l_2 = \frac{(k_G)^2}{l_1} = \frac{0.1425}{0.625} = 0.228 \text{ m}$$

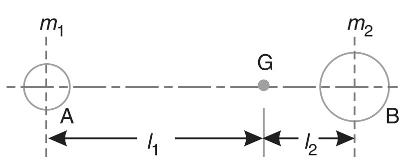


Fig. 15.19

Let m_1 = Mass placed at the small end centre A, and

m_2 = Mass placed at a distance l_2 from G, *i.e.* at B.

We know that, for a dynamically equivalent system,

$$m_1 = \frac{l_2.m}{l_1 + l_2} = \frac{0.228 \times 37.5}{0.625 + 0.228} = 10 \text{ kg } \textbf{Ans.}$$

and
$$m_2 = \frac{l_1.m}{l_1 + l_2} = \frac{0.625 \times 37.5}{0.625 + 0.228} = 27.5 \text{ kg } \textbf{Ans.}$$

Example 15.17. *The following data relate to a connecting rod of a reciprocating engine*:

Mass = 55 kg; Distance between bearing centres = 850 mm; Diameter of small end bearing = 75 mm; Diameter of big end bearing = 100 mm; Time of oscillation when the connecting rod is suspended from small end = 1.83 s; Time of oscillation when the connecting rod is suspended from big end = 1.68 s.

Determine: **1.** *the radius of gyration of the rod about an axis passing through the centre of gravity and perpendicular to the plane of oscillation;* **2.** *the moment of inertia of the rod about the same axis; and* **3.** *the dynamically equivalent system for the connecting rod, constituted of two masses, one of which is situated at the small end centre.*

Solution. Given : m = 55 kg ; l = 850 mm = 0.85 m ; d_1 = 75 mm = 0.075 m ; d_2 = 100 mm = 0.1 m ; t_{p1} = 1.83 s ; t_{p2} = 1.68 s

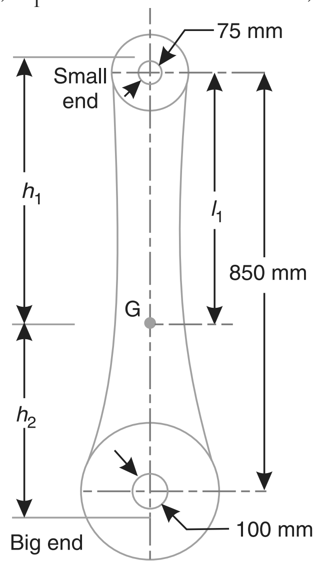


Fig. 15.20

First of all, let us find the lengths of the equivalent simple pendulum when suspended

(*a*) from the top of small end bearing; and

(*b*) from the top of big end bearing.

Let L_1 = Length of equivalent simple pendulum when suspended from the top of small end bearing,

L_2 = Length of equivalent simple pendulum when suspended from the top of big end bearing,

h_1 = Distance of centre of gravity, G, from the top of small end bearing, and

h_2 = Distance of centre of gravity, G, from the top of big end bearing.

We know that for a simple pendulum

$$t_{p1} = 2\pi\sqrt{\frac{L_1}{g}} \quad \text{or} \quad \left(\frac{t_{p1}}{2\pi}\right)^2 = \frac{L_1}{g} \qquad \text{...(Squaring both sides)}$$

$$\therefore \quad L_1 = g\left(\frac{t_{p1}}{2\pi}\right)^2 = 9.81\left(\frac{1.83}{2\pi}\right)^2 = 0.832 \text{ m}$$

Similarly,
$$L_2 = g\left(\frac{t_{p2}}{2\pi}\right)^2 = 9.81\left(\frac{1.68}{2\pi}\right)^2 = 0.7 \text{ m}$$

1. ***Radius of gyration of the rod about an axis passing through the centre of gravity and perpendicular to the plane of oscillation***

Let k_G = Required radius of gyration of the rod.

We know that the length of equivalent simple pendulum,

$$L = \frac{(k_G)^2 + h^2}{h} \quad \text{or} \quad (k_G)^2 = L.h - h^2 = h\,(L - h)$$

$\therefore$ When the rod is suspended from the top of small end bearing,

$$(k_G)^2 = h_1\,(L_1 - h_1) \qquad \text{...}(i)$$

and when the rod is suspended from the top of big end bearing,

$$(k_G)^2 = h_2 (L_2 - h_2) \quad ...(ii)$$

Also, from the geometry of the Fig. 15.20,

$$h_1 + h_2 = \frac{d_1}{2} + l + \frac{d_2}{2} = \frac{0.075}{2} + 0.85 + \frac{0.1}{2} = 0.9375 \text{ m}$$

$$\therefore \qquad h_2 = 0.9375 - h_1 \quad ...(iii)$$

From equations (*i*) and (*ii*),

$$h_1 (L_1 - h_1) = h_2 (L_2 - h_2)$$

Substituting the value of h_2 from equation (*iii*),

$$h_1 (0.832 - h_1) = (0.9375 - h_1) [0.7 - (0.9375 - h_1)]$$

$$0.832 h_1 - (h_1)^2 = -0.223 + 1.175 h_1 - (h_1)^2$$

$$0.343 h_1 = 0.233 \text{ or } h_1 = 0.223 / 0.343 = 0.65 \text{ m}$$

Now from equation (*i*),

$$(k_G)^2 = 0.65 (0.832 - 0.65) = 0.1183 \quad \text{or} \quad k_G = 0.343 \text{ m} \quad \textbf{Ans.}$$

2. ***Moment of inertia of the rod***

We know that moment of inertia of the rod,

$$I = m (k_G)^2 = 55 \times 0.1183 = 6.51 \text{ kg-m}^2 \quad \textbf{Ans.}$$

3. ***Dynamically equivalent system for the rod***

Since one of the masses (m_1) is situated at the centre of small end bearing, therefore its distance from the centre of gravity, *G*, is

$$l_1 = h_1 - 0.075 / 2 = 0.65 - 0.0375 = 0.6125 \text{ m}$$

Let m_2 = Magnitude of the second mass, and

l_2 = Distance of the second mass from the centre of gravity, *G*, towards big end bearing.

For a dynamically equivalent system,

$$l_1 . l_2 = (k_G)^2 \quad \text{or} \quad l_2 = \frac{(k_G)^2}{l_1} = \frac{0.1183}{0.6125} = 0.193 \text{ m}$$

We know that $m_1 = \dfrac{l_2 . m}{l_1 + l_2} = \dfrac{0.193 \times 55}{0.6125 + 0.193} = 13.18$ kg **Ans.**

and $m_2 = \dfrac{l_1 . m}{l_1 + l_2} = \dfrac{0.6125 \times 55}{0.6125 + 0.193} = 41.82$ kg **Ans.**

15.13. Correction Couple to be Applied to Make Two Mass System Dynamically Equivalent

In Art. 15.11, we have discussed the conditions for equivalent dynamical system of two bodies. A little consideration will show that when two masses are placed arbitrarily*, then the condi-

* When considering the inertia forces on the connecting rod in a mechanism, we replace the rod by two masses arbitrarily. This is discussed in Art. 15.14.

tions (*i*) and (*ii*) as given in Art. 15.11 will only be satisfied. But the condition (*iii*) is not possible to satisfy. This means that the mass moment of inertia of these two masses placed arbitrarily, will differ than that of mass moment of inertia of the rigid body.

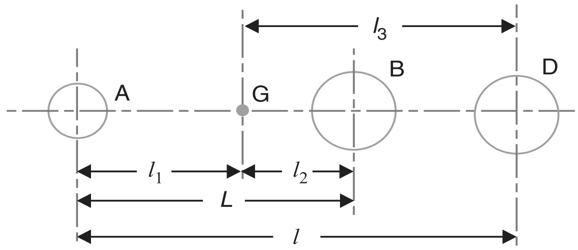


Fig. 15.21. Correction couple to be applied to make the two-mass system dynamically equivalent.

Consider two masses, one at A and the other at D be placed arbitrarily, as shown in Fig. 15.21.

Let l_3 = Distance of mass placed at D from G,

I_1 = New mass moment of inertia of the two masses;

k_1 = New radius of gyration;

α = Angular acceleration of the body;

I = Mass moment of inertia of a dynamically equivalent system;

k_G = Radius of gyration of a dynamically equivalent system.

We know that the torque required to accelerate the body,

$$T = I.\alpha = m\,(k_G)^2\,\alpha \qquad ...(i)$$

Similarly, the torque required to accelerate the two-mass system placed arbitrarily,

$$T_1 = I_1.\alpha = m\,(k_1)^2\,\alpha \qquad ...(ii)$$

$\therefore$ Difference between the torques required to accelerate the two-mass system and the torque required to accelerate the rigid body,

$$T' = T_1 - T = m\,(k_1)^2\,\alpha - m\,(k_G)^2\,\alpha = m\,[(k_1)^2 - (k_G)^2]\,\alpha \qquad ...(iv)$$

The difference of the torques T' is known as ***correction couple***. This couple must be applied, when the masses are placed arbitrarily to make the system dynamical equivalent. This, of course, will satisfy the condition (*iii*) of Art. 15.11.

Note: We know that $(k_G)^2 = l_1.l_2$, and $(k_1)^2 = l_1.l_3$

$\therefore$ Correction couple, $T' = m\,(l_1.l_3 - l_1.l_2)\,\alpha = m.l_1\,(l_3 - l_2)\,\alpha$

But $l_3 - l_2 = l - L$

$\therefore$ $T' = m.l_1\,(l - L)\,\alpha$

where l = Distance between the two arbitrarily masses, and

L = Distance between the two masses for a true dynamically equivalent system. It is the equivalent length of a simple pendulum when a body is suspended from an axis which passes through the position of mass m, and perpendicular to the plane of rotation of the two mass system.

$$= \frac{(k_G)^2 + (l_1)^2}{l_1}$$

Example 15.18. *A connecting rod of an I.C. engine has a mass of 2 kg and the distance between the centre of gudgeon pin and centre of crank pin is 250 mm. The C.G. falls at a point 100 mm from the gudgeon pin along the line of centres. The radius of gyration about an axis through the C.G. perpendicular to the plane of rotation is 110 mm. Find the equivalent dynamical system if only one of the masses is located at gudgeon pin.*

If the connecting rod is replaced by two masses, one at the gudgeon pin and the other at the crank pin and the angular acceleration of the rod is 23 000 rad/s² clockwise, determine the correction couple applied to the system to reduce it to a dynamically equivalent system.

Solution. Given : $m = 2$ kg ; $l = 250$ mm $= 0.25$ m ; $l_1 = 100$ mm $= 0.1$m ; $k_G = 110$ mm $= 0.11$ m ; $\alpha = 23\,000$ rad/s^2

Equivalent dynamical system

It is given that one of the masses is located at the gudgeon pin. Let the other mass be located at a distance l_2 from the centre of gravity. We know that for an equivalent dynamical system.

$$l_1.l_2 = (k_G)^2 \quad \text{or} \quad l_2 = \frac{(k_G)^2}{l_1} = \frac{(0.11)^2}{0.1} = 0.121 \text{ m}$$

Let m_1 = Mass placed at the gudgeon pin, and

m_2 = Mass placed at a distance l_2 from *C.G.*

We know that $$m_1 = \frac{l_2.m}{l_1 + l_2} = \frac{0.121 \times 2}{0.1 + 0.121} = 1.1 \text{ kg} \textbf{ Ans.}$$

and $$m_2 = \frac{l_1.m}{l_1 + l_2} = \frac{0.1 \times 2}{0.1 + 0.121} = 0.9 \text{ kg} \textbf{ Ans.}$$

Correction couple

Since the connecting rod is replaced by two masses located at the two centres (*i.e.* one at the gudgeon pin and the other at the crank pin), therefore,

$$l = 0.1 \text{ m}, \quad \text{and} \quad l_3 = l - l_1 = 0.25 - 0.1 = 0.15 \text{ m}$$

Let k_1 = New radius of gyration.

We know that $(k_1)^2 = l_1.l_3 = 0.1 \times 0.15 = 0.015 \text{ m}^2$

$\therefore$ Correction couple,

$$T' = m(k_1^2 - k_G^2)\,\alpha = 2\left[0.015 - (0.11)^2\right] 23\,000 = 133.4 \text{ N-m} \textbf{ Ans.}$$

Note : Since T' is positive, therefore, the direction of correction couple is same as that of angular acceleration *i.e.* clockwise.

DO YOU KNOW ?

1. Define 'inertia force' and 'inertia torque'.
2. Draw and explain Klien's construction for determining the velocity and acceleration of the piston in a slider crank mechanism.
3. Explain Ritterhaus's and Bennett's constructions for determining the acceleration of the piston of a reciprocating engine.
4. How are velocity and acceleration of the slider of a single slider crank chain determined analytically?
5. Derive an expression for the inertia force due to reciprocating mass in reciprocating engine, neglecting the mass of the connecting rod.
6. What is the difference between piston effort, crank effort and crank-pin effort?
7. Discuss the method of finding the crank effort in a reciprocating single acting, single cylinder petrol engine.
8. The inertia of the connecting rod can be replaced by two masses concentrated at two points and connected rigidly together. How to determine the two masses so that it is dynamically equivalent to the connecting rod ? Show this.
9. Given acceleration image of a link. Explain how dynamical equivalent system can be used to determine the direction of inertia force on it.
10. Describe the graphical and analytical method of finding the inertia torque on the crankshaft of a horizontal reciprocating engine.
11. Derive an expression for the correction torque to be applied to a crankshaft if the connecting rod of a reciprocating engine is replaced by two lumped masses at the piston pin and the crank pin respectively.

OBJECTIVE TYPE QUESTIONS

1. When the crank is at the inner dead centre, in a horizontal reciprocating steam engine, then the velocity of the piston will be

   (*a*) zero (*b*) minimum (*c*) maximum

2. The acceleration of the piston in a reciprocating steam engine is given by

(a) $\omega.r\left(\sin\theta + \frac{\sin 2\theta}{n}\right)$ (b) $\omega.r\left(\cos\theta + \frac{\cos 2\theta}{n}\right)$

(c) $\omega^2.r\left(\sin\theta + \frac{\sin 2\theta}{n}\right)$ (d) $\omega^2.r\left(\cos\theta + \frac{\cos 2\theta}{n}\right)$

where ω = Angular velocity of the crank,

r = Radius of the crank,

θ = Angle turned by the crank from inner dead centre, and

n = Ratio of length of connecting rod to crank radius.

3. A rigid body, under the action of external forces, can be replaced by two masses placed at a fixed distance apart. The two masses form an equivalent dynamical system, if

(a) the sum of two masses is equal to the total mass of the body

(b) the centre of gravity of the two masses coincides with that of the body

(c) the sum of mass moment of inertia of the masses about their centre of gravity is equal to the mass moment of inertia of the body

(d) all of the above

4. The essential condition of placing the two masses, so that the system becomes dynamically equivalent is

(a) $l_1 . l_2 = k_G^2$ (b) $l_1 . l_2 = k_G$ (c) $l_1 = k_G$ (d) $l_2 = k_G$

where l_1 and l_2 = Distance of two masses from the centre of gravity of the body, and

k_G = Radius of gyration of the body.

5. In an engine, the work done by inertia forces in a cycle is

(a) positive (b) zero (c) negative (d) none of these

ANSWERS

1. (a) 2. (d) 3. (d) 4. (a) 5. (a)

MODULE-IV

Friction Effects: Screw jack, friction between pivot and collars, single, multi-plate and cone clutches, anti friction bearing, film friction, friction circle, friction axis.

Flexible Mechanical Elements: Belt, rope and chain drives, initial tension, effect of centrifugal tension on power transmission, maximum power transmission capacity, belt creep and slip.

Features (Main)

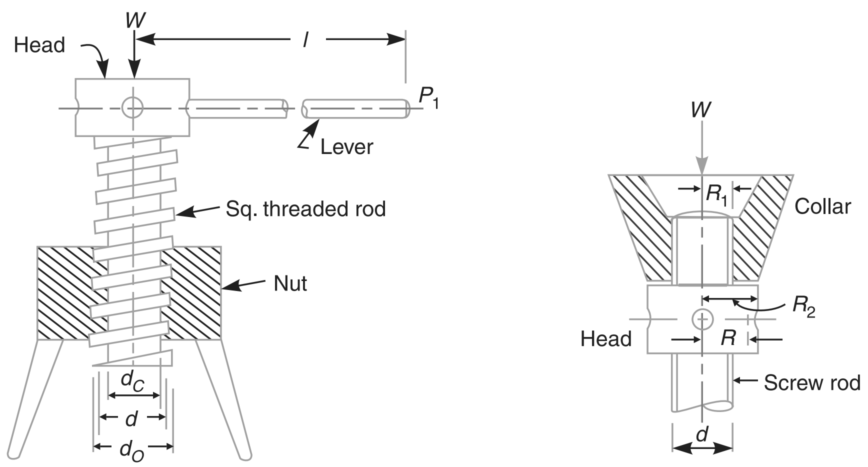
Friction

10.1. Introduction

It has been established since long, that the surfaces of the bodies are never perfectly smooth. When, even a very smooth surface is viewed under a microscope, it is found to have roughness and irregularities, which may not be detected by an ordinary touch. If a block of one substance is placed over the level surface of the same or of different material, a certain degree of interlocking of the minutely projecting particles takes place. This does not involve any force, so long as the block does not move or tends to move. But whenever one block moves or tends to move tangentially with respect to the surface, on which it rests, the interlocking property of the projecting particles opposes the motion. This opposing force, which acts in the opposite direction of the movement of the upper block, is called the ***force of friction*** or simply ***friction***. It thus follows, that at every joint in a machine, force of friction arises due to the relative motion between two parts and hence some energy is wasted in overcoming the friction. Though the friction is considered undesirable, yet it plays an important role both in nature and in engineering *e.g.* walking on a road, motion of locomotive on rails, transmission of power by belts, gears etc. The friction between the wheels and the road is essential for the car to move forward.

10.17. Screw Jack

The screw jack is a device, for lifting heavy loads, by applying a comparatively smaller effort at its handle. The principle, on which a screw jack works is similar to that of an inclined plane.



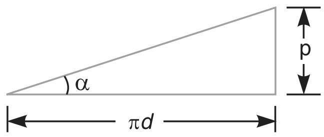
(*a*) Screw jack. (*b*) Thrust collar.

Fig. 10.11

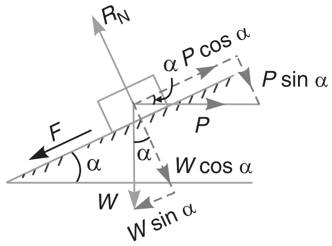
Fig. 10.11 (*a*) shows a common form of a screw jack, which consists of a square threaded rod (also called screw rod or simply screw) which fits into the inner threads of the nut. The load, to be raised or lowered, is placed on the head of the square threaded rod which is rotated by the application of an effort at the end of the lever for lifting or lowering the load.

10.18. Torque Required to Lift the Load by a Screw Jack

If one complete turn of a screw thread by imagined to be unwound, from the body of the screw and developed, it will form an inclined plane as shown in Fig. 10.12 (*a*).



(*a*) Development of a screw.



(*b*) Forces acting on the screw.

Fig. 10.12

Let p = Pitch of the screw,

d = Mean diameter of the screw,

α = Helix angle,

P = Effort applied at the circumference of the screw to lift the load,

W = Load to be lifted, and

μ = Coefficient of friction, between the screw and nut = $\tan \phi$, where ϕ is the friction angle.

From the geometry of the Fig. 10.12 (*a*), we find that

$$\tan \alpha = p/\pi d$$

Since the principle on which a screw jack works is similar to that of an inclined plane, therefore the force applied on the lever of a screw jack may be considered to be horizontal as shown in Fig. 10.12 (*b*).

Since the load is being lifted, therefore the force of friction ($F = \mu.R_N$) will act downwards. All the forces acting on the screw are shown in Fig. 10.12 (*b*).

Resolving the forces along the plane,

$$P \cos \alpha = W \sin \alpha + F = W \sin \alpha + \mu.R_N \quad ...(i)$$

and resolving the forces perpendicular to the plane,

$$R_N = P \sin \alpha + W \cos \alpha \quad ...(ii)$$

Substituting this value of R_N in equation (*i*),

$$P \cos \alpha = W \sin \alpha + \mu (P \sin \alpha + W \cos \alpha)$$

$$= W \sin \alpha + \mu P \sin \alpha + \mu W \cos \alpha$$

or $$P \cos \alpha - \mu P \sin \alpha = W \sin \alpha + \mu W \cos \alpha$$

or $$P (\cos \alpha - \mu \sin \alpha) = W (\sin \alpha + \mu \cos \alpha)$$

$$\therefore \qquad P = W \times \frac{\sin\alpha + \mu\cos\alpha}{\cos\alpha - \mu\sin\alpha}$$

Substituting the value of $\mu = \tan\phi$ in the above equation, we get

$$P = W \times \frac{\sin\alpha + \tan\phi\cos\alpha}{\cos\alpha - \tan\phi\sin\alpha}$$

Multiplying the numerator and denominator by $\cos\phi$,

$$P = W \times \frac{\sin\alpha\cos\phi + \sin\phi\cos\alpha}{\cos\alpha\cos\phi - \sin\alpha\sin\phi} = W \times \frac{\sin(\alpha+\phi)}{\cos(\alpha+\phi)}$$

$$= W\tan(\alpha+\phi)$$

$\therefore$ Torque required to overcome friction between the screw and nut,

$$T_1 = P \times \frac{d}{2} = W\tan(\alpha+\phi)\frac{d}{2}$$

When the axial load is taken up by a thrust collar or a flat surface, as shown in Fig. 10.11 (*b*), so that the load does not rotate with the screw, then the torque required to overcome friction at the collar,

$$T_2 = \mu_1.W\left(\frac{R_1 + R_2}{2}\right) = \mu_1.W.R$$

where R_1 and R_2 = Outside and inside radii of the collar,

R = Mean radius of the collar, and

μ_1 = Coefficient of friction for the collar.

$\therefore$ Total torque required to overcome friction (*i.e.* to rotate the screw),

$$T = T_1 + T_2 = P \times \frac{d}{2} + \mu_1.W.R$$

If an effort P_1 is applied at the end of a lever of arm length l, then the total torque required to overcome friction must be equal to the torque applied at the end of the lever, *i.e.*

$$T = P \times \frac{d}{2} = P_1.l$$

Notes : 1. When the *nominal diameter (d_0) and the **core diameter (d_c) of the screw thread is given, then the mean diameter of the screw,

$$d = \frac{d_0 + d_c}{2} = d_0 - \frac{p}{2} = d_c + \frac{p}{2}$$

2. Since the mechanical advantage is the ratio of load lifted (W) to the effort applied (P_1) at the end of the lever, therefore mechanical advantage,

$$M.A. = \frac{W}{P_1} = \frac{W \times 2l}{p.d} \qquad \ldots\left(\because\ P_1 = \frac{P.d}{2l}\right)$$

$$= \frac{W \times 2l}{W\tan(\alpha+\phi)\,d} = \frac{2l}{d.\tan(\alpha+\phi)}$$

Example 10.3. *An electric motor driven power screw moves a nut in a horizontal plane against a force of 75 kN at a speed of 300 mm/min. The screw has a single square thread of 6 mm pitch on a major diameter of 40 mm. The coefficient of friction at the screw threads is 0.1. Estimate power of the motor.*

---

* The **nominal diameter** of a screw thread is also known as **outside diameter** or **major diameter.**

** The **core diameter** of a screw thread is also known as inner diameter or **root diameter** or **minor diameter.**

Solution. Given : W = 75 kN = 75 × 10^3 N ; v = 300 mm/min ; p = 6 mm ; d_0 = 40 mm ; $\mu = \tan\phi = 0.1$

We know that mean diameter of the screw,

$$d = d_0 - p/2 = 40 - 6/2 = 37 \text{ mm} = 0.037 \text{ m}$$

and
$$\tan\alpha = \frac{p}{\pi d} = \frac{6}{\pi \times 37} = 0.0516$$

∴ Force required at the circumference of the screw,

$$P = W\tan(\alpha + \phi) = W\left[\frac{\tan\alpha + \tan\phi}{1 - \tan\alpha.\tan\phi}\right]$$

$$= 75 \times 10^3\left[\frac{0.0516 + 0.1}{1 - 0.0516 \times 0.1}\right] = 11.43 \times 10^3 \text{ N}$$

and torque required to overcome friction,

$$T = P \times d/2 = 11.43 \times 10^3 \times 0.037/2 = 211.45 \text{ N-m}$$

We know that speed of the screw,

$$N = \frac{\text{Speed of the nut}}{\text{Pitch of the screw}} = \frac{300}{6} = 50 \text{ r.p.m.}$$

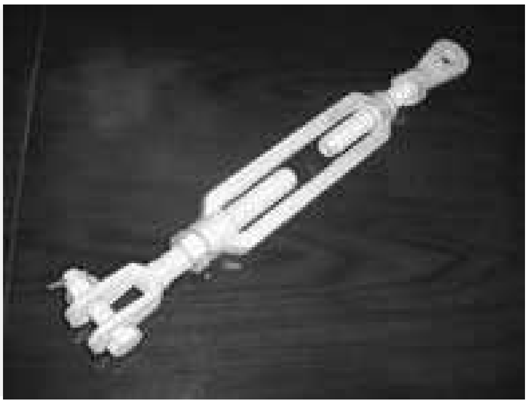
and angular speed, $\omega = 2\pi \times 50/60 = 5.24$ rad/s

∴ Power of the motor $= T.\omega = 211.45 \times 5.24 = 1108$ W = 1.108 **kW Ans.**

Example 10.4. *A turnbuckle, with right and left hand single start threads, is used to couple two wagons. Its thread pitch is 12 mm and mean diameter 40 mm. The coefficient of friction between the nut and screw is 0.16.*

1. Determine the work done in drawing the wagons together a distance of 240 mm, against a steady load of 2500 N.

2. If the load increases from 2500 N to 6000 N over the distance of 240 mm, what is the work to be done?


Turnbuckle.

Solution. Given : p = 12 mm ; d = 40 mm ; $\mu = \tan\phi = 0.16$; W = 2500 N

1. ***Work done in drawing the wagons together against a steady load of 2500 N***

We know that
$$\tan\alpha = \frac{p}{\pi d} = \frac{12}{\pi \times 40} = 0.0955$$

∴ Effort required at the circumference of the screw,

$$P = W\tan(\alpha + \phi) = W\left[\frac{\tan\alpha + \tan\phi}{1 - \tan\alpha.\tan\phi}\right]$$

$$= 2500\left[\frac{0.0955 + 0.16}{1 - 0.0955 \times 0.16}\right] = 648.7 \text{ N}$$

and torque required to overcome friction between the screw and nut,

$$T = P \times d/2 = 648.7 \times 40/2 = 12\,947 \text{ N-mm} = 12.974 \text{ N-m}$$

A little consideration will show that for one complete revolution of the screwed rod, the wagons are drawn together through a distance equal to 2 p, *i.e.* $2 \times 12 = 24$ mm. Therefore in order to draw the wagons together through a distance of 240 mm, the number of turns required are given by

$$N = 240/24 = 10$$

$\therefore$ Work done $= T \times 2\,\pi\,N = 12.974 \times 2\,\pi \times 10 = 815.36$ N-m **Ans.**

2. ***Work done in drawing the wagons together when load increases from 2500 N to 6000 N***

For an increase in load from 2500 N to 6000 N,

$$\text{Work done} = \frac{815.3(6000 - 2500)}{2500} = 114.4 \text{ N-m } \textbf{Ans.}$$

Example 10.5. *A 150 mm diameter valve, against which a steam pressure of 2 MN/m² is acting, is closed by means of a square threaded screw 50 mm in external diameter with 6 mm pitch. If the coefficient of friction is 0.12 ; find the torque required to turn the handle.*

Solution. Given : D = 150 mm = 0.15 mm = 0.15 m ; Ps = 2 MN/m² = 2×10^6 N/m² ; d_0 = 50 mm ; p = 6 mm ; $\mu = \tan\phi = 0.12$

We know that load on the valve,

$$W = \text{Pressure} \times \text{Area} = p_S \times \frac{\pi}{4} D^2 = 2 \times 10^6 \times \frac{\pi}{4}(0.15)^2 \text{ N}$$

$$= 35\,400 \text{ N}$$

Mean diameter of the screw,

$$d = d_0 - p/2 = 50 - 6/2 = 47 \text{ mm} = 0.047 \text{ m}$$

$$\therefore \quad \tan\alpha = \frac{p}{\pi d} = \frac{6}{\pi \times 47} = 0.0406$$

We know that force required to turn the handle,

$$P = W\tan(\alpha + \phi) = W\left[\frac{\tan\alpha + \tan\phi}{1 - \tan\alpha.\tan\phi}\right]$$

$$= 35\,400\left[\frac{0.0406 + 12}{1 - 0.0406 \times 0.12}\right] = 5713 \text{ N}$$

$\therefore$ Torque required to turn the handle,

$$T = P \times d/2 = 5713 \times 0.047/2 = 134.2 \text{ N-m } \textbf{Ans.}$$

Example 10.6. *A square threaded bolt of root diameter 22.5 mm and pitch 5 mm is tightened by screwing a nut whose mean diameter of bearing surface is 50 mm. If coefficient of friction for nut and bolt is 0.1 and for nut and bearing surface 0.16, find the force required at the end of a spanner 500 mm long when the load on the bolt is 10 kN.*

Solution. Given : d_c = 22.5 mm ; p = 5 mm ; D = 50 mm or R = 25 mm ; $\mu = \tan\phi = 0.1$; $\mu_1 = 0.16$; l = 500 mm ; W = 10 kN = 10×10^3 N

Let P_1 = Force required at the end of a spanner in newtons.

We know that mean diameter of the screw,

$$d = d_c + p/2 = 22.5 + 5/2 = 25 \text{ mm}$$

$$\therefore \qquad \tan\alpha = \frac{p}{\pi d} = \frac{5}{\pi \times 25} = 0.0636$$

Force requred at the circumference of the screw,

$$P = W\tan(\alpha + \phi) = W\left[\frac{\tan\alpha + \tan\phi}{1 - \tan\alpha.\tan\phi}\right]$$

$$= 10 \times 10^3\left[\frac{0.0636 + 0.1}{1 - 0.06363 \times 0.1}\right] = 1646 \text{ N}$$

We know that total torque required,

$$T = P \times \frac{d}{2} + \mu_1.W.R. = 1646 \times \frac{25}{2} + 0.16 \times 10 \times 10^3 \times 25$$

$$= 60575 \text{ N - mm} \qquad ..(i)$$

We also know that torque required at the end of a spanner,

$$T = P_1 \times l = P_1 \times 500 = 500\,P_1 \text{ N-mm} \qquad ...(ii)$$

Equating equations (*i*) and (*ii*),

$$P_1 = 60575/500 = 121.15 \text{ N} \quad \textbf{Ans.}$$

Example 10.7. *A vertical screw with single start square threads 50 mm mean diameter and 12.5 mm pitch is raised against a load of 10 kN by means of a hand wheel, the boss of which is threaded to act as a nut. The axial load is taken up by a thrust collar which supports the wheel boss and has a mean diameter of 60 mm. If the coefficient of friction is 0.15 for the screw and 0.18 for the collar and the tangential force applied by each hand to the wheel is 100 N ; find suitable diameter of the hand wheel.*

Solution. Given : d = 50 mm ; p = 12.5 mm ; W = 10 kN = 10×10^3 N ; D = 60 mm or R = 30 mm ; $\mu = \tan\phi = 0.15$; $\mu_1 = 0.18$; $P_1 = 100$ N

We know that $\tan\alpha = \dfrac{p}{\pi d} = \dfrac{12.5}{\pi \times 50} = 0.08$

and the tangential force required at the circumference of the screw,

$$P = W\tan(\alpha + \phi) = W\left[\frac{\tan\alpha + \tan\phi}{1 - \tan\alpha.\tan\phi}\right]$$

$$= 10 \times 10^3\left[\frac{0.08 + 0.15}{1 - 0.08 \times 0.15}\right] = 2328 \text{ N}$$

Also we know that the total torque required to turn the hand wheel,

$$T = P \times \frac{d}{2} + \mu_1.W.R = 2328 \times \frac{50}{2} + 0.18 \times 10 \times 10^3 \times 30$$

$$= 112200 \text{ N-mm} \qquad ...(i)$$

Let $\qquad D_1$ = Diameter of the hand wheel in mm.

We know that the torque applied to the hand wheel,

$$T = 2P_1 \times \frac{D_1}{2} = 2 \times 100 \times \frac{D_1}{2} = 100\,D_1 \text{ N-mm} \qquad ...(ii)$$

Equating equations (*i*) and (*ii*),

$$D_1 = 112\ 200/100 = 1222 \text{ mm} = 1.222 \text{ m} \quad \textbf{Ans.}$$

Example 10.8. *The cutter of a broaching machine is pulled by square threaded screw of 55 mm external diameter and 10 mm pitch. The operating nut takes the axial load of 400 N on a flat surface of 60 mm internal diameter and 90 mm external diameter. If the coefficient of firction is 0.15 for all contact surfaces on the nut, determine the power required to rotate the operating nut, when the cutting speed is 6 m/min.*

Solution. Given : d_0 = 55 mm ; p = 10 mm = 0.01 m ; W = 400 N ; D_2 = 60 mm or R_2 = 30 mm ; D_1 = 90 mm or R_1 = 45 mm ; $\mu = \tan\phi = \mu_1 = 0.15$

We know that mean diameter of the screw,

$$d = d_0 - p/2 = 55 - 10/2 = 50 \text{ mm}$$

$$\therefore \qquad \tan\alpha = \frac{p}{\pi d} = \frac{10}{\pi \times 50} = 0.0637$$

and force required at the circumference of the screw,

$$P = W\tan(\alpha + \phi) = W\left[\frac{\tan\alpha + \tan\phi}{1 - \tan\alpha.\tan\phi}\right]$$

$$= 400\left[\frac{0.0637 + 0.15}{1 - 0.0637 \times 0.15}\right] = 86.4 \text{ N}$$

We know that mean radius of the flat surface,

$$R = \frac{R_1 + R_2}{2} = \frac{45 + 30}{2} = 37.5 \text{ mm}$$

$\therefore$ Total torque required,

$$T = P \times \frac{d}{2} + \mu_1.W.R = 86.4 \times \frac{50}{2} + 0.15 \times 400 \times 37.5 \text{ N-mm}$$

$$= 4410 \text{ N-mm} = 4.41 \text{ N-m} \qquad ...(\because \mu_1 = \mu)$$

Since the cutting speed is 6 m/min, therefore speed of the screw,

$$N = \frac{\text{Cutting speed}}{\text{Pitch}} = \frac{6}{0.01} = 600 \text{ r.p.m.}$$

and angular speed, $\omega = 2\,\pi \times 600/60 = 62.84$ rad/s

We know that power required to operate the nut

$$= T.\omega = 4.41 \times 62.84 = 277 \text{ W} = 0.277 \text{ kW} \quad \textbf{Ans.}$$

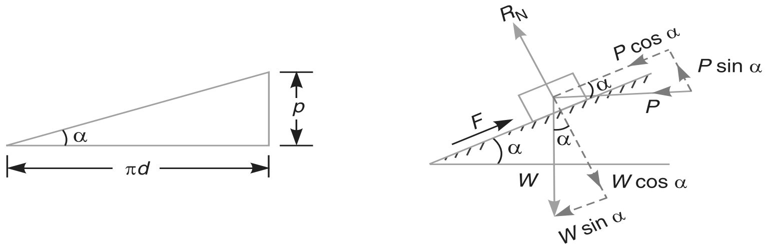
10.19. Torque Required to Lower the Load by a Screw Jack

We have discussed in Art. 10.18, that the principle on which the screw jack works is similar to that of an inclined plane. If one complete turn of a screw thread be imagined to be unwound from the body of the screw and developed, it will form an inclined plane as shown in Fig. 10.13 (*a*).

Let
p = Pitch of the screw,
d = Mean diameter of the screw,
α = Helix angle,
P = Effort applied at the circumference of the screw to lower the load,

W = Weight to be lowered, and

μ = Coefficient of friction between the screw and nut = $\tan \phi$, where ϕ is the friction angle.



(*a*) (*b*)

Fig. 10.13

From the geometry of the figure, we find that

$$\tan \alpha = p/\pi d$$

Since the load is being lowered, therefore the force of friction ($F = \mu.R_N$) will act upwards. All the forces acting on the screw are shown in Fig. 10.13 (*b*).

Resolving the forces along the plane,

$$P \cos \alpha = F - W \sin \alpha = \mu.R_N - W \sin \alpha \quad ...(i)$$

and resolving the forces perpendicular to the plane,

$$R_N = W \cos \alpha - P \sin \alpha \quad ...(ii)$$

Substituting this value of R_N in equation (*i*),

$$P \cos \alpha = \mu (W \cos \alpha - P \sin \alpha) - W \sin \alpha$$

$$= \mu.W \cos \alpha - \mu.P \sin \alpha - W \sin \alpha$$

or $$P \cos \alpha + \mu.P \sin \alpha = \mu.W \cos \alpha - W \sin \alpha$$

or $$P (\cos \alpha + \mu \sin \alpha) = W (\mu \cos \alpha - \sin \alpha)$$

$\therefore$ $$P = W \times \frac{(\mu \cos \alpha - \sin \alpha)}{(\cos \alpha + \mu \sin \alpha)}$$

Substituting the value of $\mu = \tan \phi$ in the above equation, we get

$$P = W \times \frac{(\tan \phi \cos \alpha - \sin \alpha)}{(\cos \alpha + \tan \phi \sin \alpha)}$$

Multiplying the numerator and denominator by $\cos \phi$,

$$P = W \times \frac{(\sin \phi \cos \alpha - \sin \alpha \cos \phi)}{(\cos \alpha \cos \phi + \sin \phi \sin \alpha)} = W \times \frac{\sin (\phi - \alpha)}{\cos (\phi - \alpha)}$$

$$= W \tan (\phi - \alpha)$$

$\therefore$ Torque required to overcome friction between the screw and nut,

$$T = P \times \frac{d}{2} = W \tan (\phi - \alpha) \frac{d}{2}$$

Note : When $\alpha > \phi$, then $P = \tan (\alpha - \phi)$.

Example 10.9. *The mean diameter of a square threaded screw jack is 50 mm. The pitch of the thread is 10 mm. The coefficient of friction is 0.15. What force must be applied at the end of a 0.7 m long lever, which is perpendicular to the longitudinal axis of the screw to raise a load of 20 kN and to lower it?*

Solution. Given : $d = 50$ mm $= 0.05$ m ; $p = 10$ mm ; $\mu = \tan \phi = 0.15$; $l = 0.7$ m ; $W = 20$ kN $= 20 \times 10^3$ N

We know that $$\tan \alpha = \frac{p}{\pi d} = \frac{10}{\pi \times 50} = 0.0637$$

Let P_1 = Force required at the end of the lever.

Force required to raise the load

We know that force required at the circumference of the screw,

$$P = W \tan(\alpha + \phi) = W \left[\frac{\tan \alpha + \tan \phi}{1 - \tan \alpha . \tan \phi}\right]$$

$$= 20 \times 10^3 \left[\frac{0.0637 + 0.15}{1 - 0.0637 \times 0.15}\right] = 4314 \text{ N}$$

Now the force required at the end of the lever may be found out by the relation,

$$P_1 \times l = P \times d/2$$

$$\therefore \quad P_1 = \frac{P \times d}{2\,l} = \frac{4314 \times 0.05}{2 \times 0.7} = 154 \text{ N} \quad \textbf{Ans.}$$

Force required to lower the load

We know that the force required at the circumference of the screw,

$$P = W \tan(\phi - \alpha) = W \left[\frac{\tan \phi - \tan \alpha}{1 + \tan \phi . \tan \alpha}\right]$$

$$= 20 \times 10^3 \left[\frac{0.15 - 0.0637}{1 + 0.15 \times 0.0637}\right] = 1710 \text{ N}$$

Now the force required at the end of the lever may be found out by the relation,

$$P_1 \times l = P \times \frac{d}{2} \quad \text{or} \quad P_1 = \frac{P \times d}{2l} = \frac{1710 \times 0.05}{2 \times 0.7} = 61 \text{ N} \quad \textbf{Ans.}$$

10.20. Efficiency of a Screw Jack

The efficiency of a screw jack may be defined as **the ratio between the ideal effort** (*i.e.* the effort required to move the load, neglecting friction) to **the actual effort** (*i.e.* the effort required to move the load taking friction into account).

We know that the effort required to lift the load (W) when friction is taken into account,

$$P = W \tan(\alpha + \phi) \qquad \ldots(i)$$

where α = Helix angle,

ϕ = Angle of friction, and

μ = Coefficient of friction, between the screw and nut = $\tan \phi$.

If there would have been no friction between the screw and the nut, then ϕ will be equal to zero. The value of effort P_0 necessary to raise the load, will then be given by the equation,

$$P_0 = W \tan \alpha \qquad [i.e. \text{ Putting } \phi = 0 \text{ in equation } (i)]$$

$$\therefore \text{ Efficiency, } \eta = \frac{\text{Ideal effort}}{\text{Actual effort}} = \frac{P_0}{P} = \frac{W \tan \alpha}{W \tan(\alpha + \phi)} = \frac{\tan \alpha}{\tan(\alpha + \phi)}$$

which shows that the efficiency of a screw jack, is independent of the load raised.

In the above expression for efficiency, only the screw friction is considered. However, if the screw friction and the collar friction is taken into account, then

$$\therefore \quad \eta = \frac{\text{Torque required to move the load, neglecting friction}}{\text{Torque required to move the load, including screw and collar friction}}$$

$$= \frac{T_0}{T} = \frac{P_0 \times d/2}{P \times d/2 + \mu_1.W.R}$$

Note: The efficiency of the screw jack may also be defined as **the ratio of mechanical advantage to the velocity ratio.**

We know that mechanical advantage,

$$M.A. = \frac{W}{P_1} = \frac{W \times 2l}{P \times d} = \frac{W \times 2l}{W \tan(\alpha + \phi) d} = \frac{2l}{\tan(\alpha + \phi) d} \qquad \text{...(Refer Art 10.17)}$$

and velocity ratio, $$V.R. = \frac{\text{Distance moved by the effort } (P_1)\text{, in one revolution}}{\text{Distance moved by the load } (W)\text{, in one revolution}}$$

$$= \frac{2\pi l}{p} = \frac{2\pi l}{\tan\alpha \times \pi d} = \frac{2l}{\tan\alpha \times d} \qquad \text{...}(\because \tan\alpha = p/\pi d)$$

$$\therefore \text{ Efficiency, } \eta = \frac{M.A.}{V.R.} = \frac{2l}{\tan(\alpha + \phi) d} \times \frac{\tan\alpha \times d}{2l} = \frac{\tan \times \alpha}{\tan(\alpha + \phi)}$$

10.21. Maximum Efficiency of a Screw Jack

We have seen in Art. 10.20 that the efficiency of a screw jack,

$$\eta = \frac{\tan\alpha}{\tan(\alpha + \theta)} = \frac{\dfrac{\sin\alpha}{\cos\alpha}}{\dfrac{\sin(\alpha + \phi)}{\cos(\alpha + \phi)}} = \frac{\sin\alpha \times \cos(\alpha + \phi)}{\cos\alpha \times \sin(\alpha + \phi)} \qquad \text{...}(i)$$

$$= \frac{2\sin\alpha \times \cos(\alpha + \phi)}{2\cos\alpha \times \sin(\alpha + \phi)}$$

...(Multiplying the numerator and denominator by 2)

$$= \frac{\sin(2\alpha + \phi) - \sin\phi}{\sin(2\alpha + \phi) + \sin\phi} \qquad \text{...}(ii)$$

$$\text{...}\left[\begin{array}{l} \because\ 2\sin A\cos B = \sin(A+B) + \sin(A-B) \\ \quad 2\cos A\sin B = \sin(A+B) - \sin(A-B) \end{array}\right]$$

The efficiency given by equation *(ii)* is maximum when $\sin(2\alpha + \phi)$ is maximum, *i.e.* when

$$\sin(2\alpha + \phi) = 1 \quad \text{or} \quad \text{when } 2\alpha + \phi = 90°$$

$$\therefore \qquad 2\alpha = 90° - \phi \quad \text{or} \quad \alpha = 45° - \phi/2$$

Substituting the value of 2α in equation *(ii)*, we have maximum efficiency,

$$\eta_{max} = \frac{\sin(90° - \phi + \phi) - \sin\phi}{\sin(90° - \phi + \phi) + \sin\phi} = \frac{\sin 90° - \sin\phi}{\sin 90° + \sin\phi} = \frac{1 - \sin\phi}{1 + \sin\phi}$$

Example 10.10. *The pitch of 50 mm mean diameter threaded screw of a screw jack is 12.5 mm. The coefficient of friction between the screw and the nut is 0.13. Determine the torque required on the screw to raise a load of 25 kN, assuming the load to rotate with the screw. Determine the ratio of the torque required to raise the load to the torque required to lower the load and also the efficiency of the machine.*

Solution. Given : $d = 50$ mm ; $p = 12.5$ mm ; $\mu = \tan\phi = 0.13$; $W = 25$ kN $= 25 \times 10^3$ N

We know that, $\tan\alpha = \dfrac{p}{\pi\, d} = \dfrac{12.5}{\pi \times 50} = 0.08$

and force required on the screw to raise the load,

$$P = W\tan(\alpha + \phi) = W\left[\frac{\tan\phi - \tan\alpha}{1 + \tan\phi.\tan\alpha}\right]$$

$$= 25 \times 10^3\left[\frac{0.08 + 0.13}{1 - 0.08 \times 0.13}\right] = 5305 \text{ N}$$

Torque required on the screw

We know that the torque required on the screw to raise the load,

$$T_1 = P \times d/2 = 5305 \times 50/2 = 132\,625 \text{ N-mm} \quad \textbf{Ans.}$$

Ratio of the torques required to raise and lower the load

We know that the force required on the screw to lower the load,

$$P = W\tan(\phi - \alpha) = W\left[\frac{\tan\phi - \tan\alpha}{1 + \tan\phi.\tan\alpha}\right]$$

$$= 25 \times 10^3\left[\frac{0.13 + 0.08}{1 + 0.13 \times 0.08}\right] = 1237 \text{ N}$$

and torque required to lower the load

$$T_2 = P \times d/2 = 1237 \times 50/2 = 30\,905 \text{ N-mm}$$

$\therefore$ Ratio of the torques required,

$$= T_1 / T_2 = 132625/30925 = 4.3 \quad \textbf{Ans.}$$

Efficiency of the machine

We know that the efficiency,

$$\eta = \frac{\tan\alpha}{\tan(\alpha + \phi)} = \frac{\tan\alpha\,(1 - \tan\alpha.\tan\phi)}{\tan\alpha + \tan\phi} = \frac{0.08\,(1 - 0.08 \times 0.13)}{0.08 + 0.13}$$

$$= 0.377 = 37.7\% \quad \textbf{Ans.}$$

Example 10.11. *The mean diameter of the screw jack having pitch of 10 mm is 50 mm. A load of 20 kN is lifted through a distance of 170 mm. Find the work done in lifting the load and efficiency of the screw jack when*

1. *the load rotates with the screw, and*
2. *the load rests on the loose head which does not rotate with the screw.*

The external and internal diameter of the bearing surface of the loose head are 60 mm and 10 mm respectively. The coefficient of friction for the screw as well as the bearing surface may be taken as 0.08.

Solution. Given : $p = 10$ mm ; $d = 50$ mm ; $W = 20$ kN $= 20 \times 10^3$ N ; $D_2 = 60$ mm or $R_2 = 30$ mm ; $D_1 = 10$ mm or $R_1 = 5$ mm ; $\mu = \tan\phi = \mu_1 = 0.08$

We know that $\tan\alpha = \dfrac{p}{\pi\, d} = \dfrac{10}{\pi \times 50} = 0.0637$

∴ Force required at the circumference of the screw to lift the load,

$$P = W \tan(\alpha + \phi) = W\left[\frac{\tan\alpha + \tan\phi}{1 - \tan\alpha.\tan\phi}\right]$$

$$= 20 \times 10^3 \left[\frac{0.0637 + 0.08}{1 - 0.0637 \times 0.08}\right] = 2890 \text{ N}$$

and torque required to overcome friction at the screw,

$$T = P \times d/2 = 2890 \times 50/2 = 72\,250 \text{ N-mm} = 72.25 \text{ N-m}$$

Since the load is lifted through a vertical distance of 170 mm and the distance moved by the screw in one rotation is 10 mm (equal to pitch), therefore number of rotations made by the screw,

$$N = 170/10 = 17$$

1. ***When the load rotates with the screw***

We know that work done in lifting the load

$$= T \times 2\pi N = 72.25 \times 2\pi \times 17 = 7718 \text{ N-m} \quad \textbf{Ans.}$$

and efficiency of the screw jack,

$$\eta = \frac{\tan\alpha}{\tan(\alpha + \phi)} = \frac{\tan\alpha(1 - \tan\alpha.\tan\phi)}{\tan\alpha + \tan\alpha}$$

$$= \frac{0.0637(1 - 0.0637 \times 0.08)}{0.0637 + 0.08} = 0.441 \text{ or } 44.1\% \quad \textbf{Ans.}$$

2. ***When the load does not rotate with the screw***

We know that mean radius of the bearing surface,

$$R = \frac{R_1 + R_2}{2} = \frac{30 + 5}{2} = 17.5 \text{ mm}$$

and torque required to overcome friction at the screw and the collar,

$$T = P \times d/2 + \mu_1.W.R$$

$$= 2890 \times 50/2 + 0.08 \times 20 \times 10^3 \times 17.5 = 100\ 250 \text{ N-mm}$$

$$= 100.25 \text{ N-m}$$

∴ Work done by the torque in lifting the load

$$= T \times 2\pi N = 100.25 \times 2\pi \times 17 = 10\ 710 \text{ N-m} \quad \textbf{Ans.}$$

We know that the torque required to lift the load, neglecting friction,

$$T_0 = P_0 \times d/2 = W\tan\alpha \times d/2 \qquad ...(\because P_0 = W\tan\alpha)$$

$$= 20 \times 10^3 \times 0.0637 \times 50/2 = 31\ 850 \text{ N-mm} = 31.85 \text{ N-m}$$

∴ Efficiency of the screw jack,

$$\eta = T_0/T = 31.85/100.25 = 0.318 \text{ or } 31.8\% \quad \textbf{Ans.}$$

10.22. Over Hauling and Self Locking Screws

We have seen in Art. 10.20 that the effort required at the circumference of the screw to lower the load is

$$P = W\tan(\phi - \alpha)$$

and the torque required to lower the load

$$T = P \times \frac{d}{2} = W \tan (\phi - \alpha) \frac{d}{2}$$

In the above expression, if $\phi < \alpha$, then torque required to lower the load will be ***negative***. In other words, the load will start moving downward without the application of any torque. Such a condition is known as ***over haulding of screws.*** If however, $\phi > \alpha$, the torque required to lower the load will ***positive***, indicating that an effort is applied to lower the load. Such a screw is known as ***self locking screw.*** In other words, a screw will be self locking if the friction angle is greater than helix angle or coefficient of friction is greater than tangent of helix angle *i.e.* μ or $\tan \phi > \tan \alpha$.

10.23. Efficiency of Self Locking Screws

We know that efficiency of the screw,

$$\eta = \frac{\tan \alpha}{\tan (\alpha + \phi)}$$

and for self locking screws, $\phi \geq \alpha$ or $\alpha \leq \phi$.

$\therefore$ Efficiency of self locking screws,

$$\eta \leq \frac{\tan \phi}{\tan (\phi + \phi)} \leq \frac{\tan \phi}{\tan 2\phi} \leq \frac{\tan \phi (1 - \tan^2 \phi)}{2 \tan \phi}$$

$$\leq \frac{1}{2} - \frac{\tan^2 \phi}{2} \qquad \ldots \left(\because \tan 2\phi = \frac{2 \tan \phi}{1 - \tan^2 \phi} \right)$$

From this expression we see that efficiency of self locking screws is less than $\frac{1}{2}$ or 50%. If the efficiency is more than 50%, then the screw is said to be overhauling,

Note : It can also be proved as follows :

Let W = Load to be lifted, and

h = Distance through which the load is lifted.

$\therefore$ Output = $W.h$

and $$\text{Input} = \frac{\text{Output}}{\eta} = \frac{W.h}{\eta}$$

$\therefore$ Work lost in over coming friction.

$$= \text{Input} - \text{Output} = \frac{W.h}{\eta} - W.h = W.h \left(\frac{1}{\eta} - 1 \right)$$

For self locking,, $W.h \left(\frac{1}{\eta} - 1 \right) \leq W.h$

$\therefore$ $\frac{1}{\eta} - 1 \leq 1$ or $\eta \leq \frac{1}{2}$ or 50%

Example 10.12. *A load of 10 kN is raised by means of a screw jack, having a square threaded screw of 12 mm pitch and of mean diameter 50 mm. If a force of 100 N is applied at the end of a lever to raise the load, what should be the length of the lever used? Take coefficient of friction = 0.15. What is the mechanical advantage obtained? State whether the screw is self locking.*

Solution. Given : $W = 10$ kN $= 10 \times 10^3$ N ; $p = 12$ mm ; $d = 50$ mm ; $P_1 = 100$ N ; $\mu = \tan \phi = 0.15$

Length of the lever

Let l = Length of the lever.

We know that $\tan\alpha = \dfrac{p}{\pi d} = \dfrac{12}{\pi \times 50} = 0.0764$

$\therefore$ Effort required at the circumference of the screw to raise the load,

$$P = W\tan(\alpha + \phi) = W\left[\frac{\tan\alpha + \tan\phi}{1 - \tan\alpha.\tan\phi}\right]$$

$$= 10 \times 10^3\left[\frac{0.0764 + 0.15}{1 - 0.0764 \times 0.15}\right] = 2290\text{ N}$$

and torque required to overcome friction,

$$T = P \times d/2 = 2290 \times 50/2 = 57\ 250\text{ N-mm} \qquad ...(i)$$

We know that torque applied at the end of the lever,

$$T = P_1 \times l = 100 \times l\text{ N-mm} \qquad ...(ii)$$

Equating equations (*i*) and (*ii*)

$$l = 57\ 250/100 = 572.5\text{ mm}$$ **Ans.**

Mechanical advantage

We know that mechanical advantage,

$$M.A. = \frac{W}{P_1} = \frac{10 \times 10^3}{100} = 100$$ **Ans.**

Self locking of the screw

We know that efficiency of the screw jack,

$$\eta = \frac{\tan\alpha}{\tan(\alpha + \phi)} = \frac{\tan\alpha(1 - \tan\alpha.\tan\phi)}{\tan\alpha + \tan\phi}$$

$$= \frac{0.0764(1 - 0.0764 \times 0.15)}{0.0764 + 0.15} = \frac{0.0755}{0.2264} = 0.3335 \text{ or } 33.35\%$$

Since the efficiency of the screw jack is less than 50%, therefore the screw is a self locking screw. **Ans.**

10.25. Friction in Journal Bearing-Friction Circle

A journal bearing forms a turning pair as shown in Fig. 10.15 (*a*). The fixed outer element of a turning pair is called a ***bearing*** and that portion of the inner element (*i.e.* shaft) which fits in the bearing is called a ***journal.*** The journal is slightly less in diameter than the bearing, in order to permit the free movement of the journal in a bearing.

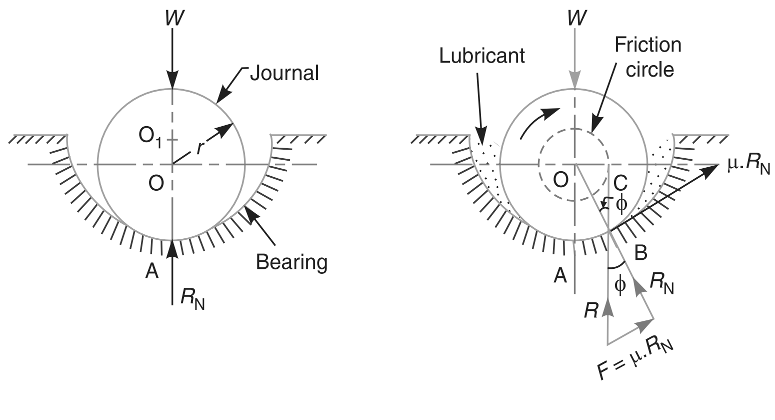


Fig. 10.15. Friction in journal bearing.

When the bearing is not lubricated (or the journal is stationary), then there is a line contact between the two elements as shown in Fig. 10.15 (*a*). The load W on the journal and normal reaction R_N (equal to W) of the bearing acts through the centre. The reaction R_N acts vertically upwards at point A. This point A is known as ***seat*** or ***point of pressure.***

Now consider a shaft rotating inside a bearing in clockwise direction as shown in Fig. 10.15 (*b*). The lubricant between the journal and bearing forms a thin layer which gives rise to a greasy friction. Therefore, the reaction R does not act vertically upward, but acts at another point of pressure B. This is due to the fact that when shaft rotates, a frictional force $F = \mu R_N$ acts at the circumference of the shaft which has a tendency to rotate the shaft in opposite direction of motion and this shifts the point A to point B.

In order that the rotation may be maintained, there must be a couple rotating the shaft.

Let ϕ = Angle between R (resultant of F and R_N) and R_N,

μ = Coefficient of friction between the journal and bearing,

T = Frictional torque in N-m, and

r = Radius of the shaft in metres.

For uniform motion, the resultant force acting on the shaft must be zero and the resultant turning moment on the shaft must be zero. In other words,

$$R = W, \text{ and } T = W \times OC = W \times OB \sin \phi = W.r \sin \phi$$

Since ϕ is very small, therefore substituting $\sin \phi = \tan \phi$

$$\therefore \quad T = W.r \tan \phi = \mu.W.r \qquad ...(\because \mu = \tan \phi)$$

If the shaft rotates with angular velocity ω rad/s, then power wasted in friction,

$$P = T.\omega = T \times 2\pi N/60 \text{ watts}$$

where N = Speed of the shaft in r.p.m.

Notes : **1.** If a circle is drawn with centre O and radius $OC = r \sin \phi$, then this circle is called the ***friction circle*** of a bearing.

2. The force R exerted by one element of a turning pair on the other element acts along a tangent to the friction circle.

Example 10.15. *A 60 mm diameter shaft running in a bearing carries a load of 2000 N. If the coefficient of friction between the shaft and bearing is 0.03, find the power transmitted when it runs at 1440 r.p.m.*

Solution. Given : $d = 60$ mm or $r = 30$ mm $= 0.03$ m ; $W = 2000$ N ; $\mu = 0.03$; $N = 1440$ r.p.m. or $\omega = 2\pi \times 1440/60 = 150.8$ rad/s

We know that torque transmitted,

$$T = \mu.W.r = 0.03 \times 2000 \times 0.03 = 1.8 \text{ N-m}$$

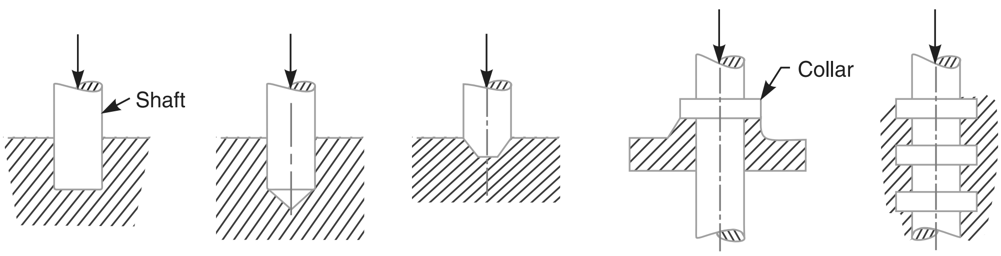
$\therefore$ Power transmitted, $P = T.\omega = 1.8 \times 150.8 = 271.4$ W **Ans.**

10.26. Friction of Pivot and Collar Bearing

The rotating shafts are frequently subjected to axial thrust. The bearing surfaces such as pivot and collar bearings are used to take this axial thrust of the rotating shaft. The propeller shafts of ships, the shafts of steam turbines, and vertical machine shafts are examples of shafts which carry an axial thrust.

The bearing surfaces placed at the end of a shaft to take the axial thrust are known as ***pivots***. The pivot may have a flat surface or conical surface as shown in Fig. 10.16 (*a*) and (*b*) respectively. When the cone is truncated, it is then known as truncated or trapezoidal pivot as shown in Fig. 10.16 (*c*).

The collar may have flat bearing surface or conical bearing surface, but the flat surface is most commonly used. There may be a single collar, as shown in Fig. 10.16 (*d*) or several collars along the length of a shaft, as shown in Fig. 10.16 (*e*) in order to reduce the intensity of pressure.



(*a*) Flat pivot. (*b*) Conical pivot. (*c*) Truncated pivot. (*d*) Single flat collar. (*e*) Multiple flat collar.

Fig. 10.16. Pivot and collar bearings.

In modern practice, ball and roller thrust bearings are used when power is being transmitted and when thrusts are large as in case of propeller shafts of ships.

A little consideration will show that in a new bearing, the contact between the shaft and bearing may be good over the whole surface. In other words, we can say that the pressure over the rubbing surfaces is uniformly distributed. But when the bearing becomes old, all parts of the rubbing surface will not move with the same velocity, because the velocity of rubbing surface increases with the distance from the axis of the bearing. This means that wear may be different at different radii and this causes to alter the distribution of pressure. Hence, in the study of friction of bearings, it is assumed that

Collar bearing.

1. The pressure is uniformly distributed throughout the bearing surface, and
2. The wear is uniform throughout the bearing surface.

10.27. Flat Pivot Bearing

When a vertical shaft rotates in a flat pivot bearing (known as **foot step bearing**), as shown in Fig. 10.17, the sliding friction will be along the surface of contact between the shaft and the bearing.

Let W = Load transmitted over the bearing surface,

R = Radius of bearing surface,

p = Intensity of pressure per unit area of bearing surface between rubbing surfaces, and

μ = Coefficient of friction.

We will consider the following two cases :

1. When there is a uniform pressure ; and
2. When there is a uniform wear.

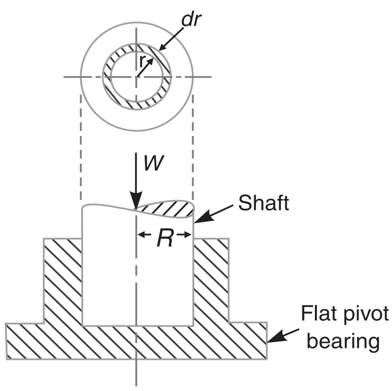


Fig. 10.17. Flat pivot or footstep bearing.

1. ***Considering unifrom pressure***

When the pressure is uniformly distributed over the bearing area, then

$$p = \frac{W}{\pi R^2}$$

Consider a ring of radius r and thickness dr of the bearing area.

$\therefore$ Area of bearing surface, $A = 2\pi r.dr$

Load transmitted to the ring,

$$\delta W = p \times A = p \times 2\,\pi\, r.dr \qquad ...(i)$$

Frictional resistance to sliding on the ring acting tangentially at radius r,

$$F_r = \mu.\delta W = \mu\, p \times 2\pi\, r.dr = 2\pi\, \mu.p.r.dr$$

$\therefore$ Frictional torque on the ring,

$$T_r = F_r \times r = 2\pi\, \mu\, p\, r.dr \times r = 2\,\pi\,\mu\, p\, r^2\, dr \qquad ...(ii)$$

Integrating this equation within the limits from 0 to R for the total frictional torque on the pivot bearing.

$\therefore$ Total frictional torque, $$T = \int_0^R 2\pi\mu\, p\, r^2\, dr = 2\pi\mu\, p \int_0^R r^2\, dr$$

$$= 2\pi\mu\, p \left[\frac{r^3}{3}\right]_0^R = 2\pi\mu\, p \times \frac{R^3}{3} = \frac{2}{3} \times \pi\mu.p.R^3$$

$$= \frac{2}{3} \times \pi\mu \times \frac{W}{\pi R^2} \times R^3 = \frac{2}{3} \times \mu.W.R \qquad \ldots\left(\because\ p = \frac{W}{\pi R^2}\right)$$

When the shaft rotates at ω rad/s, then power lost in friction,

$$P = T.\omega = T \times 2\pi N/60 \qquad \ldots(\because\ \omega = 2\pi N/60)$$

where N = Speed of shaft in r.p.m.

2. ***Considering uniform wear***

We have already discussed that the rate of wear depends upon the intensity of pressure (p) and the velocity of rubbing surfaces (v). It is assumed that the rate of wear is proportional to the product of intensity of pressure and the velocity of rubbing surfaces (*i.e.* $p.v.$.). Since the velocity of rubbing surfaces increases with the distance (*i.e.* radius r) from the axis of the bearing, therefore for uniform wear

$$p.r = C \text{ (a constant)} \quad \text{or} \quad p = C/r$$

and the load transmitted to the ring,

$$\delta W = p \times 2\pi r.dr \qquad \ldots\text{[From equation } (i)]$$

$$= \frac{C}{r} \times 2\pi r.dr = 2\pi C.dr$$

$\therefore$ Total load transmitted to the bearing

$$W = \int_0^R 2\pi\, C.dr = 2\pi\, C[r]_0^R = 2\pi C.R \quad \text{or} \quad C = \frac{W}{2\pi R}$$

We know that frictional torque acting on the ring,

$$T_r = 2\pi\mu\, p\, r^2\, dr = 2\pi\mu \times \frac{C}{r} \times r^2\, dr \qquad \ldots\left(\because\ p = \frac{C}{r}\right)$$

$$= 2\pi\, \mu.C.r\, dr \qquad \ldots(iii)$$

$\therefore$ Total frictional torque on the bearing,

$$T = \int_0^R 2\pi\, \mu.C.r.dr = 2\pi\mu.C\left[\frac{r^2}{2}\right]_0^R$$

$$= 2\pi\mu.C \times \frac{R^2}{2} = \pi\mu.C.R^2$$

$$= \pi\mu \times \frac{W}{2\pi R} \times R^2 = \frac{1}{2} \times \mu.W.R \qquad \ldots\left(\because\ C = \frac{W}{2\pi R}\right)$$

Example 10.16. *A vertical shaft 150 mm in diameter rotating at 100 r.p.m. rests on a flat end footstep bearing. The shaft carries a vertical load of 20 kN. Assuming uniform pressure distribution and coefficient of friction equal to 0.05, estimate power lost in friction.*

Solution. Given : D = 150 mm or R = 75 mm = 0.075 m ; N = 100 r.p.m or $\omega = 2\,\pi \times 100/60$ = 10.47 rad/s ; W = 20 kN = 20×10^3 N ; μ = 0.05

We know that for uniform pressure distribution, the total frictional torque,

$$T = \frac{2}{3} \times \mu.W.R = \frac{2}{3} \times 0.05 \times 20 \times 10^3 \times 0.075 = 50 \text{ N-m}$$

∴ Power lost in friction,

$$P = T.\omega = 50 \times 10.47 = 523.5 \text{ W} \quad \textbf{Ans.}$$

10.28. Conical Pivot Bearing

The conical pivot bearing supporting a shaft carrying a load W is shown in Fig. 10.18.

Let P_n = Intensity of pressure normal to the cone,

α = Semi angle of the cone,

μ = Coefficient of friction between the shaft and the bearing, and

R = Radius of the shaft.

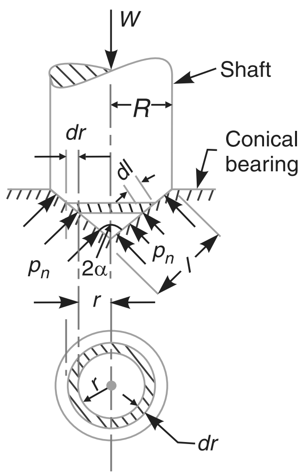


Fig. 10.18.
Conical pivot bearing.

Consider a small ring of radius r and thickness dr. Let dl is the length of ring along the cone, such that

$$dl = dr \text{ cosec } \alpha$$

∴ Area of the ring,

$$A = 2\pi r.dl = 2\pi r.dr \text{ cosec } \alpha \quad ...(\because dl = dr \text{ cosec } \alpha)$$

1. ***Considering uniform pressure***

We know that normal load acting on the ring,

$$\delta W_n = \text{Normal pressure} \times \text{Area} = p_n \times 2\pi r.dr \text{ cosec } \alpha$$

and vertical load acting on the ring,

$$^*\delta W = \text{Vertical component of } \delta W_n = \delta W_n.\sin\alpha$$
$$= p_n \times 2\pi r.dr \text{ cosec } \alpha.\ \sin\alpha = p_n \times 2\pi\ r.dr$$

∴ Total vertical load transmitted to the bearing,

$$W = \int_0^R p_n \times 2\pi r.dr = 2\pi\, p_n \left[\frac{r^2}{2}\right]_0^R = 2\pi\, p_n \times \frac{R^2}{2} = \pi R^2.p_n$$

or $$p_n = W/\pi R^2$$

We know that frictional force on the ring acting tangentially at radius r,

$$F_r = \mu.\delta W_n = \mu.p_n.2\pi r.dr \text{ cosec } \alpha = 2\pi\mu.p_n.\text{cosec } \alpha.r.dr$$

and frictional torque acting on the ring,

$$T_r = F_r \times r = 2\pi\mu.p_n.\text{cosec } \alpha.r.dr \times r = 2\pi\mu.p_n \text{ cosec}\alpha.r^2.dr$$

---

* The vertical load acting on the ring is also given by

$$\delta W = \text{Vertical component of } p_n \times \text{Area of the ring}$$
$$= p_n \sin\alpha \times 2\pi r.dr.\text{cosec } \alpha = p_n \times 2\pi r.dr$$

Integrating the expression within the limits from 0 to R for the total frictional torque on the conical pivot bearing.

∴ Total frictional torque,

$$T = \int_0^R 2\,\pi\,\mu.p_n \text{ cosec } \alpha.r^2\ dr = 2\,\pi\mu.p_n.\text{cosec } \alpha\left[\frac{r^3}{3}\right]_0^R$$

$$= 2\pi\mu.p_n.\text{cosec } \alpha \times \frac{R^3}{3} = \frac{2\pi\ R^3}{3} \times \mu.p_n.\text{cosec } \alpha \qquad \text{...}(i)$$

Substituting the value of p_n in equation (i),

$$T = \frac{2\pi R^3}{3} \times \pi \times \frac{W}{\pi R^2} \times \text{cosec } \alpha = \frac{2}{3} \times \mu.W.R.\text{ cosec } \alpha$$

Note : If slant length (l) of the cone is known, then

$$T = \frac{2}{3} \times \mu.W.l \qquad \text{...}(\because\ l = R \text{ cosec } \alpha)$$

2. ***Considering uniform wear***

In Fig. 10.18, let p_r be the normal intensity of pressure at a distance r from the central axis. We know that, in case of uniform wear, the intensity of pressure varies inversely with the distance.

$$\therefore \qquad p_r.r = C \text{ (a constant)} \quad \text{or} \quad p_r = C/r$$

and the load transmitted to the ring,

$$\delta W = p_r \times 2\pi r.dr = \frac{C}{r} \times 2\pi r.dr = 2\pi C.dr$$

∴ Total load transmitted to the bearing,

$$W = \int_0^R 2\pi C.dr = 2\pi C\,[r]_0^R = 2\pi C.R \quad \text{or} \quad C = \frac{W}{2\pi R}$$

We know that frictional torque acting on the ring,

$$T_r = 2\pi\mu.p_r.\text{cosec } \alpha.r^2.dr = 2\pi\mu \times \frac{C}{r} \times \text{cosec}\,\alpha.r^2.dr$$

$$= 2\pi\mu.C.\text{cosec } \alpha.r.dr$$

∴ Total frictional torque acting on the bearing,

$$T = \int_0^R 2\pi\ \mu.C.\text{cosec } \alpha.r.dr = 2\pi\mu.C.\text{cosec } \alpha\left[\frac{r^2}{2}\right]_0^R$$

$$= 2\pi\ \mu.C.\text{cosec } \alpha \times \frac{R^2}{2} = \pi\mu.C.\text{cosec } \alpha.R^2$$

Substituting the value of C, we have

$$T = \pi\mu \times \frac{W}{2\pi R} \times \text{cosec } \alpha.R^2 = \frac{1}{2} \times \mu.W.R \text{ cosec } \alpha = \frac{1}{2} \times \mu.W.l$$

10.29. Trapezoidal or Truncated Conical Pivot Bearing

If the pivot bearing is not conical, but a frustrum of a cone with r_1 and r_2, the external and internal radius respectively as shown in Fig. 10.19, then

Area of the bearing surface,

$$A = \pi[(r_1)^2 - (r_2)^2]$$

$\therefore$ Intensity of uniform pressure,

$$p_n = \frac{W}{A} = \frac{W}{\pi[(r_1)^2 - (r_2)^2]} \qquad ...(i)$$

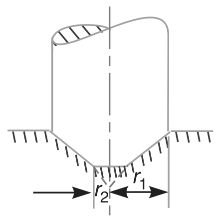


Fig.10.19. Trapezoidal pivot bearing.

1. ***Considering uniform pressure***

The total torque acting on the bearing is obtained by integrating the value of T_r (as discussed in Art. 10.27) within the limits r_1 and r_2.

$\therefore$ Total torque acting on the bearing,

$$T = \int_{r_2}^{r_1} 2\pi \propto . p_n \text{ cosec } \alpha r^2.dr = 2\pi \propto . p_n.\text{cosec } \alpha \left[\frac{r^3}{3}\right]_{r_2}^{r_1}$$

$$= 2\pi \propto . p_n.\text{cosec } \alpha \left[\frac{(r_1)^3 - (r_2)^3}{3}\right]$$

Substituting the value of p_n from equation (*i*),

$$T = 2\pi.\propto \cdot \frac{W}{\pi[(r_1)^2 - (r_2)^2]} \cdot \text{ cosec}\alpha \left[\frac{(r_1)^3 - (r_2)^3}{3}\right]$$

$$= \frac{2}{3} \cdot \propto .W.\text{cosec } \alpha \left[\frac{(r_1)^3 - (r_2)^3}{(r_1)^2 - (r_2)^2}\right]$$

2. ***Considering uniform wear***

We have discussed in Art. 10.26 that the load transmitted to the ring,

$$\delta W = 2\pi C.dr$$

$\therefore$ Total load transmitted to the ring,

$$W = \int_{r_2}^{r_1} 2\pi\, C.dr = 2\pi C[r]_{r_2}^{r_1} = 2\pi C(r_1 - r_2)$$

or

$$C = \frac{W}{2\pi(r_1 - r_2)} \qquad ...(ii)$$

We know that the torque acting on the ring, considering uniform wear, is

$$T_r = 2\pi \propto .C \text{ cosec } \alpha.r.dr$$

$\therefore$ Total torque acting on the bearing,

$$T = \int_{r_2}^{r_1} 2\pi \propto .C \text{ cosec } \alpha r.dr = 2\pi \propto C.\text{cosec } \alpha \left[\frac{r^2}{2}\right]_{r_2}^{r_1}$$

$$= \pi \propto C.\text{cosec } \alpha\left[(r_1)^2 - (r_2)^2\right]$$

Substituting the value of C from equation ***(ii)***, we get

$$T = \pi \propto \cdot \frac{W}{2\pi(r_1 - r_2)} \cdot \operatorname{cosec} \alpha \; [(r_1)^2 - (r_2)^2]$$

$$= \frac{1}{2} \cdot \propto .W\,(r_1 + r_2) \operatorname{cosec} \alpha = \propto .W.R \operatorname{cosec} \alpha$$

where R = Mean radius of the bearing $= \dfrac{r_1 + r_2}{2}$

Example 10.17. *A conical pivot supports a load of 20 kN, the cone angle is 120º and the intensity of normal pressure is not to exceed 0.3 N/mm². The external diameter is twice the internal diameter. Find the outer and inner radii of the bearing surface. If the shaft rotates at 200 r.p.m. and the coefficient of friction is 0.1, find the power absorbed in friction. Assume uniform pressure.*

Solution. Given : $W = 20$ kN $= 20 \times 10^3$ N ; $2\,\alpha = 120°$ or $\alpha = 60°$; $p_n = 0.3$ N/mm² ; $N = 200$ r.p.m. or $\omega = 2\,\pi \times 200/60 = 20.95$ rad/s ; $\propto = 0.1$

Outer and inner radii of the bearing surface

Let r_1 and r_2 = Outer and inner radii of the bearing surface, in mm.

Since the external diameter is twice the internal diameter, therefore

$$r_1 = 2\,r_2$$

We know that intensity of normal pressure (p_n),

$$0.3 = \frac{W}{\pi[(r_1)^2 - (r_2)^2]} = \frac{20 \cdot 10^3}{\pi[(2r_2)^2 - (r_2)^2]} = \frac{2.12 \cdot 10^3}{(r_2)^2}$$

$\therefore$ $(r_2)^2 = 2.12 \cdot 10^3 / 0.3 = 7.07 \cdot 10^3$ or $r_2 = 84$ mm **Ans.**

and $r_1 = 2\,r_2 = 2 \times 84 = 168$ mm **Ans.**

Power absorbed in friction

We know that total frictional torque (assuming uniform pressure),

$$T = \frac{2}{3} \cdot \propto .W.\operatorname{cosec} \alpha \left[\frac{(r_1)^3 - (r_2)^3}{(r_1)^2 - (r_2)^2}\right]$$

$$= \frac{2}{3} \cdot 0.1 \cdot 20 \; 10^3 \; \operatorname{cosec} 60° \left[\frac{(168)^3 - (84)^3}{(168)^2 - (84)^2}\right] \text{N-mm}$$

$$= 301760 \text{ N-mm} = 301.76 \text{ N-m}$$

$\therefore$ Power absorbed in friction,

$$P = T.\omega = 301.76 \times 20.95 = 6322 \text{ W} = 6.322 \text{ kW} \quad \textbf{Ans.}$$

Example 10.18. *A conical pivot bearing supports a vertical shaft of 200 mm diameter. It is subjected to a load of 30 kN. The angle of the cone is 120º and the coefficient of friction is 0.025. Find the power lost in friction when the speed is 140 r.p.m., assuming* **1.** *uniform pressure ; and* **2.** *uniform wear.*

Solution. Given : $D = 200$ mm or $R = 100$ mm $= 0.1$ m ; $W = 30$ kN $= 30 \times 10^3$ N ; $2\,\alpha = 120°$ or $\alpha = 60°$; $\propto = 0.025$; $N = 140$ r.p.m. or $\omega = 2\,\pi \times 140/160 = 14.66$ rad/s

1. ***Power lost in friction assuming uniform pressure***

We know that total frictional torque,

$$T = \frac{2}{3} \cdot \propto .W.R. \operatorname{cosec} \alpha$$

$$= \frac{2}{3} \times 0.025 \times 30 \times 10^3 \times 0.1 \times \text{cosec } 60^\circ = 57.7 \text{ N-m}$$

$\therefore$ Power lost in friction,

$$P = T.\omega = 57.7 \times 14.66 = 846 \text{ W } \textbf{Ans.}$$

2. ***Power lost in friction assuming uniform wear***

We know that total frictional torque,

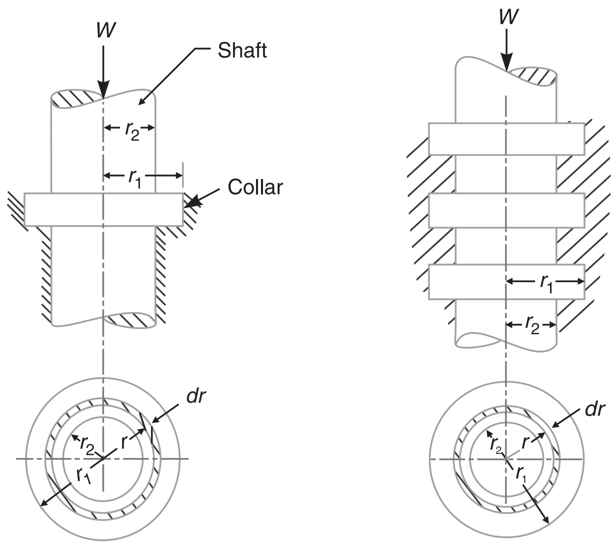
$$T = \frac{1}{2} \cdot \propto .W.R. \text{ cosec}\,\alpha$$

$$= \frac{1}{2} \cdot \; 0.025 \cdot \; 30 \quad 10^3 \quad 0.1 \quad \text{cosec } 60^\circ \quad 43.3 \text{ N-m}$$

$\therefore$ Power lost in friction, $P = T.\omega = 43.3 \times 14.66 = 634.8$ W **Ans.**

10.30. Flat Collar Bearing

We have already discussed that collar bearings are used to take the axial thrust of the rotating shafts. There may be a single collar or multiple collar bearings as shown in Fig. 10.20 (*a*) and (*b*) respectively. The collar bearings are also known as ***thrust bearings.*** The friction in the collar bearings may be found as discussed below :



(*a*) Single collar bearing (*b*) Multiple collar bearing.

Fig. 10.20. Flat collar bearings.

Consider a single flat collar bearing supporting a shaft as shown in Fig. 10.20 (*a*).

Let r_1 = External radius of the collar, and

r_2 = Internal radius of the collar.

$\therefore$ Area of the bearing surface,

$$A = \pi\,[(r_1)^2 - (r_2)^2]$$

1. ***Considering uniform pressure***

When the pressure is uniformly distributed over the bearing surface, then the intensity of pressure,

$$p = \frac{W}{A} = \frac{W}{\pi[(r_1)^2 - (r_2)^2]} \qquad \text{...}(i)$$

We have seen in Art. 10.25, that the frictional torque on the ring of radius r and thickness dr,

$$T_r = 2\pi\mu.p.r^2.dr$$

Integrating this equation within the limits from r_2 to r_1 for the total frictional torque on the collar.

$\therefore$ Total frictional torque,

$$T = \int_{r_2}^{r_1} 2\pi\mu.p.r^2.dr = 2\pi\mu.p\left[\frac{r^3}{3}\right]_{r_2}^{r_1} = 2\pi\mu.p\left[\frac{(r_1)^3 - (r_2)^3}{3}\right]$$

Substituting the value of p from equation (i),

$$T = 2\pi\mu \cdot \frac{W}{\pi[(r_1)^2 - (r_2)^2]}\left[\frac{(r_1)^3 - (r_2)^3}{3}\right]$$

$$= \frac{2}{3} \cdot \mu.W\left[\frac{(r_1)^3 - (r_2)^3}{(r_1)^2 - (r_2)^2}\right]$$

Notes: 1. In order to increase the amount of rubbing surfaces so as to reduce the intensity of pressure, it is better to use two or more collars, as shown in Fig. 10.20 (b), rather than one larger collar.

2. In case of a multi-collared bearings with, say n collars, the intensity of the uniform pressure,

$$p = \frac{\text{Load}}{\text{No. of collars} \times \text{Bearing area of one collar}} = \frac{W}{n\pi[(r_1)^2 - (r_2)^2]}$$

3. The total torque transmitted in a multi collared shaft remains constant *i.e.*

$$T = \frac{2}{3} \cdot \mu.W\left[\frac{(r_1)^3 - (r_2)^3}{(r_1)^2 - (r_2)^2}\right]$$

2. ***Considering unifrom wear***

We have seen in Art. 10.25 that the load transmitted on the ring, considering uniform wear is,

$$\delta W = p_r.2\pi r.dr = \frac{C}{r} \cdot 2\pi\, r.dr = 2\pi\, C.dr$$

$\therefore$ Total load transmitted to the collar,

$$W = \int_{r_2}^{r_1} 2\pi C.dr = 2\pi C[r]_{r_2}^{r_1} = 2\pi C(r_1 - r_2)$$

or

$$C = \frac{W}{2\pi(r_1 - r_2)} \qquad \text{...}(ii)$$

We also know that frictional torque on the ring,

$$T_r = \mu.\delta W.r = \propto \cdot 2\pi C.dr.r = 2\pi\propto.C.r.dr$$

∴ Total frictional torque on the bearing,

$$T = \int_{r_2}^{r_1} 2\pi\propto C.r.dr = 2\pi\propto C\left[\frac{r^2}{2}\right]_{r_2}^{r_1} = 2\pi\propto C\left[\frac{(r_1)^2 - (r_2)^2}{2}\right]$$

$$= \pi\propto C[(r_1)^2 - (r_2)^2]$$

Substituting the value of C from equation **(*ii*)**,

$$T = \pi\propto \cdot \frac{W}{2\pi(r_1 - r_2)}[(r_1)^2 - (r_2)^2] = \frac{1}{2}\cdot\propto .W(r_1 + r_2)$$

Example 10.19. *A thrust shaft of a ship has 6 collars of 600 mm external diameter and 300 mm internal diameter. The total thrust from the propeller is 100 kN. If the coefficient of friction is 0.12 and speed of the engine 90 r.p.m., find the power absorbed in friction at the thrust block, assuming* **1.** *uniform pressure ; and* **2.** *uniform wear.*

Solution. Given : $n = 6$; $d_1 = 600$ mm or $r_1 = 300$ mm ; $d_2 = 300$ mm or $r_2 = 150$ mm ; $W = 100$ kN $= 100 \times 10^3$ N ; $\propto = 0.12$; $N = 90$ r.p.m. or $\omega = 2\pi \times 90/60 = 9.426$ rad/s

Ship propeller.

1. ***Power absorbed in friction, assuming uniform pressure***

We know that total frictional torque transmitted,

$$T = \frac{2}{3}\cdot\propto .W\left[\frac{(r_1)^3 - (r_2)^3}{(r_1)^2 - (r_2)^2}\right]$$

$$= \frac{2}{3}\cdot 0.12 \cdot 100 \quad 10^3\left[\frac{(300)^3 - (150)^3}{(300)^2 - (150)^2}\right] = 2800 \quad 10^3 \text{ N-mm}$$

$$= 2800 \text{ N-m}$$

∴ Power absorbed in friction,

$$P = T.\omega = 2800 \cdot 9.426 = 26\,400 \text{ W} = 26.4 \text{ kW} \quad \textbf{Ans.}$$

2. ***Power absorbed in friction assuming uniform wear***

We know that total frictional torque transmitted,

$$T = \frac{1}{2}\cdot\propto .W(r_1 + r_2) = \frac{1}{2}\cdot 0.12 \quad 100 \quad 10^3 \ (300 \quad 150) \text{ N-mm}$$

$$= 2700 \cdot 10^3 \text{ N-mm} = 2700 \text{ N-m}$$

∴ Power absorbed in friction,

$$P = T.\omega = 2700 \times 9.426 = 25\,450 \text{ W} = 25.45 \text{ kW} \quad \textbf{Ans.}$$

Example 10.20. *A shaft has a number of a collars integral with it. The external diameter of the collars is 400 mm and the shaft diemater is 250 mm. If the intensity of pressure is 0.35 N/mm² (uniform) and the coefficient of friction is 0.05, estimate :* **1.** *power absorbed when the shaft runs at 105 r.p.m. carrying a load of 150 kN ; and* **2.** *number of collars required.*

Solution. Given : d_1 = 400 mm or r_1 = 200 mm ; d_2 = 250 mm or r_2 = 125 mm ; p = 0.35 N/mm² ; μ = 0.05 ; N = 105 r.p.m or $\omega = 2\,\pi \times 105/60 = 11$ rad/s ; W = 150 kN = 150×10^3 N

1. ***Power absorbed***

We know that for uniform pressure, total frictional torque transmitted,

$$T = \frac{2}{3} \cdot \propto .W \left[\frac{(r_1)^3 - (r_2)^3}{(r_1)^2 - (r_2)^2}\right] = \frac{2}{3} \cdot 0.05 \quad 150 \quad 10^3 \left[\frac{(200)^3 - (125)^3}{(200)^2 - (125)^2}\right] \text{N-mm}$$

$$= 5000 \cdot 248 = 1240 \cdot 10^3 \text{ N-mm} = 1240 \text{ N-m}$$

$\therefore$ Power absorbed,

$$P = T.\omega = 1240 \cdot 11 = 13640 \text{ W} = 13.64 \text{ kW} \quad \textbf{Ans.}$$

2. ***Number of collars required***

Let $\quad n$ = Number of collars required.

We know that the intensity of uniform pressure (p),

$$0.35 = \frac{W}{n.\pi[(r_1)^2 - (r_2)^2]} = \frac{150 \cdot 10^3}{n.\pi[(200)^2 - (125)^2]} = \frac{1.96}{n}$$

$\therefore \quad n = 1.96/0.35 = 5.6$ say 6 **Ans.**

Example 10.21. *The thrust of a propeller shaft in a marine engine is taken up by a number of collars integral with the shaft which is 300 mm in diameter. The thrust on the shaft is 200 kN and the speed is 75 r.p.m. Taking ∝ constant and equal to 0.05 and assuming intensity of pressure as uniform and equal to 0.3 N/mm², find the external diameter of the collars and the number of collars required, if the power lost in friction is not to exceed 16 kW.*

Solution. Given : d_2 = 300 mm or r_2 = 150 mm = 0.15 m ; W = 200 kN = 200×10^3 N ; N = 75 r.p.m. or $\omega = 2\,\pi \times 75/60 = 7.86$ rad/s ; $\propto$ = 0.05 ; p = 0.3 N/mm² ; P = 16 kW = 16×10^3 W

Let $\quad T$ = Total frictional torque transmitted in N-m.

We know that power lost in friction (P),

$$16 \times 10^3 = T.\omega = T \times 7.86 \text{ or } T = 16 \times 10^3/7.86 = 2036 \text{ N-m}$$

External diameter of the collar

Let $\quad d_1$ = External diameter of the collar in metres = 2 r_1.

We know that for uniform pressure, total frictional torque transmitted (T),

$$2036 = \frac{2}{3} \cdot \propto .W \left[\frac{(r_1)^3 - (r_2)^3}{(r_1)^2 - (r_2)^2}\right] = \frac{2}{3} \cdot \propto \cdot W \left[\frac{(r_1)^2 + (r_2)^2 + r_1.r_2}{r_1 + r_2}\right]^{*}$$

$$= \frac{2}{3} \cdot 0.05 \cdot 200 \quad 10^3 \left[\frac{(r_1)^2 + (0.15)^2 + r_1 \cdot 0.15}{r_1 + 0.15}\right]$$

$$2036 \cdot 3(r_1 + 0.15) = 20 \cdot 10^3 [(r_1)^2 + 0.15\, r_1 + 0.0225]$$

---

* $\dfrac{(r_1)^3 - (r_2)^3}{(r_1)^2 - (r_2)^3} = \dfrac{(r_1 - r_2)\,[(r_1)^2 + (r_2)^2 + r_1.r_2]}{(r_1 + r_2)(r_1 - r_2)} = \dfrac{(r_1)^2 + (r_2)^2 + r_1.r_2}{r_1 + r_2}$

Dividing throughout by 20×10^3,

$$0.305\,(r_1 + 0.15) = (r_1)^2 + 0.15\,r_1 + 0.0225$$

$$(r_1)^2 - 0.155\,r_1 - 0.0233 = 0$$

Solving this as a quadratic equation,

$$r_1 = \frac{0.155 \pm \sqrt{(0.155)^2 + 4 \times 0.0233}}{2} = \frac{0.155 \pm 0.342}{2}$$

$$= 0.2485 \text{ m} = 248.5 \text{ mm} \qquad \text{...(Taking + ve sign)}$$

$$\therefore \quad d_1 = 2\,r_1 = 2 \times 248.5 = 497 \text{ mm} \quad \textbf{Ans.}$$

Number of collars

Let n = Number of collars.

We know that intensity of pressure (p),

$$0.3 = \frac{W}{n\pi[(r_1)^2 - (r_2)^2]} = \frac{200 \cdot 10^3}{n\pi[(248.5)^2 - (150)^2]} = \frac{1.62}{n}$$

$$\therefore \quad n = 1.62/0.3 = 5.4 \quad \text{or} \quad 6 \quad \textbf{Ans.}$$

10.31. Friction Clutches

A friction clutch has its principal application in the transmission of power of shafts and machines which must be started and stopped frequently. Its application is also found in cases in which power is to be delivered to machines partially or fully loaded. The force of friction is used to start the driven shaft from rest and gradually brings it up to the proper speed without excessive slipping of the friction surfaces. In automobiles, friction clutch is used to connect the engine to the driven shaft. In operating such a clutch, care should be taken so that the friction surfaces engage easily and gradually brings the driven shaft up to proper speed. The proper alignment of the bearing must be maintained and it should be located as close to the clutch as possible. It may be noted that

1. The contact surfaces should develop a frictional force that may pick up and hold the load with reasonably low pressure between the contact surfaces.

2. The heat of friction should be rapidly dissipated and tendency to grab should be at a minimum.

3. The surfaces should be backed by a material stiff enough to ensure a reasonably uniform distribution of pressure.

The friction clutches of the following types are important from the subject point of view :

1. Disc or plate clutches (single disc or multiple disc clutch),

2. Cone clutches, and

3. Centrifugal clutches.

We shall now discuss, these clutches, in detail, in the following pages. It may be noted that the disc and cone clutches are based on the same theory as the pivot and collar bearings.

10.32. Single Disc or Plate Clutch

A single disc or plate clutch, as shown in Fig. 10.21, consists of a clutch plate whose both sides are faced with a friction material (usually of Ferrodo). It is mounted on the hub which is free to move axially along the splines of the driven shaft. The pressure plate is mounted inside the clutch body which is bolted to the flywheel. Both the pressure plate and the flywheel rotate with the engine crankshaft or the driving shaft. The pressure plate pushes the clutch plate towards the flywheel by a set of strong springs which are arranged radially inside the body. The three levers (also known as release levers or fingers) are carried on pivots suspended from the case of the body. These are arranged in such a manner so that the pressure plate moves away from the flywheel by the inward movement of a thrust bearing. The bearing is mounted upon a forked shaft and moves forward when the clutch pedal is pressed.

Single disc clutch

When the clutch pedal is pressed down, its linkage forces the thrust release bearing to move in towards the flywheel and pressing the longer ends of the levers inward. The levers are forced to turn on their suspended pivot and the pressure plate moves away from the flywheel by the knife edges, thereby compressing the clutch springs. This action removes the pressure from the clutch plate and thus moves back from the flywheel and the driven shaft becomes stationary. On the other hand, when the foot is taken off from the clutch pedal, the thrust bearing moves back by the levers. This allows the springs to extend and thus the pressure plate pushes the clutch plate back towards the flywheel.

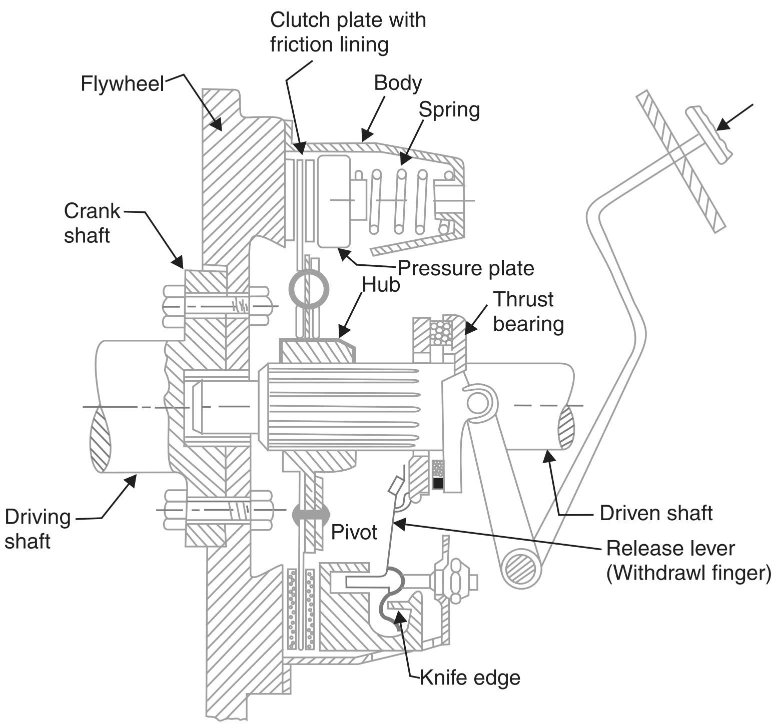


Fig. 10.21. Single disc or plate clutch.

The axial pressure exerted by the spring provides a frictional force in the circumferential direction when the relative motion between the driving and driven members tends to take place. If the torque due to this frictional force exceeds the torque to be transmitted, then no slipping takes place and the power is transmitted from the driving shaft to the driven shaft.

Now consider two friction surfaces, maintained in contact by an axial thrust W, as shown in Fig. 10.22 (*a*).

Let T = Torque transmitted by the clutch,

p = Intensity of axial pressure with which the contact surfaces are held together,

r_1 and r_2 = External and internal radii of friction faces, and

μ = Coefficient of friction.

Consider an elementary ring of radius r and thickness dr as shown in Fig. 10.22 (*b*).

We know that area of contact surface or friction surface,

$$= 2\,\pi\, r.dr$$

∴ Normal or axial force on the ring,

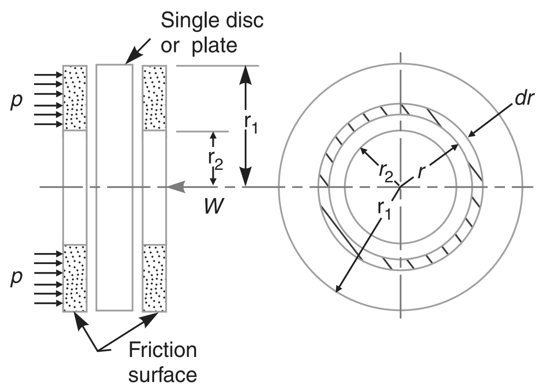
$$\delta W = \text{Pressure} \times \text{Area} = p \times 2\,\pi\, r.dr$$

and the frictional force on the ring acting tangentially at radius r,

$$F_r = \propto.\delta W = \propto.p \times 2\,\pi\, r.dr$$

∴ Frictional torque acting on the ring,

$$T_r = F_r \times r = \propto.p \times 2\,\pi\, r.dr \times r = 2\,\pi \times \propto .p.r^2\, dr$$



(*a*) (*b*)

Fig. 10.22. Forces on a single disc or plate clutch.

We shall now consider the following two cases :

1. When there is a uniform pressure, and

2. When there is a uniform wear.

1. ***Considering uniform pressure***

When the pressure is uniformly distributed over the entire area of the friction face, then the intensity of pressure,

$$p = \frac{W}{\pi[(r_1)^2 - (r_2)^2]} \quad \text{...}(i)$$

where W = Axial thrust with which the contact or friction surfaces are held together.

We have discussed above that the frictional torque on the elementary ring of radius r and thickness dr is

$$T_r = 2\,\pi \propto.p.r^2\, dr$$

Integrating this equation within the limits from r_2 to r_1 for the total frictional torque.

∴ Total frictional torque acting on the friction surface or on the clutch,

$$T = \int_{r_1}^{r_2} 2\pi \propto p.r^2.dr = 2\pi \propto p \left[\frac{r_3}{3}\right]_{r_2}^{r_1} = 2\pi \propto p \left[\frac{(r_1)^3 - (r_2)^3}{3}\right]$$

Substituting the value of p from equation (*i*),

$$T = 2\pi \propto \cdot \frac{W}{\pi[(r_1)^2 - (r_2)^2]} \cdot \frac{(r_1)^3 - (r_2)^3}{3}$$

$$= \frac{2}{3} \cdot \propto .W \left[\frac{(r_1)^3 - (r_2)^3}{(r_1)^2 - (r_2)^2}\right] = \propto .W.R$$

where R = Mean radius of friction surface

$$= \frac{2}{3}\left[\frac{(r_1)^3 - (r_2)^3}{(r_1)^2 - (r_2)^2}\right]$$

2. ***Considering uniform wear***

In Fig. 10.22, let p be the normal intensity of pressure at a distance r from the axis of the clutch. Since the intensity of pressure varies inversely with the distance, therefore

$$p.r. = C \text{ (a constant)} \quad \text{or} \quad p = C/r \qquad ...(i)$$

and the normal force on the ring,

$$\delta W = p.2\pi r.dr = \frac{C}{r} \cdot 2\pi\, C.dr = 2\pi\, C.dr$$

∴ Total force acting on the friction surface,

$$W = \int_{r_2}^{r_1} 2\pi C\, dr = 2\pi C \left[r\right]_{r_2}^{r_1} = 2\pi C(r_1 - r_2)$$

or

$$C = \frac{W}{2\pi(r_1 - r_2)}$$

We know that the frictional torque acting on the ring,

$$T_r = 2\pi \propto .p\, r^2 .dr = 2\pi \propto \cdot \frac{C}{r} \cdot r^2 .dr = 2\pi \propto .C.r.dr$$

$$...(\because \; p = C/r)$$

∴ Total frictional torque on the friction surface,

$$T = \int_{r_2}^{r_1} 2\pi \propto .C.r.dr = 2\pi \propto .C \left[\frac{r^2}{2}\right]_{r_2}^{r_1} = 2\pi \propto .C \left[\frac{(r_1)^2 - (r_2)^2}{2}\right]$$

$$= \pi \propto .C[(r_1)^2 - (r_2)^2] = \pi \propto \cdot \frac{W}{2\pi(r_1 - r_2)}\left[(r_1)^2 - (r_2)^2\right]$$

$$= \frac{1}{2} \cdot \propto .W(r_1 + r_2) = \propto .W.R$$

where R = Mean radius of the friction surface $= \dfrac{r_1 + r_2}{2}$

Notes : 1. In general, total frictional torque acting on the friction surface (or on the clutch) is given by

$$T = n.\propto .W.R$$

where n = Number of pairs of friction or contact surfaces, and

R = Mean radius of friction surface

$$= \frac{2}{3}\left[\frac{(r_1)^3 - (r_2)^3}{(r_1)^2 - (r_2)^2}\right] \quad \text{...(For uniform pressure)}$$

$$= \frac{r_1 + r_2}{2} \quad \text{...(For uniform wear)}$$

2. For a single disc or plate clutch, normally both sides of the disc are effective. Therefore, a single disc clutch has two pairs of surfaces in contact, *i.e.* $n = 2$.

3. Since the intensity of pressure is maximum at the inner radius (r_2) of the friction or contact surface, therefore equation (*i*) may be written as

$$p_{max} \times r_2 = C \quad \text{or} \quad p_{max} = C/r_2$$

4. Since the intensity of pressure is minimum at the outer radius (r_1) of the friction or contact surface, therefore equation (*i*) may be written as

$$p_{min} \times r_1 = C \quad \text{or} \quad p_{min} = C/r_1$$

5. The average pressure (p_{av}) on the friction or contact surface is given by

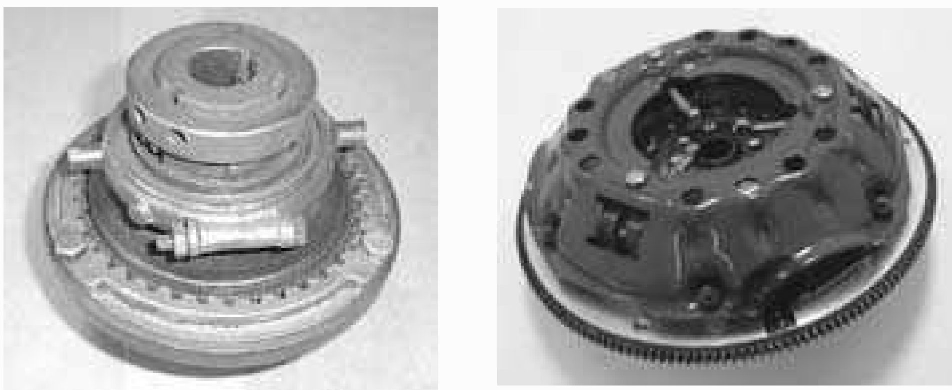
$$p_{av} = \frac{\text{Total force on friction surface}}{\text{Cross-sectional area of friction surface}} = \frac{W}{\pi[(r_1)^2 - (r_2)^2]}$$

6. In case of a new clutch, the intensity of pressure is approximately uniform but in an old clutch the uniform wear theory is more approximate.

7. The uniform pressure theory gives a higher frictional torque than the uniform wear theory. Therefore in case of friction clutches, uniform wear should be considered, unless otherwise stated.

10.33. Multiple Disc Clutch

A multiple disc clutch, as shown in Fig. 10.23, may be used when a large torque is to be transmitted. The inside discs (usually of steel) are fastened to the driven shaft to permit axial motion



Dual Disc Clutches.

(except for the last disc). The outside discs (usually of bronze) are held by bolts and are fastened to the housing which is keyed to the driving shaft. The multiple disc clutches are extensively used in motor cars, machine tools etc.

Let n_1 = Number of discs on the driving shaft, and

n_2 = Number of discs on the driven shaft.

∴ Number of pairs of contact surfaces,

$$n = n_1 + n_2 - 1$$

and total frictional torque acting on the friction surfaces or on the clutch,

$$T = n.\mu.W.R$$

where R = Mean radius of the friction surfaces

$$= \frac{2}{3}\left[\frac{(r_1)^3 - (r_2)^3}{(r_1)^2 - (r_2)^2}\right] \qquad \text{...(For uniform pressure)}$$

$$= \frac{r_1 + r_2}{2} \qquad \text{...(For uniform wear)}$$

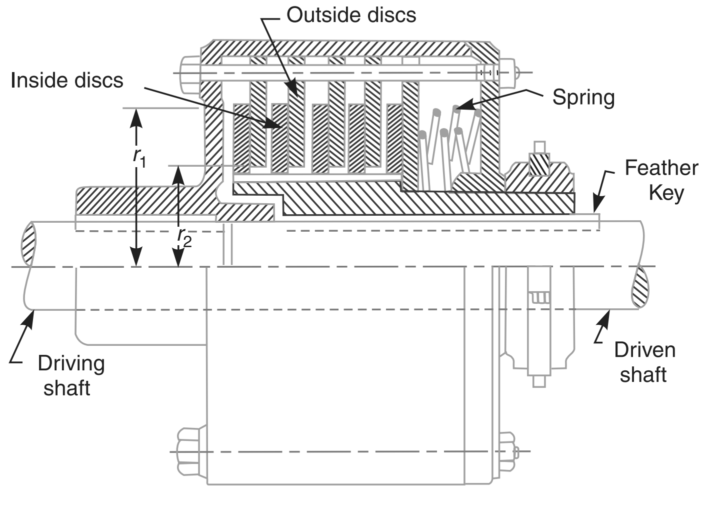


Fig. 10.23. Multiple disc clutch.

Example 10.22. *Determine the maximum, minimum and average pressure in plate clutch when the axial force is 4 kN. The inside radius of the contact surface is 50 mm and the outside radius is 100 mm. Assume uniform wear.*

Solution. Given : $W = 4$ kN $= 4 \times 10^3$ N ; $r_2 = 50$ mm ; $r_1 = 100$ mm

Maximum pressure

Let p_{max} = Maximum pressure.

Since the intensity of pressure is maximum at the inner radius (r_2), therefore

$$p_{max} \times r_2 = C \quad \text{or} \quad C = 50\, p_{max}$$

We know that the total force on the contact surface (W),

$$4 \times 10^3 = 2\pi C (r_1 - r_2) = 2\pi \times 50\, p_{max} (100 - 50) = 15\,710\, p_{max}$$

$$\therefore \quad p_{max} = 4 \times 10^3/15\,710 = 0.2546 \text{ N/mm}^2 \quad \textbf{Ans.}$$

Minimum pressure

Let p_{min} = Minimum pressure.

Since the intensity of pressure is minimum at the outer radius (r_1), therefore

$$p_{min} \times r_1 = C \quad \text{or} \quad C = 100\, p_{min}$$

We know that the total force on the contact surface (W),

$$4 \times 10^3 = 2\,\pi\, C\,(r_1 - r_2) = 2\pi \times 100\, p_{min}\,(100 - 50) = 31\,420\, p_{min}$$

$$\therefore \quad p_{min} = 4 \times 10^3/31\,420 = 0.1273 \text{ N/mm}^2 \quad \textbf{Ans.}$$

Average pressure

We know that average pressure,

$$p_{av} = \frac{\text{Total normal force on contact surface}}{\text{Cross-sectional area of contact surfaces}}$$

$$= \frac{W}{\pi[(r_1)^2 - (r_2)^2]} = \frac{4 \cdot 10^3}{\pi[(100)^2 - (50)^2]} = 0.17 \text{ N/mm}^2 \quad \textbf{Ans.}$$

Example 10.23. *A single plate clutch, with both sides effective, has outer and inner diameters 300 mm and 200 mm respectively. The maximum intensity of pressure at any point in the contact surface is not to exceed 0.1 N/mm². If the coefficient of friction is 0.3, determine the power transmitted by a clutch at a speed 2500 r.p.m.*

Solution. Given : $d_1 = 300$ mm or $r_1 = 150$ mm ; $d_2 = 200$ mm or $r_2 = 100$ mm ; $p = 0.1$ N/mm² ; $\propto = 0.3$; $N = 2500$ r.p.m. or $\omega = 2\pi \times 2500/60 = 261.8$ rad/s

Since the intensity of pressure (p) is maximum at the inner radius (r_2), therefore for uniform wear,

$$p.r_2 = C \quad \text{or} \quad C = 0.1 \times 100 = 10 \text{ N/mm}$$

We know that the axial thrust,

$$W = 2\,\pi\, C\,(r_1 - r_2) = 2\,\pi \times 10\,(150 - 100) = 3142 \text{ N}$$

and mean radius of the friction surfaces for uniform wear,

$$R = \frac{r_1 + r_2}{2} = \frac{150 + 100}{2} = 125 \text{ mm} = 0.125 \text{ m}$$

We know that torque transmitted,

$$T = n.\propto.W.R = 2 \times 0.3 \times 3142 \times 0.125 = 235.65 \text{ N-m}$$

...($\because$ $n = 2$, for both sides of plate effective)

$\therefore$ Power transmitted by a clutch,

$$P = T.\omega = 235.65 \times 261.8 = 61\,693 \text{ W} = 61.693 \text{ kW} \quad \textbf{Ans.}$$

Example 10.24. *A single plate clutch, effective on both sides, is required to transmit 25 kW at 3000 r.p.m. Determine the outer and inner radii of frictional surface if the coefficient of friction is 0.255, the ratio of radii is 1.25 and the maximum pressure is not to exceed 0.1 N/mm². Also determine the axial thrust to be provided by springs. Ass ume the theory of uniform wear.*

Solution. Given: $n = 2$; $P = 25$ kW $= 25 \times 10^3$ W ; $N = 3000$ r.p.m. or $\omega = 2\pi \times 3000/60 = 314.2$ rad/s ; $\propto = 0.255$; $r_1/r_2 = 1.25$; $p = 0.1$ N/mm²

Outer and inner radii of frictional surface

Let r_1 and r_2 = Outer and inner radii of frictional surfaces, and

T = Torque transmitted.

Since the ratio of radii (r_1/r_2) is 1.25, therefore

$$r_1 = 1.25\, r_2$$

We know that the power transmitted (P),

$$25 \times 10^3 = T.\omega = T \times 314.2$$

$$\therefore \quad T = 25 \times 10^3/314.2 = 79.6 \text{ N-m} = 79.6 \times 10^3 \text{ N-mm}$$

Since the intensity of pressure is maximum at the inner radius (r_2), therefore

$$p.r_2 = C \qquad \text{or} \qquad C = 0.1\, r_2 \text{ N/mm}$$

and the axial thrust transmitted to the frictional surface,

$$W = 2\,\pi\, C\,(r_1 - r_2) = 2\,\pi \times 0.1\, r_2\,(1.25\, r_2 - r_2) = 0.157\,(r_2)^2 \qquad \text{...}(i)$$

We know that mean radius of the frictional surface for uniform wear,

$$R = \frac{r_1 + r_2}{2} = \frac{1.25\, r_2 + r_2}{2} = 1.125\, r_2$$

We know that torque transmitted (T),

$$79.6 \times 10^3 = n.\propto.W.R = 2 \times 0.255 \times 0.157\,(r_2)^2 \times 1.125\, r_2 = 0.09\,(r_2)^3$$

$\therefore \qquad (r_2)^3 = 79.6 \times 10^3/0.09 = 884 \times 10^3$ or $r_2 = 96$ mm **Ans.**

and $\qquad r_1 = 1.25\, r_2 = 1.25 \times 96 = 120$ mm **Ans.**

Axial thrust to be provided by springs

We know that axial thrust to be provided by springs,

$$W = 2\,\pi\, C\,(r_1 - r_2) = 0.157\,(r_2)^2 \qquad \text{...[From equation } (i)]$$

$$= 0.157\,(96)^2 = 1447 \text{ N} \quad \textbf{Ans.}$$

Example 10.25. *A single dry plate clutch transmits 7.5 kW at 900 r.p.m. The axial pressure is limited to 0.07 N/mm². If the coefficient of friction is 0.25, find* **1.** *Mean radius and face width of the friction lining assuming the ratio of the mean radius to the face width as 4, and* **2.** *Outer and inner radii of the clutch plate.*

Solution. Given : $P = 7.5$ kW $= 7.5 \times 10^3$ W ; $N = 900$ r.p.m or $\omega = 2\,\pi \times 900/60 = 94.26$ rad/s ; $p = 0.07$ N/mm² ; $\propto = 0.25$

1. ***Mean radius and face width of the friction lining***

Let $\qquad R$ = Mean radius of the friction lining in mm, and

$\qquad w$ = Face width of the friction lining in mm,

Ratio of mean radius to the face width,

$$R/w = 4 \qquad \text{...(Given)}$$

We know that the area of friction faces,

$$A = 2\,\pi\, R.w$$

$\therefore$ Normal or the axial force acting on the friction faces,

$$W = A \times p = 2\,\pi\, R.w.p$$

We know that torque transmitted (considering uniform wear),

$$T = n.\propto.W.R = n.\propto(2\,\pi R.w.p)\,R$$

$$= n.\propto\left(2\,\pi\, R \cdot \frac{R}{4} \cdot p\right) R = \frac{\pi}{2} \cdot n\propto.p.R^3 \qquad \text{...}(\because\ w = R/4)$$

$$= \frac{\pi}{2} \cdot 2 \cdot 0.25 \quad 0.07\, R^3 = 0.055\, R^3 \text{ N-mm} \qquad \text{...}(i)$$

$$\text{...}(\because\ n = 2, \text{ for single plate clutch})$$

We also know that power transmitted (P),

$$7.5 \times 10^3 = T.\omega = T \times 94.26$$

$$\therefore \quad T = 7.5 \times 10^3/94.26 = 79.56 \text{ N-m} = 79.56 \times 10^3 \text{ N-mm} \quad \ldots(ii)$$

From equations (*i*) and (*ii*),

$$R^3 = 79.56 \times 10^3/0.055 = 1446.5 \times 10^3 \text{ or } R = 113 \text{ mm } \textbf{Ans.}$$

and $\quad w = R/4 = 113/4 = 28.25\text{mm}$ **Ans.**

2. ***Outer and inner radii of the clutch plate***

Let $\quad r_1$ and r_2 = Outer and inner radii of the clutch plate respectively.

Since the width of the clutch plate is equal to the difference of the outer and inner radii, therefore

$$w = r_1 - r_2 = 28.25 \text{ mm} \quad \ldots(iii)$$

Also for uniform wear, the mean radius of the clutch plate,

$$R = \frac{r_1 + r_2}{2} \quad \text{or} \quad r_1 + r_2 = 2R = 2 \cdot 113 = 226 \text{ mm} \quad \ldots(iv)$$

From equations (*iii*) and (*iv*),

$$r_1 = 127.125 \text{ mm ; and } r_2 = 98.875 \ \textbf{Ans.}$$

Example 10.26. *A dry single plate clutch is to be designed for an automotive vehicle whose engine is rated to give 100 kW at 2400 r.p.m. and maximum torque 500 N-m. The outer radius of friction plate is 25% more than the inner radius. The intensity of pressure between the plate is not to exceed 0.07 N/mm². The coefficient of friction may be assumed equal to 0.3. The helical springs required by this clutch to provide axial force necessary to engage the clutch are eight. If each spring has stiffness equal to 40 N /mm, determine the initial compression in the springs and dimensions of the friction plate.*

Solution. Given : P = 100 kW = 100 × 10^3 W ; T = 500 N-m = 500 × 10^3 N-mm ; $p = 0.07 \text{ N/mm}^2$; $\propto = 0.3$; Number of springs = 8 ; Stiffness = 40 N/mm

Dimensions of the friction plate

Let $\quad r_1$ and r_2 = Outer and inner radii of the friction plate respectively.

Since the outer radius of the friction plate is 25% more than the inner radius, therefore

$$r_1 = 1.25\ r_2$$

We know that, for uniform wear,

$$p.r_2 = C \quad \text{or} \quad C = 0.07\ r_2 \text{ N/mm}$$

and load transmitted to the friction plate,

$$W = 2\pi C (r_1 - r_2) = 2\pi \times 0.07\ r^2 (1.125\ r_2 - r_2) = 0.11\ (r_2)^2 \text{ N} \quad \ldots(i)$$

We know that mean radius of the plate for uniform wear,

$$R = \frac{r_1 + r_2}{2} = \frac{1.25\ r_2 + r_2}{2} = 1.125\ r_2$$

$\therefore$ Torque transmitted (T),

$$500 \times 10^3 = n.\propto.W.R = 2 \times 0.3 \times 0.11\ (r_2)^2 \times 1.125\ r_2 = 0.074\ (r_2)^3 \quad \ldots(\because\ n = 2)$$

$$\therefore \quad (r_2)^3 = 500 \times 10^3/0.074 = 6757 \times 10^3 \text{ or } r_2 = 190 \text{ mm} \ \textbf{Ans.}$$

and $r_1 = 1.25\, r_2 = 1.25 \times 190 = 273.5$ mm **Ans.**

Initial compression of the springs

We know that total stiffness of the springs,

$$s = \text{Stiffness per spring} \times \text{No. of springs} = 40 \times 8 = 320 \text{ N/mm}$$

Axial force required to engage the clutch,

$$W = 0.11\,(r_2)^2 = 0.11\,(190)^2 = 3970 \text{ N} \qquad \text{...[From equation } (i)]$$

∴ Initial compression in the springs

$$= W/s = 3970/320 = 12.5 \text{ mm}$$ **Ans.**

Example 10.28. *A multiple disc clutch has five plates having four pairs of active friction surfaces. If the intensity of pressure is not to exceed 0.127 N/mm², find the power transmitted at 500 r.p.m. The outer and inner radii of friction surfaces are 125 mm and 75 mm respectively. Assume uniform wear and take coefficient of friction = 0.3.*

Solution. Given : $n_1 + n_2 = 5$; $n = 4$; $p = 0.127$ N/mm² ; $N = 500$ r.p.m. or $\omega = 2\pi \times 500/60 = 52.4$ rad/s ; $r_1 = 125$ mm ; $r_2 = 75$ mm ; $\propto = 0.3$

Since the intensity of pressure is maximum at the inner radius r_2, therefore

$$p.r_2 = C \quad \text{or} \quad C = 0.127 \times 75 = 9.525 \text{ N/mm}$$

We know that axial force required to engage the clutch,

$$W = 2\,\pi\,C\,(r_1 - r_2) = 2\,\pi \times 9.525\,(125 - 75) = 2990 \text{ N}$$

and mean radius of the friction surfaces,

$$R = \frac{r_1 + r_2}{2} = \frac{125 + 75}{2} = 100 \text{ mm} = 0.1 \text{ m}$$

We know that torque transmitted,

$$T = n.\propto.W.R = 4 \times 0.3 \times 2990 \times 0.1 = 358.8 \text{ N-m}$$

$\therefore$ Power transmitted,

$$P = T.\omega = 358.8 \times 52.4 = 18\,800 \text{ W} = 18.8 \text{ kW}$$ **Ans.**

Example 10.29. *A multi-disc clutch has three discs on the driving shaft and two on the driven shaft. The outside diameter of the contact surfaces is 240 mm and inside diameter 120 mm. Assuming uniform wear and coefficient of friction as 0.3, find the maximum axial intensity of pressure between the discs for transmitting 25 kW at 1575 r.p.m.*

Solution. Given : $n_1 = 3$; $n_2 = 2$; $d_1 = 240$ mm or $r_1 = 120$ mm ; $d_2 = 120$ mm or $r_2 = 60$ mm ; $\propto = 0.3$; $P = 25$ kW $= 25 \times 10^3$ W ; $N = 1575$ r.p.m. or $\omega = 2\,\pi \times 1575/60 = 165$ rad/s

Let T = Torque transmitted in N-m, and

W = Axial force on each friction surface.

We know that the power transmitted (P),

$$25 \times 10^3 = T.\omega = T \times 165 \quad \text{or} \quad T = 25 \times 10^3/165 = 151.5 \text{ N-m}$$

Number of pairs of friction surfaces,

$$n = n_1 + n_2 - 1 = 3 + 2 - 1 = 4$$

and mean radius of friction surfaces for uniform wear,

$$R = \frac{r_1 + r_2}{2} = \frac{120 + 60}{2} = 90 \text{ mm} = 0.09 \text{ m}$$

We know that torque transmitted (T),

$$151.5 = n.\mu.W.R = 4 \times 0.3 \times W \times 0.09 = 0.108\ W$$

$$\therefore \quad W = 151.5/0.108 = 1403 \text{ N}$$

Let p = Maximum axial intensity of pressure.

Since the intensity of pressure (p) is maximum at the inner radius (r_2), therefore for uniform wear

$$p.r_2 = C \quad \text{or} \quad C = p \times 60 = 60\,p \text{ N/mm}$$

We know that the axial force on each friction surface (W),

$$1403 = 2\,\pi.C\,(r_1 - r_2) = 2\,\pi \times 60\,p\,(120 - 60) = 22\,622\,p$$

$$\therefore \quad p = 1403/22\,622 = 0.062 \text{ N/mm}^2$$ **Ans.**

Example 10.30. *A plate clutch has three discs on the driving shaft and two discs on the driven shaft, providing four pairs of contact surfaces. The outside diameter of the contact surfaces is 240 mm and inside diameter 120 mm. Assuming uniform pressure and $\propto = 0.3$; find the total spring load pressing the plates together to transmit 25 kW at 1575 r.p.m.*

If there are 6 springs each of stiffness 13 kN/m and each of the contact surfaces has worn away by 1.25 mm, find the maximum power that can be transmitted, assuming uniform wear.

Solution. Given : $n_1 = 3$; $n_2 = 2$; $n = 4$; $d_1 = 240$ mm or $r_1 = 120$ mm ; $d_2 = 120$ mm or $r_2 = 60$ mm ; $\propto = 0.3$; $P = 25$ kW $= 25 \times 10^3$ W ; $N = 1575$ r.p.m. or $\omega = 2\,\pi \times 1575/60 = 165$ rad/s

Total spring load

Let W = Total spring load, and

T = Torque transmitted.

We know that power transmitted (P),

$$25 \times 10^3 = T.\omega = T \times 165 \quad \text{or} \quad T = 25 \times 10^3/165 = 151.5 \text{ N-m}$$

Mean radius of the contact surface, for uniform pressure,

$$R = \frac{2}{3}\left[\frac{(r_1)^3 - (r_2)^3}{(r_1)^2 - (r_2)^2}\right] = \frac{2}{3}\left[\frac{(120)^3 - (60)^3}{(120)^2 - (60)^2}\right] = 93.3 \text{ mm} = 0.0933 \text{ m}$$

and torque transmitted (T),

$$151.5 = n.\propto.W.R = 4 \times 0.3\ W \times 0.0933 = 0.112\ W$$

$$\therefore \quad W = 151.5/0.112 = 1353 \text{ N}$$ **Ans.**

Maximum power transmitted

Given : No of springs = 6

$\therefore$ Contact surfaces of the spring

= 8

Wear on each contact surface

= 1.25 mm

$\therefore$ Total wear $= 8 \times 1.25 = 10$ mm $= 0.01$ m

Stiffness of each spring $= 13$ kN/m $= 13 \times 10^3$ N/m

$\therefore$ Reduction in spring force

= Total wear × Stiffness per spring × No. of springs

$= 0.01 \times 13 \times 10^3 \times 6 = 780$ N

∴ New axial load, $W = 1353 - 780 = 573$ N

We know that mean radius of the contact surfaces for uniform wear,

$$R = \frac{r_1 + r_2}{2} = \frac{120 + 60}{2} = 90 \text{ mm} = 0.09 \text{ m}$$

∴ Torque transmitted,

$$T = n.\mu.W.R. = 4 \times 0.3 \times 573 \times 0.09 = 62 \text{ N-m}$$

and maximum power transmitted,

$$P = T.\,\omega = 62 \times 155 = 10\,230 \text{ W} = 10.23 \text{ kW} \quad \textbf{Ans.}$$

10.34. Cone Clutch

A cone clutch, as shown in Fig. 10.24, was extensively used in automobiles but now-a-days it has been replaced completely by the disc clutch.

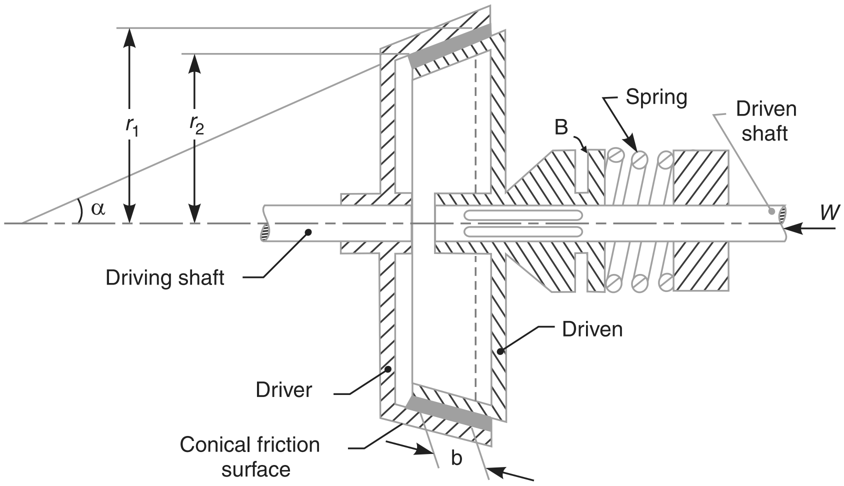


Fig. 10.24. Cone clutch.

It consists of one pair of friction surface only. In a cone clutch, the driver is keyed to the driving shaft by a sunk key and has an inside conical surface or face which exactly fits into the outside conical surface of the driven. The driven member resting on the feather key in the driven shaft, may be shifted along the shaft by a forked lever provided at B, in order to engage the clutch by bringing the two conical surfaces in contact. Due to the frictional resistance set up at this contact surface, the torque is transmitted from one shaft to another. In some cases, a spring is placed around the driven shaft in contact with the hub of the driven. This spring holds the clutch faces in contact and maintains the pressure between them, and the forked lever is used only for disengagement of the clutch. The contact surfaces of the clutch may be metal to metal contact, but more often the driven member is lined with some material like wood, leather, cork or asbestos etc. The material of the clutch faces (*i.e.* contact surfaces) depends upon the allowable normal pressure and the coefficient of friction.

Consider a pair of friction surface as shown in Fig. 10.25 (*a*). Since the area of contact of a pair of friction surface is a frustrum of a cone, therefore the torque transmitted by the cone clutch may be determined in the similar manner as discussed for conical pivot bearings in Art. 10.28.

Let p_n = Intensity of pressure with which the conical friction surfaces are held together (*i.e.* normal pressure between contact surfaces),

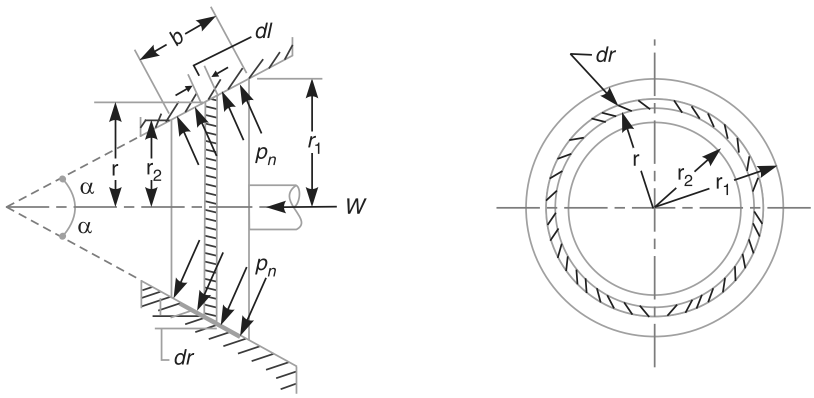
r_1 and r_2 = Outer and inner radius of friction surfaces respectively.

R = Mean radius of the friction surface $= \dfrac{r_1 + r_2}{2}$,

α = Semi angle of the cone (also called face angle of the cone) or the angle of the friction surface with the axis of the clutch,

$\propto$ = Coefficient of friction between contact surfaces, and

b = Width of the contact surfaces (also known as face width or clutch face).



(*a*) (*b*)

Fig. 10.25. Friction surfaces as a frustrum of a cone.

Consider a small ring of radius r and thickness dr, as shown in Fig. 10.25 (*b*). Let dl is length of ring of the friction surface, such that

$$dl = dr.\text{cosec}\ \alpha$$

$\therefore$ Area of the ring,

$$A = 2\pi\ r.dl = 2\pi r.dr\ \text{cosec}\ \alpha$$

We shall consider the following two cases :

1. When there is a uniform pressure, and
2. When there is a uniform wear.

1. ***Considering uniform pressure***

We know that normal load acting on the ring,

$$\delta W_n = \text{Normal pressure} \times \text{Area of ring} = p_n \times 2\,\pi\ r.dr.\text{cosec}\ \alpha$$

and the axial load acting on the ring,

$$\delta W = \text{Horizontal component of } \delta W_n \text{ (\textit{i.e.} in the direction of } W)$$

$$= \delta W_n \times \sin\alpha = p_n \times 2\pi\ r.dr.\ \text{cosec}\ \alpha \times \sin\alpha = 2\pi \times p_n.r.dr$$

$\therefore$ Total axial load transmitted to the clutch or the axial spring force required,

$$W = \int_{r_2}^{r_1} 2\,\pi\,p_n.r.dr = 2\,\pi\,p_n\left[\frac{r^2}{2}\right]_{r_2}^{r_1} = 2\,\pi\,p_n\left[\frac{(r_1)^2 - (r_2)^2}{2}\right]$$

$$= \pi\,p_n\left[(r_1)^2 - (r_2)^2\right]$$

$$\therefore \qquad p_n = \frac{W}{\pi[(r_1)^2 - (r_2)^2]} \qquad \ldots(i)$$

We know that frictional force on the ring acting tangentially at radius r,

$$F_r = \mu.\delta W_n = \propto .p_n \times 2\,\pi\, r.dr.\text{cosec}\,\alpha$$

∴ Frictional torque acting on the ring,

$$T_r = F_r \times r = \propto .p_n \times 2\,\pi\, r.dr.\ \text{cosec}\,\alpha.r = 2\,\pi \propto .p_n.\text{cosec}\,\alpha.r^2\, dr$$

Integrating this expression within the limits from r_2 to r_1 for the total frictional torque on the clutch.

∴ Total frictional torque,

$$T = \int_{r_2}^{r_1} 2\pi \propto .p_n.\text{cosec}\ \alpha r^2.dr = 2\,\pi \propto p_n.\text{cosec}\ \alpha\left[\frac{r^3}{3}\right]_{r_2}^{r_1}$$

$$= 2\pi \propto p_n.\text{cosec}\ \alpha\left[\frac{(r_1)^3 - (r_2)^3}{3}\right]$$

Substituting the value of p_n from equation (*i*), we get

$$T = 2\pi \propto \cdot \frac{W}{\pi[(r_1)^2 - (r_2)^2]} \cdot \text{cosec}\alpha \left[\frac{(r_1)^3 - (r_2)^3}{3}\right]$$

$$= \frac{2}{3} \cdot \propto .W.\text{cosec}\,\alpha \left[\frac{(r_1)^3 - (r_2)^3}{(r_1)^2 - (r_2)^2}\right] \quad ..(ii)$$

2. *Considering uniform wear*

In Fig. 10.25, let p_r be the normal intensity of pressure at a distance r from the axis of the clutch. We know that, in case of uniform wear, the intensity of pressure varies inversely with the distance.

∴ $p_r.r = C$ (a constant) or $p_r = C/r$

We know that the normal load acting on the ring,

$$\delta W_n = \text{Normal pressure} \times \text{Area of ring} = p_r \times 2\pi r.dr\ \text{cosec}\ \alpha$$

and the axial load acting on the ring ,

$$\delta W = \delta W_n \times \sin\alpha = p_r.2\,\pi\, r.dr.\text{cosec}\ \alpha\ .\sin\alpha = p_r \times 2\,\pi\, r.dr$$

$$= \frac{C}{r} \cdot 2\pi\ r.dr = 2\pi\ C.dr \quad ...(\because\ p_r = C/r)$$

∴ Total axial load transmitted to the clutch,

$$W = \int_{r_2}^{r_1} 2\pi C.dr = 2\pi C[r]_{r_2}^{r_1} = 2\pi C(r_1 - r_2)$$

or
$$C = \frac{W}{2\pi(r_1 - r_2)} \quad ...(iii)$$

We know that frictional force acting on the ring,

$$F_r = \propto .\delta W_n = \propto .p_r \times 2\,\pi\, r \times dr\ \text{cosec}\ \alpha$$

and frictional torque acting on the ring,

$$T_r = F_r \times r = \propto .p_r \times 2\,\pi\, r.dr.\text{cosec}\,\alpha \times r$$

$$= \propto \cdot \frac{C}{r} \cdot 2\pi\ r^2.dr.\text{cosec}\alpha = 2\pi \propto .C\ \text{cosec}\alpha \cdot r\ dr$$

$\therefore$ Total frictional torque acting on the clutch,

$$T = \int_{r_2}^{r_1} 2\pi \propto .C.\text{cosec } \alpha\, r\, dr = 2\pi \propto .C.\text{cosec } \alpha \left[\frac{r^2}{2}\right]_{r_2}^{r_1}$$

$$= 2\pi \propto .C.\text{cosec } \alpha \left[\frac{(r_1)^2 - (r_2)^2}{2}\right]$$

Substituting the value of C from equation (*i*), we have

$$T = 2\pi \propto \cdot \frac{W}{2\pi(r_1 - r_2)} \cdot \text{cosec}\alpha \left[\frac{(r_1)^2 - (r_2)^2}{2}\right]$$

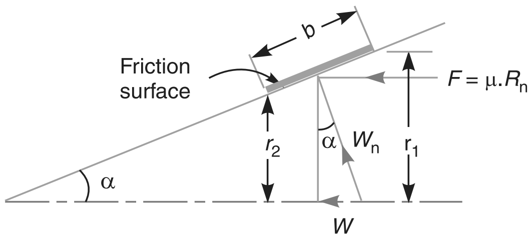
$$= \propto W \text{ cosec } \alpha\left(\frac{r_1 + r_2}{2}\right) = \propto W.R \text{ cosec } \alpha \qquad \text{...}(iv)$$

where $\quad R = \dfrac{r_1 + r_2}{2} =$ Mean radius of friction surface

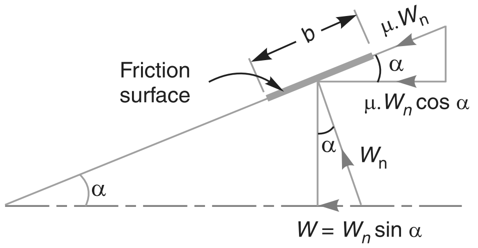
Since the normal force acting on the friction surface, $W_n = W/\sin \alpha$, therefore the equation (*iv*) may be written as

$$T = \propto .W_n.R \qquad \text{...}(v)$$

The forces on a friction surface, for steady operation of the clutch and after the clutch is engaged, is shown in Fig. 10.26.



(*a*) For steady operation of the clutch.



(*b*) During engagement of the clutch.

Fig. 10.26. Forces on a friction surface.

From Fig. 10.26 (*a*), we find that

$$r_1 - r_2 = b\sin\alpha; \text{ and } R = \frac{r_1 + r_2}{2} \quad \text{or} \quad r_1 + r_2 = 2R$$

∴ From equation, (*i*), normal pressure acting on the friction surface,

$$p_n = \frac{W}{\pi[(r_1)^2 - (r_2)^2]} = \frac{W}{\pi(r_1 + r_2)(r_1 - r_2)} = \frac{W}{2\pi R.b.\sin\alpha}$$

or $\quad W = p_n \times 2\,\pi\,R.b \sin\alpha = W_n \sin\alpha$

where $\quad W_n$ = Normal load acting on the friction surface = $p_n \times 2\,\pi\,R.b$

Now the equation (*iv*) may be written as,

$$T = \propto(p_n \cdot 2\pi\,R.b \sin\alpha)\,R\,\text{cosec}\,\alpha = 2\pi\propto.p_n.R^2 b$$

The following points may be noted for a cone clutch :

1. The above equations are valid for steady operation of the clutch and after the clutch is engaged.

2. If the clutch is engaged when one member is stationary and the other rotating (*i.e.* during engagement of the clutch) as shown in Fig. 10.26 (*b*), then the cone faces will tend to slide on each other due to the presence of relative motion. Thus an additional force (of magnitude equal to $\propto.W_n.\cos\alpha$) acts on the clutch which resists the engagement and the axial force required for engaging the clutch increases.

∴ Axial force required for engaging the clutch,

$$W_e = W + \propto.W_n \cos\alpha = W_n \sin\alpha + \propto.W_n \cos\alpha$$
$$= W_n (\sin\alpha + \propto \cos\alpha)$$

3. Under steady operation of the clutch, a decrease in the semi-cone angle (α) increases the torque produced by the clutch (T) and reduces the axial force (W). During engaging period, the axial force required for engaging the clutch (W_e) increases under the influence of friction as the angle α decreases. The value of α can not be decreased much because smaller semi-cone angle (α) requires larger axial force for its disengagement.

For free disengagement of the clutch, the value of tan α must be greater than ∝. In case the value of tan α is less than ∝, the clutch will not disengage itself and the axial force required to disengage the clutch is given by

$$W_d = W_n (\propto \cos\alpha - \sin\alpha)$$

Example 10.31. *A conical friction clutch is used to transmit 90 kW at 1500 r.p.m. The semi-cone angle is 20º and the coefficient of friction is 0.2. If the mean diameter of the bearing surface is 375 mm and the intensity of normal pressure is not to exceed 0.25 N/mm², find the dimensions of the conical bearing surface and the axial load required.*

Solution. Given : P = 90 kW = 90 × 10³ W ; N = 1500 r.p.m. or ω = 2 π × 1500/60 = 156 rad/s ; α = 20º ; ∝ = 0.2 ; D = 375 mm or R = 187.5 mm ; p_n = 0.25 N/mm²

Dimensions of the conical bearing surface

Let $\quad r_1$ and r_2 = External and internal radii of the bearing surface respectively,

b = Width of the bearing surface in mm, and

T = Torque transmitted.

We know that power transmitted (P),

$$90 \times 10^3 = T.\omega = T \times 156$$

∴ $$T = 90 \times 10^3/156 = 577 \text{ N-m} = 577 \times 10^3 \text{ N-mm}$$

and the torque transmitted (T),

$$577 \times 10^3 = 2\,\pi \propto p_n.R^2.b = 2\pi \times 0.2 \times 0.25\,(187.5)^2\,b = 11\,046\,b$$

$$\therefore \quad b = 577 \times 10^3/11\,046 = 52.2 \text{ mm} \quad \textbf{Ans.}$$

We know that $r_1 + r_2 = 2R = 2 \times 187.5 = 375$ mm ...(*i*)

and $r_1 - r_2 = b \sin \alpha = 52.2 \sin 20° = 18$ mm ...(*ii*)

From equations (*i*) and (*ii*),

$$r_1 = 196.5 \text{ mm, and } r_2 = 178.5 \text{ mm} \quad \textbf{Ans.}$$

Axial load required

Since in case of friction clutch, uniform wear is considered and the intensity of pressure is maximum at the minimum contact surface radius (r_2), therefore

$$p_n.r_2 = C \text{ (a constant) or } C = 0.25 \times 178.5 = 44.6 \text{ N/mm}$$

We know that the axial load required,

$$W = 2\pi C\,(r_1 - r_2) = 2\pi \times 44.6\,(196.5 - 178.5) = 5045 \text{ N} \quad \textbf{Ans.}$$

Example 10.32. *An engine developing 45 kW at 1000 r.p.m. is fitted with a cone clutch built inside the flywheel. The cone has a face angle of 12.5º and a maximum mean diameter of 500 mm. The coefficient of friction is 0.2. The normal pressure on the clutch face is not to exceed 0.1 N/mm^2. Determine : 1. the axial spring force necessary to engage to clutch, and 2. the face width required.*

Solution. Given : $P = 45$ kW $= 45 \times 10^3$ W ; $N = 1000$ r.p.m. or $\omega = 2\pi \times 1000/60 = 104.7$ rad/s ; $\alpha = 12.5°$; $D = 500$ mm or $R = 250$ mm $= 0.25$ m ; $\propto = 0.2$; $p_n = 0.1$ N/mm^2

1. ***Axial spring force necessary to engage the clutch***

First of all, let us find the torque (T) developed by the clutch and the normal load (W_n) acting on the friction surface.

We know that power developed by the clutch (P),

$$45 \times 10^3 = T.\omega = T \times 104.7 \quad \text{or} \quad T = 45 \times 10^3/104.7 = 430 \text{ N-m}$$

We also know that the torque developed by the clutch (T),

$$430 = \propto.W_n.R = 0.2 \times W_n \times 0.25 = 0.05\,W_n$$

$$\therefore \quad W_n = 430/0.05 = 8600 \text{ N}$$

and axial spring force necessary to engage the clutch,

$$W_e = W_n\,(\sin \alpha + \propto \cos \alpha)$$

$$= 8600\,(\sin 12.5° + 0.2 \cos 12.5°) = 3540 \text{ N} \quad \textbf{Ans.}$$

2. ***Face width required***

Let b = Face width required.

We know that normal load acting on the friction surface (W_n),

$$8600 = p_n \times 2\,\pi\,R.b = 0.1 \times 2\pi \times 250 \times b = 157\,b$$

$$\therefore \quad b = 8600/157 = 54.7 \text{ mm} \quad \textbf{Ans.}$$

OBJECTIVE TYPE QUESTIONS

1. The angle of inclination of the plane, at which the body begins to move down the plane, is called

   (*a*) angle of friction (*b*) angle of repose (*c*) angle of projection

2. In a screw jack, the effort required to lift the load W is given by

   (*a*) $P = W \tan (\alpha - \phi)$ (*b*) $P = W \tan (\alpha + \phi)$

   (*c*) $P = W \cos (\alpha - \phi)$ (*d*) $P = W \cos (\alpha + \phi)$

   where α = Helix angle, and

   ϕ = Angle of friction.

3. The efficiency of a screw jack is given by

   (*a*) $\dfrac{\tan (\alpha + \phi)}{\tan \alpha}$ (*b*) $\dfrac{\tan \alpha}{\tan (\alpha + \phi)}$

   (*c*) $\dfrac{\tan (\alpha - \phi)}{\tan \alpha}$ (*d*) $\dfrac{\tan \alpha}{\tan (\alpha - \phi)}$

4. The radius of a friction circle for a shaft of radius r rotating inside a bearing is

   (*a*) $r \sin \phi$ (*b*) $r \cos \phi$ (*c*) $r \tan \phi$ (*d*) $r \cot \phi$

5. The efficiency of a screw jack is maximum, when

(*a*) $\alpha = 45° + \frac{\phi}{2}$ (*b*) $\alpha = 45° - \frac{\phi}{2}$ (*c*) $\alpha = 90° + \phi$ (*d*) $\alpha = 90° - \phi$

6. The maximum efficiency of a screw jack is

(*a*) $\frac{1 - \sin\phi}{1 + \sin\phi}$ (*b*) $\frac{1 + \sin\phi}{1 - \sin\phi}$ (*c*) $\frac{1 - \tan\phi}{1 + \tan\phi}$ (*d*) $\frac{1 + \tan\phi}{1 - \tan\phi}$

7. The frictional torque transmitted in a flat pivot bearing, considering uniform pressure, is

(*a*) $\frac{1}{2} \cdot \propto .W.R$ (*b*) $\frac{2}{3} \cdot \propto .W.R$ (*c*) $\frac{3}{4} \cdot \propto .W.R$ (*d*) $\propto .W.R$

where $\propto$ = Coefficient of friction,

W = Load over the bearing, and

R = Radius of the bearing surface.

8. The frictional torque transmitted in a conical pivot bearing, considering uniform wear, is

(*a*) $\frac{1}{2} \cdot \propto .W.R \operatorname{cosec}\alpha$ (*b*) $\frac{2}{3} \cdot \propto .W.R \operatorname{cosec}\alpha$

(*c*) $\frac{3}{4} \cdot \propto .W.R \operatorname{cosec}\alpha$ (*d*) $\propto .\, W.R \operatorname{cosec}\alpha$

where R = Radius of the shaft, and

α = Semi-angle of the cone.

9. The frictional torque transmitted by a disc or plate clutch is same as that of

(*a*) flat pivot bearing (*b*) flat collar bearing

(*c*) conical pivot bearing (*d*) trapezoidal pivot bearing

10. The frictional torque transmitted by a cone clutch is same as that of

(*a*) flat pivot bearing (*b*) flat collar bearing

(*c*) conical pivot bearing (*d*) trapezoidal pivot bearing

ANSWERS

1. (*a*)	**2.** (*b*)	**3.** (*b*)	**4.** (*a*)	**5.** (*b*)
6. (*a*)	**7.** (*b*)	**8.** (*a*)	**9.** (*b*)	**10.** (*d*)

Belt, Rope and Chain Drives

Features (Main)

11.1. Introduction

The belts or ropes are used to transmit power from one shaft to another by means of pulleys which rotate at the same speed or at different speeds. The amount of power transmitted depends upon the following factors :

1. The velocity of the belt.
2. The tension under which the belt is placed on the pulleys.
3. The arc of contact between the belt and the smaller pulley.
4. The conditions under which the belt is used.

It may be noted that

(*a*) The shafts should be properly in line to insure uniform tension across the belt section.

(*b*) The pulleys should not be too close together, in order that the arc of contact on the smaller pulley may be as large as possible.

(*c*) The pulleys should not be so far apart as to cause the belt to weigh heavily on the shafts, thus increasing the friction load on the bearings.

(*d*) A long belt tends to swing from side to side, causing the belt to run out of the pulleys, which in turn develops crooked spots in the belt.

(*e*) The tight side of the belt should be at the bottom, so that whatever sag is present on the loose side will increase the arc of contact at the pulleys.

(*f*) In order to obtain good results with flat belts, the maximum distance between the shafts should not exceed 10 metres and the minimum should not be less than 3.5 times the diameter of the larger pulley.

11.2. Selection of a Belt Drive

Following are the various important factors upon which the selection of a belt drive depends:

1. Speed of the driving and driven shafts,
2. Speed reduction ratio,
3. Power to be transmitted,
4. Centre distance between the shafts,
5. Positive drive requirements,
6. Shafts layout,
7. Space available, and
8. Service conditions.

11.3. Types of Belt Drives

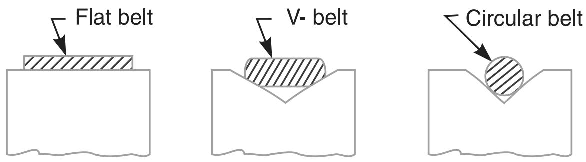
The belt drives are usually classified into the following three groups :

1. ***Light drives***. These are used to transmit small powers at belt speeds upto about 10 m/s, as in agricultural machines and small machine tools.

2. ***Medium drives***. These are used to transmit medium power at belt speeds over 10 m/s but up to 22 m/s, as in machine tools.

3. ***Heavy drives***. These are used to transmit large powers at belt speeds above 22 m/s, as in compressors and generators.

11.4. Types of Belts



(*a*) Flat belt. (*b*) V-belt. (*c*) Circular belt.

Fig. 11.1. Types of belts.

Though there are many types of belts used these days, yet the following are important from the subject point of view :

1. ***Flat belt.*** The flat belt, as shown in Fig. 11.1 (*a*), is mostly used in the factories and workshops, where a moderate amount of power is to be transmitted, from one pulley to another when the two pulleys are not more than 8 metres apart.

2. ***V-belt***. The V-belt, as shown in Fig. 11.1 (*b*), is mostly used in the factories and workshops, where a moderate amount of power is to be transmitted, from one pulley to another, when the two pulleys are very near to each other.

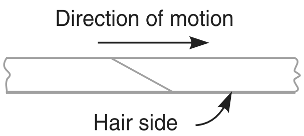
3. ***Circular belt or rope***. The circular belt or rope, as shown in Fig. 11.1 (*c*), is mostly used in the factories and workshops, where a great amount of power is to be transmitted, from one pulley to another, when the two pulleys are more than 8 meters apart.

If a huge amount of power is to be transmitted, then a single belt may not be sufficient. In such a case, wide pulleys (for V-belts or circular belts) with a number of grooves are used. Then a belt in each groove is provided to transmit the required amount of power from one pulley to another.

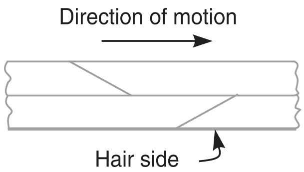
11.5. Material used for Belts

The material used for belts and ropes must be strong, flexible, and durable. It must have a high coefficient of friction. The belts, according to the material used, are classified as follows :

1. ***Leather belts***. The most important material for the belt is leather. The best leather belts are made from 1.2 metres to 1.5 metres long strips cut from either side of the back bone of the top grade steer hides. The hair side of the leather is smoother and harder than the flesh side, but the flesh side is stronger. The fibres on the hair side are perpendicular to the surface, while those on the flesh side are interwoven and parallel to the surface. Therefore for these reasons, the hair side of a belt should be in contact with the pulley surface, as shown in Fig. 11.2. This gives a more intimate contact between the belt and the pulley and places the greatest tensile strength of the belt section on the outside, where the tension is maximum as the belt passes over the pulley.



(*a*) Single layer belt.



(*b*) Double layer belt.

Fig. 11.2. Leather belts.

The leather may be either oak-tanned or mineral salt tanned *e.g.* chrome tanned. In order to increase the thickness of belt, the strips are cemented together. The belts are specified according to the number of layers *e.g.* single, double or triple ply and according to the thickness of hides used *e.g.* light, medium or heavy.

The leather belts must be periodically cleaned and dressed or treated with a compound or dressing containing neats foot or other suitable oils so that the belt will remain soft and flexible.

2. ***Cotton or fabric belts***. Most of the fabric belts are made by folding canvass or cotton duck to three or more layers (depending upon the thickness desired) and stitching together. These belts are woven also into a strip of the desired width and thickness. They are impregnated with some filler like linseed oil in order to make the belts water proof and to prevent injury to the fibres. The cotton belts are cheaper and suitable in warm climates, in damp atmospheres and in exposed positions. Since the cotton belts require little attention, therefore these belts are mostly used in farm machinery, belt conveyor etc.

3. ***Rubber belt***. The rubber belts are made of layers of fabric impregnated with rubber composition and have a thin layer of rubber on the faces. These belts are very flexible but are quickly destroyed if allowed to come into contact with heat, oil or grease. One of the principal advantage of these belts is that they may be easily made endless. These belts are found suitable for saw mills, paper mills where they are exposed to moisture.

4. ***Balata belts***. These belts are similar to rubber belts except that balata gum is used in place of rubber. These belts are acid proof and water proof and it is not effected by animal oils or alkalies. The balata belts should not be at temperatures above 40° C because at this temperature the balata begins to soften and becomes sticky. The strength of balata belts is 25 per cent higher than rubber belts.

11.6. Types of Flat Belt Drives

The power from one pulley to another may be transmitted by any of the following types of belt drives:

1. ***Open belt drive***. The open belt drive, as shown in Fig. 11.3, is used with shafts arranged parallel and rotating in the same direction. In this case, the driver A pulls the belt from one side (*i.e.* lower side RQ) and delivers it to the other side (*i.e.* upper side LM). Thus the tension in the lower side belt will be more than that in the upper side belt. The lower side belt (because of more tension) is known as ***tight side*** whereas the upper side belt (because of less tension) is known as ***slack side***, as shown in Fig. 11.3.

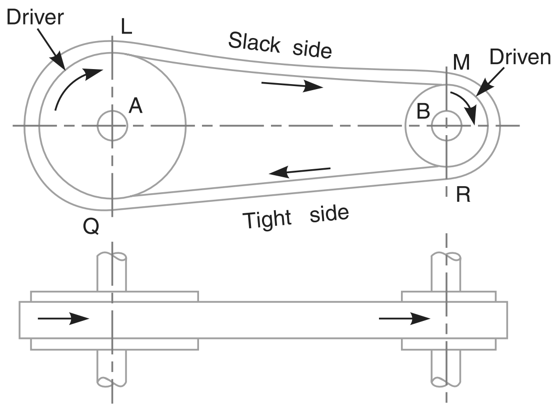


Fig. 11.3. Open belt drive.

2. ***Crossed or twist belt drive***. The crossed or twist belt drive, as shown in Fig. 11.4, is used with shafts arranged parallel and rotating in the opposite directions.

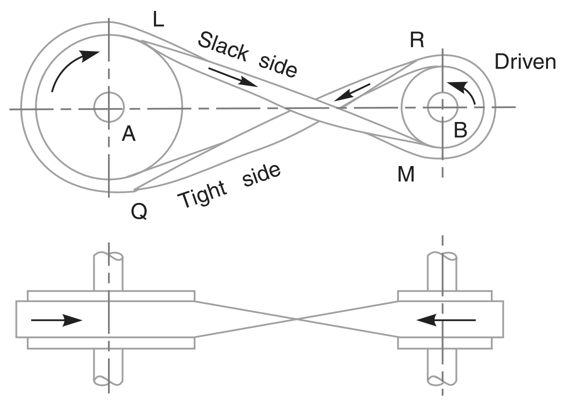


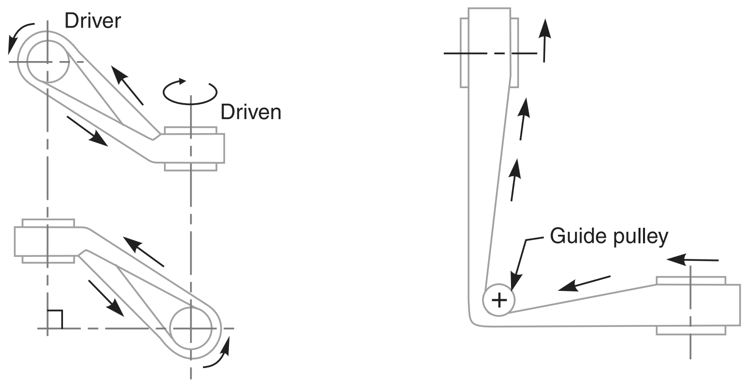
Fig. 11.4. Crossed or twist belt drive.

In this case, the driver pulls the belt from one side (*i.e.* RQ) and delivers it to the other side (*i.e.* LM). Thus the tension in the belt RQ will be more than that in the belt LM. The belt RQ (because of more tension) is known as ***tight side***, whereas the belt LM (because of less tension) is known as ***slack side***, as shown in Fig. 11.4.

A little consideration will show that at a point where the belt crosses, it rubs against each other and there will be excessive wear and tear. In order to avoid this, the shafts should be placed at a maximum distance of 20 *b*, where *b* is the width of belt and the speed of the belt should be less than 15 m/s.

3. ***Quarter turn belt drive.*** The quarter turn belt drive also known as right angle belt drive, as shown in Fig. 11.5 (*a*), is used with shafts arranged at right angles and rotating in one definite direction. In order to prevent the belt from leaving the pulley, the width of the face of the pulley should be greater or equal to 1.4 *b*, where *b* is the width of belt.

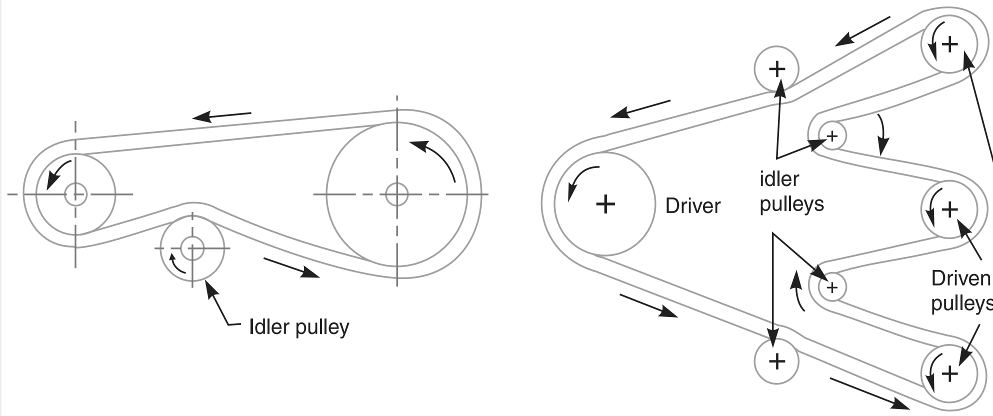
In case the pulleys cannot be arranged, as shown in Fig. 11.5 (*a*), or when the reversible motion is desired, then a ***quarter turn belt drive with guide pulley***, as shown in Fig. 11.5 (*b*), may be used.



(*a*) Quarter turn belt drive. (*b*) Quarter turn belt drive with guide pulley.

Fig. 11.5

4. ***Belt drive with idler pulleys***. A belt drive with an idler pulley, as shown in Fig. 11.6 (*a*), is used with shafts arranged parallel and when an open belt drive cannot be used due to small angle of contact on the smaller pulley. This type of drive is provided to obtain high velocity ratio and when the required belt tension cannot be obtained by other means.



(*a*) Belt drive with single idler pulley. (*b*) Belt drive with many idler pulleys.

Fig. 11.6

When it is desired to transmit motion from one shaft to several shafts, all arranged in parallel, a belt drive with many idler pulleys, as shown in Fig. 11.6 (*b*), may be employed.

5. ***Compound belt drive***. A compound belt drive, as shown in Fig. 11.7, is used when power is transmitted from one shaft to another through a number of pulleys.

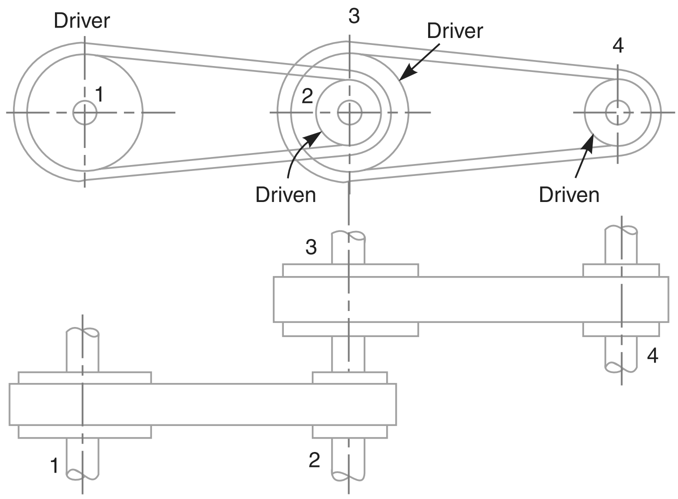


Fig. 11.7. Compound belt brive.

6. ***Stepped or cone pulley drive***. A stepped or cone pulley drive, as shown in Fig. 11.8, is used for changing the speed of the driven shaft while the main or driving shaft runs at constant speed. This is accomplished by shifting the belt from one part of the steps to the other.

7. ***Fast and loose pulley drive***. A fast and loose pulley drive, as shown in Fig. 11.9, is used when the driven or machine shaft is to be started or stopped when ever desired without interfering with the driving shaft. A pulley which is keyed to the machine shaft is called ***fast pulley*** and runs at the same speed as that of machine shaft. A loose pulley runs freely over the machine shaft and is incapable of transmitting any power. When the driven shaft is required to be stopped, the belt is pushed on to the loose pulley by means of sliding bar having belt forks.

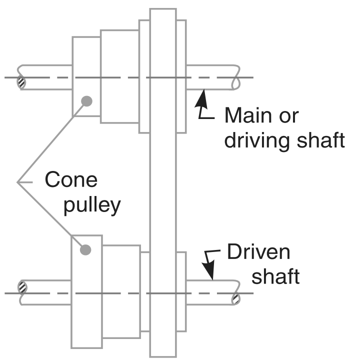


Fig. 11.8. Stepped or cone pulley drive.

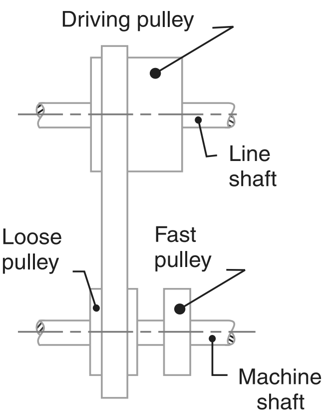


Fig. 11.9. Fast and loose pulley drive.

11.7. Velocity Ratio of Belt Drive

It is the **ratio between the velocities of the driver and the follower or driven.** It may be expressed, mathematically, as discussed below :

Let d_1 = Diameter of the driver,

d_2 = Diameter of the follower,

N_1 = Speed of the driver in r.p.m., and

N_2 = Speed of the follower in r.p.m.

∴ Length of the belt that passes over the driver, in one minute

$$= \pi\, d_1 . N_1$$

Similarly, length of the belt that passes over the follower, in one minute

$$= \pi\, d_2 . N_2$$

Since the length of belt that passes over the driver in one minute is equal to the length of belt that passes over the follower in one minute, therefore

$$\pi\, d_1 . N_1 = \pi\, d_2 . N_2$$

∴ Velocity ratio, $\dfrac{N_2}{N_1} = \dfrac{d_1}{d_2}$

When the thickness of the belt (t) is considered, then velocity ratio,

$$\frac{N_2}{N_1} = \frac{d_1 + t}{d_2 + t}$$

Note: The velocity ratio of a belt drive may also be obtained as discussed below :

We know that peripheral velocity of the belt on the driving pulley,

$$v_1 = \frac{\pi\, d_1 . N_1}{60} \text{ m/s}$$

and peripheral velocity of the belt on the driven or follower pulley,

$$v_2 = \frac{\pi\, d_2 . N_2}{60} \text{ m/s}$$

When there is no slip, then $v_1 = v_2$.

$$\therefore \quad \frac{\pi d_1 . N_1}{60} = \frac{\pi d_2 . N_2}{60} \quad \text{or} \quad \frac{N_2}{N_1} = \frac{d_1}{d_2}$$

11.8. Velocity Ratio of a Compound Belt Drive

Sometimes the power is transmitted from one shaft to another, through a number of pulleys as shown in Fig. 11.7. Consider a pulley 1 driving the pulley 2. Since the pulleys 2 and 3 are keyed to the same shaft, therefore the pulley 1 also drives the pulley 3 which, in turn, drives the pulley 4.

Let d_1 = Diameter of the pulley 1,

N_1 = Speed of the pulley 1 in r.p.m.,

d_2, d_3, d_4, and N_2, N_3, N_4 = Corresponding values for pulleys 2, 3 and 4.

We know that velocity ratio of pulleys 1 and 2,

$$\frac{N_2}{N_1} = \frac{d_1}{d_2} \quad \text{...}(i)$$

Similarly, velocity ratio of pulleys 3 and 4,

$$\frac{N_4}{N_3} = \frac{d_3}{d_4} \quad \text{...}(ii)$$

Multiplying equations (*i*) and (*ii*),

$$\frac{N_2}{N_1} \cdot \frac{N_4}{N_3} = \frac{d_1}{d_2} \cdot \frac{d_3}{d_4}$$

or $$\frac{N_4}{N_1} = \frac{d_1 \times d_3}{d_2 \times d_4} \qquad \ldots (\because N_2 = N_3, \text{ being keyed to the same shaft})$$

A little consideration will show, that if there are six pulleys, then

$$\frac{N_6}{N_1} = \frac{d_1 \times d_3 \times d_5}{d_2 \times d_4 \times d_6}$$

or $$\frac{\text{Speed of last driven}}{\text{Speed of first driver}} = \frac{\text{Product of diameters of drivers}}{\text{Product of diameters of drivens}}$$

11.9. Slip of Belt

In the previous articles, we have discussed the motion of belts and shafts assuming a firm frictional grip between the belts and the shafts. But sometimes, the frictional grip becomes insufficient. This may cause some forward motion of the driver without carrying the belt with it. This may also cause some forward motion of the belt without carrying the driven pulley with it. This is called ***slip of the belt*** and is generally expressed as a percentage.

The result of the belt slipping is to reduce the velocity ratio of the system. As the slipping of the belt is a common phenomenon, thus the belt should never be used where a definite velocity ratio is of importance (as in the case of hour, minute and second arms in a watch).

Let $s_1\,\%$ = Slip between the driver and the belt, and

$s_2\,\%$ = Slip between the belt and the follower.

∴ Velocity of the belt passing over the driver per second

$$v = \frac{\pi d_1 . N_1}{60} - \frac{\pi d_1 . N_1}{60} \cdot \frac{s_1}{100} = \frac{\pi d_1 . N_1}{60}\left(1 - \frac{s_1}{100}\right) \qquad \ldots (i)$$

and velocity of the belt passing over the follower per second,

$$\frac{\pi d_2 . N_2}{60} = v - v \cdot \frac{s_2}{100} = v\left(1 - \frac{s_2}{100}\right)$$

Substituting the value of v from equation (i),

$$\frac{\pi d_2 \, N_2}{60} = \frac{\pi d_1 \, N_1}{60}\left(1 - \frac{s_1}{100}\right)\left(1 - \frac{s_2}{100}\right)$$

$$\frac{N_2}{N_1} = \frac{d_1}{d_2}\left(1 - \frac{s_1}{100} - \frac{s_2}{100}\right) \qquad \ldots \left(\text{Neglecting } \frac{s_1 \cdot s_2}{100 \cdot 100}\right)$$

$$= \frac{d_1}{d_2}\left(1 - \frac{s_1 + s_2}{100}\right) = \frac{d_1}{d_2}\left(1 - \frac{s}{100}\right)$$

... (where $s = s_1 + s_2$, *i.e.* total percentage of slip)

If thickness of the belt (t) is considered, then

$$\frac{N_2}{N_1} = \frac{d_1 + t}{d_2 + t}\left(1 - \frac{s}{100}\right)$$

Example 11.1. *An engine, running at 150 r.p.m., drives a line shaft by means of a belt. The engine pulley is 750 mm diameter and the pulley on the line shaft being 450 mm. A 900 mm diameter pulley on the line shaft drives a 150 mm diameter pulley keyed to a dynamo shaft. Find the speed of the dynamo shaft, when* **1.** *there is no slip, and* **2.** *there is a slip of 2% at each drive.*

Solution. Given : $N_1 = 150$ r.p.m. ; $d_1 = 750$ mm ; $d_2 = 450$ mm ; $d_3 = 900$ mm ; $d_4 = 150$ mm

The arrangement of belt drive is shown in Fig. 11.10.

Let $\quad N_4$ = Speed of the dynamo shaft .

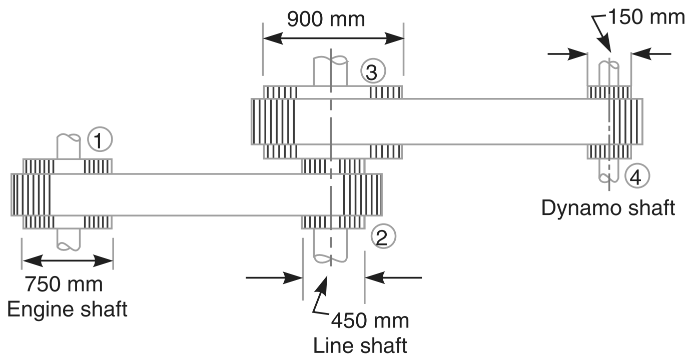


Fig. 11.10

1. ***When there is no slip***

We know that
$$\frac{N_4}{N_1} = \frac{d_1 \times d_3}{d_2 \times d_4} \quad \text{or} \quad \frac{N_4}{150} = \frac{750 \times 900}{450 \times 150} = 10$$

$\therefore \quad N_4 = 150 \times 10 = 1500$ r.p.m. **Ans.**

2. ***When there is a slip of 2% at each drive***

We know that
$$\frac{N_4}{N_1} = \frac{d_1 \cdot d_3}{d_2 \cdot d_4}\left(1 - \frac{s_1}{100}\right)\left(1 - \frac{s_2}{100}\right)$$

$$\frac{N_4}{150} = \frac{750 \cdot 900}{450 \cdot 150}\left(1 - \frac{2}{100}\right)\left(1 - \frac{2}{100}\right) = 9.6$$

$\therefore \quad N_4 = 150 \times 9.6 = 1440$ r.p.m. **Ans.**

11.10. Creep of Belt

When the belt passes from the slack side to the tight side, a certain portion of the belt extends and it contracts again when the belt passes from the tight side to slack side. Due to these changes of length, there is a relative motion between the belt and the pulley surfaces. This relative motion is termed as ***creep***. The total effect of creep is to reduce slightly the speed of the driven pulley or follower. Considering creep, the velocity ratio is given by

$$\frac{N_2}{N_1} = \frac{d_1}{d_2} \cdot \frac{E + \sqrt{\sigma_2}}{E + \sqrt{\sigma_1}}$$

where $\quad \sigma_1$ and σ_2 = Stress in the belt on the tight and slack side respectively, and

E = Young's modulus for the material of the belt.

Example 11.2. *The power is transmitted from a pulley 1 m diameter running at 200 r.p.m. to a pulley 2.25 m diameter by means of a belt. Find the speed lost by the driven pulley as a result of creep, if the stress on the tight and slack side of the belt is 1.4 MPa and 0.5 MPa respectively. The Young's modulus for the material of the belt is 100 MPa.*

Solution. Given : d_1 = 1 m ; N_1 = 200 r.p.m. ; d_2 = 2.25 m ; σ_1 = 1.4 MPa = 1.4 × 10^6 N/m^2; σ_2 = 0.5 MPa = 0.5 × 10^6 N/m^2 ; E = 100 MPa = 100 × 10^6 N/m^2

Let N_2 = Speed of the driven pulley.

Neglecting creep, we know that

$$\frac{N_2}{N_1} = \frac{d_1}{d_2} \quad \text{or} \quad N_2 = N_1 \cdot \frac{d_1}{d_2} = 200 \cdot \frac{1}{2.25} = 88.9 \text{ r.p.m.}$$

Considering creep, we know that

$$\frac{N_2}{N_1} = \frac{d_1}{d_2} \cdot \frac{E + \sqrt{\sigma_2}}{E + \sqrt{\sigma_1}}$$

or

$$N_2 = 200 \cdot \frac{1}{2.25} \cdot \frac{100 \cdot 10^6 + \sqrt{0.5 \cdot 10^6}}{100 \cdot 10^6 + \sqrt{1.4 \cdot 10^6}} = 88.7 \text{ r.p.m.}$$

∴ Speed lost by driven pulley due to creep

= 88.9 – 88.7 = 0.2 r.p.m. **Ans.**

11.13. Power Transmitted by a Belt

Fig. 11.14 shows the driving pulley (or driver) A and the driven pulley (or follower) B. We have already discussed that the driving pulley pulls the belt from one side and delivers the same to the other side. It is thus obvious that the tension on the former side (*i.e.* tight side) will be greater than the latter side (*i.e.* slack side) as shown in Fig. 11.14.

Let T_1 and T_2 = Tensions in the tight and slack side of the belt respectively in newtons,

r_1 and r_2 = Radii of the driver and follower respectively, and

v = Velocity of the belt in m/s.

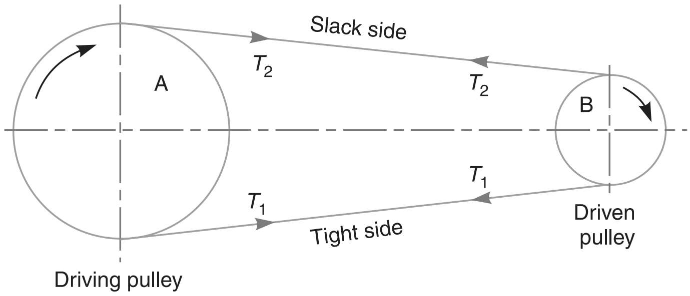


Fig. 11.14. Power transmitted by a belt.

The effective turning (driving) force at the circumference of the follower is the difference between the two tensions (*i.e.* $T_1 - T_2$).

$\therefore$ Work done per second = $(T_1 - T_2)\, v$ N-m/s

and power transmitted, $P = (T_1 - T_2)\, v$ W ...($\because$ 1 N-m/s = 1 W)

A little consideration will show that the torque exerted on the driving pulley is $(T_1 - T_2)\, r_1$. Similarly, the torque exerted on the driven pulley *i.e.* follower is $(T_1 - T_2)\, r_2$.

11.14. Ratio of Driving Tensions For Flat Belt Drive

Consider a driven pulley rotating in the clockwise direction as shown in Fig. 11.15.

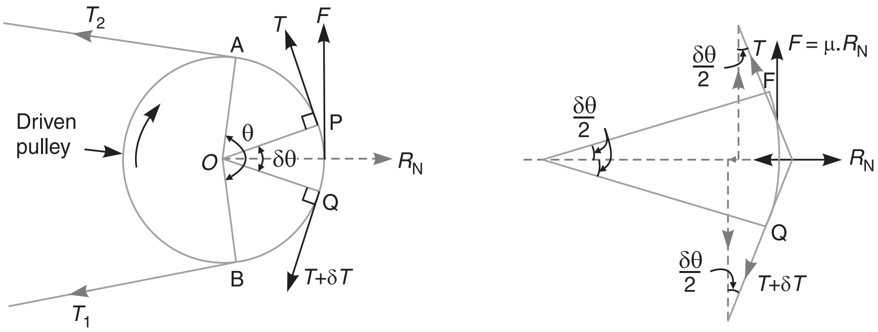


Fig. 11.15. Ratio of driving tensions for flat belt.

Let T_1 = Tension in the belt on the tight side,

T_2 = Tension in the belt on the slack side, and

θ = Angle of contact in radians (*i.e.* angle subtended by the arc AB, along which the belt touches the pulley at the centre).

Now consider a small portion of the belt PQ, subtending an angle $\delta\theta$ at the centre of the pulley as shown in Fig. 11.15. The belt PQ is in equilibrium under the following forces :

1. Tension T in the belt at P,
2. Tension $(T + \delta T)$ in the belt at Q,
3. Normal reaction R_N, and
4. Frictional force, $F = \mu \times R_N$, where μ is the coefficient of friction between the belt and pulley.

Resolving all the forces horizontally and equating the same,

$$R_N = (T + \delta T)\sin\frac{\delta\theta}{2} + T\sin\frac{\delta\theta}{2} \quad ...(i)$$

Since the angle $\delta\theta$ is very small, therefore putting $\sin \delta\theta/2 = \delta\theta/2$ in equation (*i*),

$$R_N = (T + \delta T)\frac{\delta\theta}{2} + T\cdot\frac{\delta\theta}{2} = \frac{T.\delta\theta}{2} + \frac{\delta T.\delta\theta}{2} + \frac{T.\delta\theta}{2} = T\delta\theta \quad ...(ii)$$

$$...\left(\text{Neglecting } \frac{\delta T.\delta\theta}{2}\right)$$

Now resolving the forces vertically, we have

$$\propto \cdot R_N = (T + \delta T)\cos\frac{\delta\theta}{2} - T\cos\frac{\delta\theta}{2} \quad ...(iii)$$

Since the angle $\delta\theta$ is very small, therefore putting $\cos\delta\theta/2 = 1$ in equation (*iii*),

$$\mu \times R_N = T + \delta T - T = \delta T \text{ or } R_N = \frac{\delta T}{\propto} \quad ...(iv)$$

Equating the values of R_N from equations (*ii*) and (*iv*),

$$T.\delta\theta = \frac{\delta T}{\propto} \text{ or } \frac{\delta T}{T} = \propto.\delta\theta$$

Integrating both sides between the limits T_2 and T_1 and from 0 to θ respectively,

i.e. $$\int_{T_2}^{T_1}\frac{\delta T}{T} = \propto\int_0^{\theta}\delta\theta \quad \text{or} \quad \log_e\left(\frac{T_1}{T_2}\right) = \propto.\theta \text{ or } \frac{T_1}{T_2} = e^{\propto.\theta} \quad ...(v)$$

Equation (*v*) can be expressed in terms of corresponding logarithm to the base 10, *i.e.*

$$2.3\log\left(\frac{T_1}{T_2}\right) = \propto.\theta$$

The above expression gives the relation between the tight side and slack side tensions, in terms of coefficient of friction and the angle of contact.

11.15. Determination of Angle of Contact

When the two pulleys of different diameters are connected by means of an open belt as shown in Fig. 11.16 (*a*), then the angle of contact or lap (θ) at the smaller pulley must be taken into consideration.

Let r_1 = Radius of larger pulley,

r_2 = Radius of smaller pulley, and

x = Distance between centres of two pulleys (*i.e.* $O_1 O_2$).

From Fig. 11.16 (*a*),

$$\sin\alpha = \frac{O_1 M}{O_1 O_2} = \frac{O_1 E - ME}{O_1 O_2} = \frac{r_1 - r_2}{x} \quad ...(\because ME = O_2 F = r_2)$$

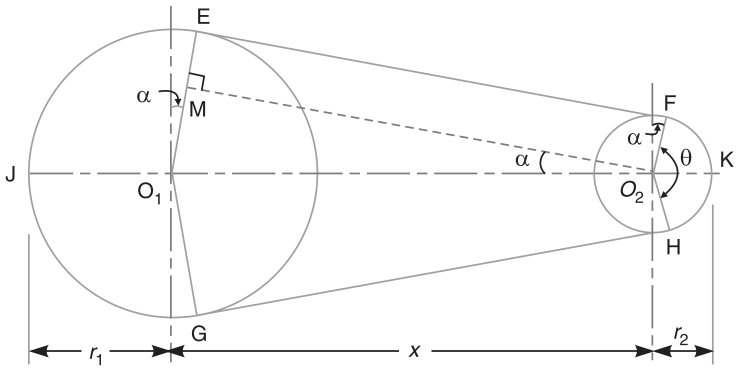
$\therefore$ Angle of contact or lap,

$$\theta = (180° - 2\alpha)\frac{\pi}{180} \text{ rad}$$

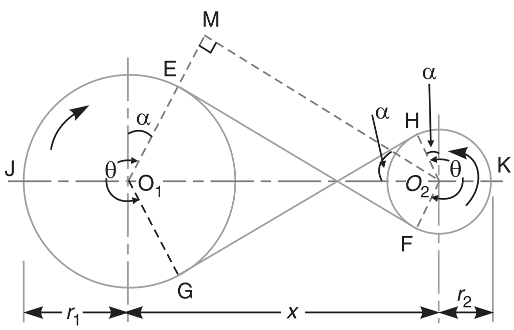
A little consideration will show that when the two pulleys are connected by means of a crossed belt as shown in Fig. 11.16 (*b*), then the angle of contact or lap (θ) on both the pulleys is same. From Fig. 11.16 (*b*),

$$\sin\alpha = \frac{O_1 M}{O_1 O_2} = \frac{O_1 E + ME}{O_1 O_2} = \frac{r_1 + r_2}{x}$$

∴ Angle of contact or lap, $\theta = (180° + 2\alpha)\dfrac{\pi}{180}$ rad



(*a*) Open belt drive.



(*b*) Crossed belt drive.

Fig. 11.16

Example 11.4. *Find the power transmitted by a belt running over a pulley of 600 mm diameter at 200 r.p.m. The coefficient of friction between the belt and the pulley is 0.25, angle of lap 160° and maximum tension in the belt is 2500 N.*

Solution. Given : d = 600 mm = 0.6 m ; N = 200 r.p.m. ; μ = 0.25 ; θ = 160° = 160 × π / 180 = 2.793 rad ; T_1 = 2500 N

We know that velocity of the belt,

$$v = \frac{\pi d . N}{60} = \frac{\pi \cdot 0.6 \cdot 200}{60} = 6.284 \text{ m/s}$$

Let T_2 = Tension in the slack side of the belt.

We know that $2.3 \log\left(\dfrac{T_1}{T_2}\right) = \propto . \theta = 0.25 \cdot 2.793 = 0.6982$

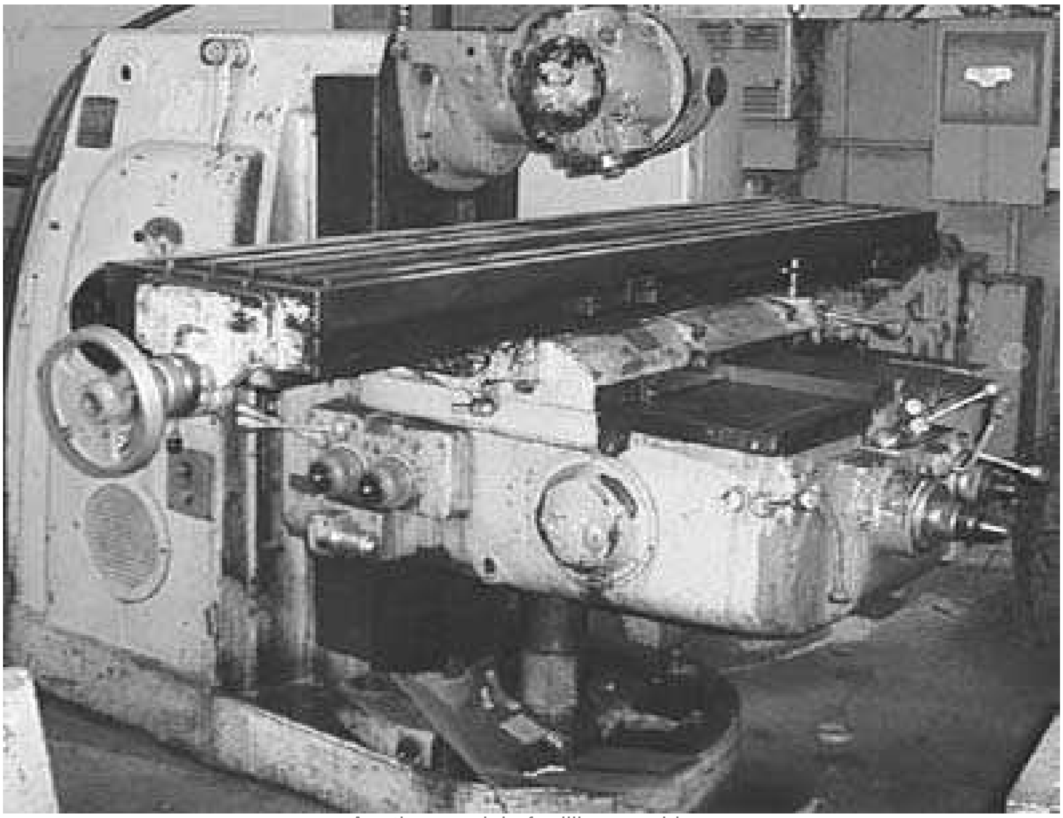
$$\log\left(\frac{T_1}{T_2}\right) = \frac{0.6982}{2.3} = 0.3036$$

$\therefore$ $$\frac{T_1}{T_2} = 2.01 \qquad \text{...(Taking antilog of 0.3036)}$$

and $$T_2 = \frac{T_1}{2.01} = \frac{2500}{2.01} = 1244 \text{ N}$$

We know that power transmitted by the belt,

$$P = (T_1 - T_2)\, v = (2500 - 1244)\, 6.284 = 7890 \text{ W}$$
$$= 7.89 \text{ kW} \quad \textbf{Ans.}$$



Another model of milling machine.

Note : This picture is given as additional information and is not a direct example of the current chapter.

Example 11.5. *A casting weighing 9 kN hangs freely from a rope which makes 2.5 turns round a drum of 300 mm diameter revolving at 20 r.p.m. The other end of the rope is pulled by a man. The coefficient of friction is 0.25. Determine* **1.** *The force required by the man, and* **2.** *The power to raise the casting.*

Solution. Given : $W = T_1 = 9$ kN = 9000 N ; $d = 300$ mm = 0.3 m ; $N = 20$ r.p.m. ; $\mu = 0.25$

1. ***Force required by the man***

Let T_2 = Force required by the man.

Since the rope makes 2.5 turns round the drum, therefore angle of contact,

$$\theta = 2.5 \times 2\,\pi = 5\,\pi \text{ rad}$$

We know that $$2.3 \log \left(\frac{T_1}{T_2} \right) = \propto . \theta = 0.25 \cdot 5\pi = 3.9275$$

$$\log \left(\frac{T_1}{T_2} \right) = \frac{3.9275}{2.3} = 1.71 \quad \text{or} \quad \frac{T_1}{T_2} = 51$$

...(Taking antilog of 1.71)

$\therefore$ $$T_2 = \frac{T_1}{51} = \frac{9000}{51} = 176.47 \text{ N} \quad \textbf{Ans.}$$

2. ***Power to raise the casting***

We know that velocity of the rope,

$$v = \frac{\pi\, d.N}{60} = \frac{\pi \cdot 0.3 \cdot 20}{60} = 0.3142 \text{ m/s}$$

$\therefore$ Power to raise the casting,

$$P = (T_1 - T_2)\, v = (9000 - 176.47)\, 0.3142 = 2772 \text{ W}$$
$$= 2.772 \text{ kW} \quad \textbf{Ans.}$$

Example 11.6. *Two pulleys, one 450 mm diameter and the other 200 mm diameter are on parallel shafts 1.95 m apart and are connected by a crossed belt. Find the length of the belt required and the angle of contact between the belt and each pulley.*

What power can be transmitted by the belt when the larger pulley rotates at 200 rev/min, if the maximum permissible tension in the belt is 1 kN, and the coefficient of friction between the belt and pulley is 0.25 ?

Solution. Given : d_1 = 450 mm = 0.45 m or r_1 = 0.225 m ; d_2 = 200 mm = 0.2 m or r_2 = 0.1 m ; x = 1.95 m ; N_1 = 200 r.p.m. ; T_1 = 1 kN = 1000 N ; μ = 0.25

We know that speed of the belt,

$$v = \frac{\pi\, d_1 . N_1}{60} = \frac{\pi \cdot 0.45 \cdot 200}{60} = 4.714 \text{ m/s}$$

Length of the belt

We know that length of the crossed belt,

$$L = \pi (r_1 + r_2) + 2x + \frac{(r_1 + r_2)^2}{x}$$

$$= \pi (0.225 + 0.1) + 2 \cdot 1.95 + \frac{(0.225 + 0.1)^2}{1.95} = 4.975 \text{ m} \quad \textbf{Ans.}$$

Angle of contact between the belt and each pulley

Let θ = Angle of contact between the belt and each pulley.

We know that for a crossed belt drive,

$$\sin \alpha = \frac{r_1 + r_2}{x} = \frac{0.225 + 0.1}{1.95} = 0.1667 \quad \text{or} \quad \alpha = 9.6°$$

$\therefore$ $$\theta = 180° + 2\,\alpha = 180° + 2 \times 9.6° = 199.2°$$

$$= 199.2 \cdot \frac{\pi}{180} = 3.477 \text{ rad} \quad \textbf{Ans.}$$

Power transmitted

Let T_2 = Tension in the slack side of the belt.

We know that

$$2.3 \log\left(\frac{T_1}{T_2}\right) = \propto.\ \theta = 0.25 \cdot 3.477 = 0.8692$$

$$\log\left(\frac{T_1}{T_2}\right) = \frac{0.8692}{2.3} = 0.378 \quad \text{or} \quad \frac{T_1}{T_2} = 2.387 \qquad \text{...(Taking antilog of 0.378)}$$

$$\therefore \qquad T_2 = \frac{T_1}{2.387} = \frac{1000}{2.387} = 419 \text{ N}$$

We know that power transmitted,

$$P = (T_1 - T_2)\, v = (1000 - 419)\ 4.714 = 2740 \text{ W} = 2.74 \text{ kW} \quad \textbf{Ans.}$$

11.16. Centrifugal Tension

Since the belt continuously runs over the pulleys, therefore, some centrifugal force is caused, whose effect is to increase the tension on both, tight as well as the slack sides. The tension caused by centrifugal force is called ***centrifugal tension***. At lower belt speeds (less than 10 m/s), the centrifugal tension is very small, but at higher belt speeds (more than 10 m/s), its effect is considerable and thus should be taken into account.

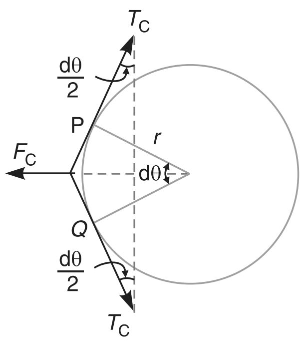


Fig. 11.17. Centrifugal tension.

Consider a small portion PQ of the belt subtending an angle $d\theta$ the centre of the pulley as shown in Fig. 11.17.

Let m = Mass of the belt per unit length in kg,

v = Linear velocity of the belt in m/s,

r = Radius of the pulley over which the belt runs in metres, and

T_C = Centrifugal tension acting tangentially at P and Q in newtons.

We know that length of the belt PQ

$$= r.\, d\theta$$

and mass of the belt PQ $= m.\, r.\, d\theta$

$\therefore$ Centrifugal force acting on the belt PQ,

$$F_C = (m.r.d\theta)\frac{v^2}{r} = m.d\theta.v^2$$

The centrifugal tension T_C acting tangentially at P and Q keeps the belt in equilibrium.

Now resolving the forces (*i.e.* centrifugal force and centrifugal tension) horizontally and equating the same, we have

$$T_C \sin\left(\frac{d\theta}{2}\right) + T_C \sin\left(\frac{d\theta}{2}\right) = F_C = m.d\theta.v^2$$

Since the angle $d\theta$ is very small, therefore, putting $\sin\left(\frac{d\theta}{2}\right) = \frac{d\theta}{2}$, in the above expression,

$$2T_C\left(\frac{d\theta}{2}\right) = m.d\theta.v^2 \quad \text{or} \quad T_C = m\,.\,v^2$$

Notes : 1. When the centrifugal tension is taken into account, then total tension in the tight side,

$$T_{t1} = T_1 + T_C$$

and total tension in the slack side,

$$T_{t2} = T_2 + T_C$$

2. Power transmitted, $P = (T_{t1} - T_{t2})\,v$ ...(in watts)

$= [(T_1 + T_C) - (T_2 + T_C)]\,v = (T_1 - T_2)\,v$ ...(same as before)

Thus we see that centrifugal tension has no effect on the power transmitted.

3. The ratio of driving tensions may also be written as

$$2.3\log\left(\frac{T_{t1} - T_C}{T_{t2} - T_C}\right) = \propto.\,\theta$$

where T_{t1} = Maximum or total tension in the belt.

11.17. Maximum Tension in the Belt

A little consideration will show that the maximum tension in the belt (T) is equal to the total tension in the tight side of the belt (T_{t1}).

Let σ = Maximum safe stress in N/mm^2,

b = Width of the belt in mm, and

t = Thickness of the belt in mm.

We know that maximum tension in the belt,

T = Maximum stress × cross-sectional area of belt = $\sigma.\,b.\,t$

When centrifugal tension is neglected, then

T (or T_{t1}) = T_1, *i.e.* Tension in the tight side of the belt

and when centrifugal tension is considered, then

$$T \text{ (or } T_{t1}) = T_1 + T_C$$

11.18. Condition For the Transmission of Maximum Power

We know that power transmitted by a belt,

$$P = (T_1 - T_2)\,v \qquad ...(i)$$

where T_1 = Tension in the tight side of the belt in newtons,

T_2 = Tension in the slack side of the belt in newtons, and

v = Velocity of the belt in m/s.

From Art. 11.14, we have also seen that the ratio of driving tensions is

$$\frac{T_1}{T_2} = e^{\propto.\,\theta} \quad \text{or} \quad T_2 = \frac{T_1}{e^{\propto.\,\theta}} \qquad ...(ii)$$

Substituting the value of T_2 in equation (*i*),

$$P = \left(T_1 - \frac{T_1}{e^{\propto.\,\theta}}\right)v = T_1\left(1 - \frac{1}{e^{\propto.\,\theta}}\right)v = T_1.v.C \qquad ...(iii)$$

where $$C = 1 - \frac{1}{e^{\mu.\theta}}$$

We know that $T_1 = T - T_C$

where T = Maximum tension to which the belt can be subjected in newtons, and

T_C = Centrifugal tension in newtons.

Substituting the value of T_1 in equation ***(iii)***,

$$P = (T - T_C)\, v.C$$

$$= (T - m.v^2)\, v.C = (T.v - m\, v^3)\, C \quad \text{... (Substituting } T_C = m.\ v^2\text{)}$$

For maximum power, differentiate the above expression with respect to v and equate to zero, *i.e.*

$$\frac{dP}{dv} = 0 \qquad \text{or} \qquad \frac{d}{dv}(T.v - mv^3)\, C = 0$$

$\therefore$ $$T - 3\, m\, .\, v^2 = 0$$

or $$T - 3\, T_C = 0 \text{ or } T = 3\, T_C \quad ...(iv)$$

It shows that when the power transmitted is maximum, 1/3rd of the maximum tension is absorbed as centrifugal tension.

Notes : 1. We know that $T_1 = T - T_C$ and for maximum power, $T_C = \frac{T}{3}$.

$\therefore$ $$T_1 = T - \frac{T}{3} = \frac{2T}{3}$$

2. From equation ***(iv)***, the velocity of the belt for the maximum power,

$$v = \sqrt{\frac{T}{3m}}$$

Example. 11.7. *A shaft rotating at 200 r.p.m. drives another shaft at 300 r.p.m. and transmits 6 kW through a belt. The belt is 100 mm wide and 10 mm thick. The distance between the shafts is 4m. The smaller pulley is 0.5 m in diameter. Calculate the stress in the belt, if it is* **1.** *an open belt drive, and* **2.** *a cross belt drive. Take μ = 0.3.*

Solution. Given : N_1 = 200 r.p.m. ; N_2 = 300 r.p.m. ; P = 6 kW = 6×10^3 W ; b = 100 mm ; t = 10 mm ; x = 4 m ; d_2 = 0.5 m ; μ = 0.3

Let σ = Stress in the belt.

1. ***Stress in the belt for an open belt drive***

First of all, let us find out the diameter of larger pulley (d_1). We know that

$$\frac{N_2}{N_1} = \frac{d_1}{d_2} \quad \text{or} \quad d_1 = \frac{N_2.d_2}{N_1} = \frac{300 \cdot 0.5}{200} = 0.75\,\text{m}$$

and velocity of the belt, $$v = \frac{\pi d_2.N_2}{60} = \frac{\pi \cdot 0.5 \cdot 300}{60} = 7.855 \text{ m/s}$$

Now let us find the angle of contact on the smaller pulley. We know that, for an open belt drive,

$$\sin\alpha = \frac{r_1 - r_2}{x} = \frac{d_1 - d_2}{2x} = \frac{0.75 - 0.5}{2 \cdot 4} = 0.03125 \quad \text{or} \quad \alpha = 1.8°$$

∴ Angle of contact, $\theta = 180° - 2\,\alpha = 180 - 2 \times 1.8 = 176.4°$

$= 176.4 \times \pi / 180 = 3.08$ rad

Let T_1 = Tension in the tight side of the belt, and

T_2 = Tension in the slack side of the belt.

We know that

$$2.3 \log\left(\frac{T_1}{T_2}\right) = \propto.\,\theta = 0.3 \cdot 3.08 = 0.924$$

∴ $$\log\left(\frac{T_1}{T_2}\right) = \frac{0.924}{2.3} = 0.4017 \quad \text{or} \quad \frac{T_1}{T_2} = 2.52 \qquad ...(i)$$

...(Taking antilog of 0.4017)

We also know that power transmitted (P),

$$6 \times 10^3 = (T_1 - T_2)\,v = (T_1 - T_2)\,7.855$$

∴ $$T_1 - T_2 = 6 \times 10^3 / 7.855 = 764 \text{ N} \qquad ...(ii)$$

From equations (*i*) and (*ii*),

$$T_1 = 1267 \text{ N, and } T_2 = 503 \text{ N}$$

We know that maximum tension in the belt (T_1),

$$1267 = \sigma\,.\,b.\,t = \sigma \times 100 \times 10 = 1000\,\sigma$$

∴ $$\sigma = 1267 / 1000 = 1.267 \text{ N/mm}^2 = 1.267 \text{ MPa}$$ **Ans.**

...[∵ 1 MPa = 1 MN/m^2 = 1 N/mm^2]

Stress in the belt for a cross belt drive

We know that for a cross belt drive,

$$\sin\alpha = \frac{r_1 + r_2}{x} = \frac{d_1 + d_2}{2x} = \frac{0.75 + 0.5}{2 \cdot 4} = 0.1562 \quad \text{or} \quad \alpha = 9°$$

∴ Angle of contact, $\theta = 180° + 2\alpha = 180 + 2 \times 9 = 198°$

$= 198 \times \pi / 180 = 3.456$ rad

We know that

$$2.3 \log\left(\frac{T_1}{T_2}\right) = \propto.\,\theta = 0.3 \cdot 3.456 = 1.0368$$

$$\log\left(\frac{T_1}{T_2}\right) = \frac{1.0368}{2.3} = 0.4508 \quad \text{or} \quad \frac{T_1}{T_2} = 2.82 \qquad ...(iii)$$

...(Taking antilog of 0.4508)

From equations (*ii*) and (*iii*),

$$T_1 = 1184 \text{ N and } T_2 = 420 \text{ N}$$

We know that maximum tension in the belt (T_1),

$$1184 = \sigma.\,b.\,t = \sigma \times 100 \times 10 = 1000\,\sigma$$

∴ $$\sigma = 1184 / 1000 = 1.184 \text{ N/mm}^2 = 1.184 \text{ MPa}$$ **Ans.**

Example 11.8. *A leather belt is required to transmit 7.5 kW from a pulley 1.2 m in diameter, running at 250 r.p.m. The angle embraced is 165° and the coefficient of friction between the belt and the pulley is 0.3. If the safe working stress for the leather belt is 1.5 MPa, density of leather 1 Mg/m^3 and thickness of belt 10 mm, determine the width of the belt taking centrifugal tension into account.*

Solution. Given : $P = 7.5$ kW = 7500 W ; $d = 1.2$ m ; $N = 250$ r.p.m. ; $\theta = 165° = 165 \times \pi / 180 = 2.88$ rad ; $\mu = 0.3$; $\sigma = 1.5$ MPa = 1.5×10^6 * N/m^2 ; $\rho = 1$ Mg/m^3 = 1×10^6 g/m^3 = 1000 kg/m^3; $t = 10$ mm = 0.01 m

Let b = Width of belt in metres,

T_1 = Tension in the tight side of the belt in N, and

T_2 = Tension in the slack side of the belt in N.

We know that velocity of the belt,

$$v = \pi\, d\,.\, N / 60 = \pi \times 1.2 \times 250/60 = 15.71 \text{ m/s}$$

and power transmitted (P),

$$7500 = (T_1 - T_2)\, v = (T_1 - T_2)\, 15.71$$

$$\therefore \quad T_1 - T_2 = 7500 / 15.71 = 477.4 \text{ N} \quad ...(i)$$

We know that

$$2.3 \log\left(\frac{T_1}{T_2}\right) = \propto .\, \theta = 0.3 \cdot 2.88 = 0.864$$

$$\log\left(\frac{T_1}{T_2}\right) = \frac{0.864}{2.3} = 0.3756 \quad \text{or} \quad \frac{T_1}{T_2} = 2.375 \quad ...(ii)$$

...(Taking antilog of 0.3756)

From equations (*i*) and (*ii*),

$$T_1 = 824.6 \text{ N}, \quad \text{and} \quad T_2 = 347.2 \text{ N}$$

We know that mass of the belt per metre length,

$$m = \text{Area} \times \text{length} \times \text{density} = b.t.l.\rho$$

$$= b \times 0.01 \times 1 \times 1000 = 10\, b \text{ kg}$$

$\therefore$ Centrifugal tension,

$$T_C = m\,.\, v^2 = 10\, b\, (15.71)^2 = 2468\, b \text{ N}$$

and maximum tension in the belt,

$$T = \sigma .\, b.\, t = 1.5 \times 10^6 \times b \times 0.01 = 15\,000\, b \text{ N}$$

We know that $T = T_1 + T_C$ or $15000\, b = 824.6 + 2468\, b$

$$15\,000\, b - 2468\, b = 824.6 \quad \text{or} \quad 12\,532\, b = 824.6$$

$$\therefore \quad b = 824.6 / 12532 = 0.0658 \text{ m} = 65.8 \text{ mm} \quad \textbf{Ans.}$$

Example. 11.9. *Determine the width of a 9.75 mm thick leather belt required to transmit 15 kW from a motor running at 900 r.p.m. The diameter of the driving pulley of the motor is 300 mm. The driven pulley runs at 300 r.p.m. and the distance between the centre of two pulleys is 3 metres. The density of the leather is 1000 kg/m^3. The maximum allowable stress in the leather is 2.5 MPa. The coefficient of friction between the leather and pulley is 0.3. Assume open belt drive and neglect the sag and slip of the belt.*

Solution. Given : $t = 9.75$ mm = 9.75×10^{-3} m ; $P = 15$ kW = 15×10^3 W ; $N_1 = 900$ r.p.m. ; $d_1 = 300$ mm = 0.3 m ; $N_2 = 300$ r.p.m. ; $x = 3$m ; $\rho = 1000$ kg/m^3 ; $\sigma = 2.5$ MPa = 2.5×10^6 N/m^2 ; $\mu = 0.3$

* 1 MPa = 1×10^6 N/m^2

First of all, let us find out the diameter of the driven pulley (d_2). We know that

$$\frac{N_2}{N_1} = \frac{d_1}{d_2} \quad \text{or} \quad d_2 = \frac{N_1 \times d_1}{N_2} = \frac{900 \times 0.3}{300} = 0.9 \text{ m}$$

and velocity of the belt,
$$v = \frac{\pi d_1 N_1}{60} = \frac{\pi \cdot 0.3 \cdot 900}{60} = 14.14 \text{ m/s}$$

For an open belt drive,

$$\sin\alpha = \frac{r_2 - r_1}{x} = \frac{d_2 - d_1}{2x} = \frac{0.9 - 0.3}{2 \cdot 3} = 0.1 \qquad \text{...}(\because d_2 > d_1)$$

or
$$\alpha = 5.74°$$

$\therefore$ Angle of lap, $\theta = 180° - 2\,\alpha = 180 - 2 \times 5.74 = 168.52°$

$= 168.52 \times \pi / 180 = 2.94$ rad

Let T_1 = Tension in the tight side of the belt, and

T_2 = Tension in the slack side of the belt.

We know that

$$2.3 \log\left(\frac{T_1}{T_2}\right) = \propto .\,\theta = 0.3 \cdot 2.94 = 0.882$$

$$\log\left(\frac{T_1}{T_2}\right) = \frac{0.882}{2.3} = 0.3835 \quad \text{or} \quad \frac{T_1}{T_2} = 2.42 \qquad \text{...}(i)$$

... (Taking antilog of 0.3835)

We also know that power transmitted (P),

$$15 \times 10^3 = (T_1 - T_2)\, v = (T_1 - T_2)\, 14.14$$

$$\therefore \quad T_1 - T_2 = 15 \times 10^3 / 14.14 = 1060 \text{ N} \qquad \text{....} (ii)$$

From equations (*i*) and (*ii*),

$$T_1 = 1806 \text{ N}$$

Let b = Width of the belt in metres.

We know that mass of the belt per metre length,

$$m = \text{Area} \times \text{length} \times \text{density} = b.t.l.\rho$$

$$= b \times 9.75 \times 10^{-3} \times 1 \times 1000 = 9.75\, b \text{ kg}$$

$\therefore$ Centrifugal tension,

$$T_C = m.v^2 = 9.75\, b\, (14.14)^2 = 1950\ b \text{ N}$$

Maximum tension in the belt,

$$T = \sigma.\, b.\, t = 2.5 \times 10^6 \times b \times 9.75 \times 10^{-3} = 24\,400\, b \text{ N}$$

We know that $T = T_1 + T_C$ or $T - T_C = T_1$

$$24\,400\, b - 1950\, b = 1806 \quad \text{or} \quad 22\,450\, b = 1806$$

$\therefore$ $b = 1806 / 22\,450 = 0.080$ m = 80 mm **Ans.**

Example. 11.10. *A pulley is driven by a flat belt, the angle of lap being 120°. The belt is 100 mm wide by 6 mm thick and density1000 kg/m³. If the coefficient of friction is 0.3 and the maximum stress in the belt is not to exceed 2 MPa, find the greatest power which the belt can transmit and the corresponding speed of the belt.*

Solution. Given : $\theta = 120° = 120 \times \pi / 180 = 2.1$ rad ; $b = 100$ mm $= 0.1$ m ; $t = 6$ mm $= 0.006$ m ; $\rho = 1000$ kg / m^3 ; $\mu = 0.3$; $\sigma = 2$ MPa $= 2 \times 10^6$ N/m^2

Speed of the belt for greatest power

We know that maximum tension in the belt,

$$T = \sigma.\, b.\, t = 2 \times 10^6 \times 0.1 \times 0.006 = 1200 \text{ N}$$

and mass of the belt per metre length,

$$m = \text{Area} \times \text{length} \times \text{density} = b.\, t.\, l.\, \rho$$

$$= 0.1 \times 0.006 \times 1 \times 1000 = 0.6 \text{ kg/m}$$

$\therefore$ Speed of the belt for greatest power,

$$v = \sqrt{\frac{T}{3m}} = \sqrt{\frac{1200}{3 \cdot 0.6}} = 25.82 \text{ m/s} \qquad \textbf{Ans.}$$

Greatest power which the belt can transmit

We know that for maximum power to be transmitted, centrifugal tension,

$$T_C = T/3 = 1200/3 = 400 \text{ N}$$

and tension in the tight side of the belt,

$$T_1 = T - T_C = 1200 - 400 = 800 \text{ N}$$

Let T_2 = Tension in the slack side of the belt.

We know that

$$2.3 \log\left(\frac{T_1}{T_2}\right) = \propto .\, \theta = 0.3 \cdot 2.1 = 0.63$$

$$\log\left(\frac{T_1}{T_2}\right) = \frac{0.63}{2.3} = 0.2739 \quad \text{or} \quad \frac{T_1}{T_2} = 1.88 \qquad \text{...(Taking antilog of 0.2739)}$$

and $$T_2 = \frac{T_1}{1.88} = \frac{800}{1.88} = 425.5 \text{ N}$$

$\therefore$ Greatest power which the belt can transmit,

$$P = (T_1 - T_2)\, v = (800 - 425.5)\, 25.82 = 9670 \text{ W} = 9.67 \text{ kW} \quad \textbf{Ans.}$$

11.19. Initial Tension in the Belt

When a belt is wound round the two pulleys (*i.e.* driver and follower), its two ends are joined together ; so that the belt may continuously move over the pulleys, since the motion of the belt from the driver and the follower is governed by a firm grip, due to friction between the belt and the pulleys. In order to increase this grip, the belt is tightened up. At this stage, even when the pulleys are stationary, the belt is subjected to some tension, called ***initial tension***.

When the driver starts rotating, it pulls the belt from one side (increasing tension in the belt on this side) and delivers it to the other side (decreasing the tension in the belt on that side). The increased tension in one side of the belt is called tension in tight side and the decreased tension in the other side of the belt is called tension in the slack side.

Let T_0 = Initial tension in the belt,

T_1 = Tension in the tight side of the belt,

T_2 = Tension in the slack side of the belt, and

α = Coefficient of increase of the belt length per unit force.

A little consideration will show that the increase of tension in the tight side

$$= T_1 - T_0$$

and increase in the length of the belt on the tight side

$$= \alpha\,(T_1 - T_0) \qquad \text{...}(i)$$

Similarly, decrease in tension in the slack side

$$= T_0 - T_2$$

and decrease in the length of the belt on the slack side

$$= \alpha\,(T_0 - T_2) \qquad \text{...}(ii)$$

Assuming that the belt material is perfectly elastic such that the length of the belt remains constant, when it is at rest or in motion, therefore increase in length on the tight side is equal to decrease in the length on the slack side. Thus, equating equations (*i*) and (*ii*),

$$\alpha\,(T_1 - T_0) = \alpha\,(T_0 - T_2) \quad \text{or} \quad T_1 - T_0 = T_0 - T_2$$

$$\therefore \qquad T_0 = \frac{T_1 + T_2}{2} \qquad \text{...(Neglecting centrifugal tension)}$$

$$= \frac{T_1 + T_2 + 2T_C}{2} \qquad \text{...(Considering centrifugal tension)}$$

Example. 11.12. *In a flat belt drive the initial tension is 2000 N. The coefficient of friction between the belt and the pulley is 0.3 and the angle of lap on the smaller pulley is 150°. The smaller pulley has a radius of 200 mm and rotates at 500 r.p.m. Find the power in kW transmitted by the belt.*

Solution. Given : $T_0 = 2000$ N ; $\mu_0 = 0.3$; $\theta = 150° = 150° \times \pi / 180 = 2.618$ rad ; $r_2 = 200$ mm or $d_2 = 400$ mm $= 0.4$ m ; $N_2 = 500$ r.p.m.

We know that velocity of the belt,

$$v = \frac{\pi d_2 . N_2}{60} = \frac{\pi \cdot\ 0.4 \cdot\ 500}{60} = 10.47 \text{ m/s}$$

Let T_1 = Tension in the tight side of the belt, and

T_2 = Tension in the slack side of the belt.

We know that initial tension (T_0),

$$2000 = \frac{T_1 + T_2}{2} \qquad \text{or} \qquad T_1 + T_2 = 4000 \text{ N} \qquad \ldots(i)$$

We also know that

$$2.3 \log\left(\frac{T_1}{T_2}\right) = \propto.\,\theta = 0.3 \cdot 2.618 = 0.7854$$

$$\log\left(\frac{T_1}{T_2}\right) = \frac{0.7854}{2.3} = 0.3415$$

or

$$\frac{T_1}{T_2} = 2.2 \qquad \ldots(ii)$$

...(Taking antilog of 0.3415)

A military tank uses chain, belt and gear drives for its movement and operation.

From equations (*i*) and (*ii*),

$T_1 = 2750$ N ;

and $T_2 = 1250$ N

$\therefore$ Power transmitted, $P = (T_1 - T_2)\, v$

$= (2750 - 1250)\, 10.47$

$= 15\,700$ W $= 15.7$ kW **Ans.**

Example 11.13. *Two parallel shafts whose centre lines are 4.8 m apart, are connected by open belt drive. The diameter of the larger pulley is 1.5 m and that of smaller pulley 1 m. The initial tension in the belt when stationary is 3 kN. The mass of the belt is 1.5 kg / m length. The coefficient of friction between the belt and the pulley is 0.3. Taking centrifugal tension into account, calculate the power transmitted, when the smaller pulley rotates at 400 r.p.m.*

Solution. Given : $x = 4.8$ m ; $d_1 = 1.5$ m ; $d_2 = 1$ m ; $T_0 = 3$ kN $= 3000$ N ; $m = 1.5$ kg / m ; $\mu = 0.3$; $N_2 = 400$ r.p.m.

We know that velocity of the belt,

$$v = \frac{\pi d_2 . N_2}{60} = \frac{\pi \cdot 1 \cdot 400}{60} = 21 \text{ m/s}$$

and centrifugal tension, $T_C = m.v^2 = 1.5\,(21)^2 = 661.5$ N

Let T_1 = Tension in the tight side, and

T_2 = Tension in the slack side.

We know that initial tension (T_0),

$$3000 = \frac{T_1 + T_2 + 2T_C}{2} = \frac{T_1 + T_2 + 2 \cdot 661.5}{2}$$

$$\therefore \quad T_1 + T_2 = 3000 \times 2 - 2 \times 661.5 = 4677 \text{ N} \qquad \ldots(i)$$

For an open belt drive,

$$\sin\alpha = \frac{r_1 - r_2}{x} = \frac{d_1 - d_2}{2x} = \frac{1.5 - 1}{2 \cdot 4.8} = 0.0521 \quad \text{or} \quad \alpha = 3°$$

$\therefore$ Angle of lap on the smaller pulley,

$$\theta = 180° - 2\,\alpha = 180° - 2 \times 3° = 174°$$

$$= 174° \times \pi / 180 = 3.04 \text{ rad}$$

We know that

$$2.3 \log\left(\frac{T_1}{T_2}\right) = \propto . \theta = 0.3 \cdot 3.04 = 0.912$$

$$\log\left(\frac{T_1}{T_2}\right) = \frac{0.912}{2.3} = 0.3965 \quad \text{or} \quad \frac{T_1}{T_2} = 2.5 \qquad ...(ii)$$

...(Taking antilog of 0.3965)

From equations *(i)* and *(ii)*,

$$T_1 = 3341 \text{ N}; \text{ and } T_2 = 1336 \text{ N}$$

∴ Power transmitted,

$$P = (T_1 - T_2)\, v = (3341 - 1336)\, 21 = 42\,100 \text{ W} = 42.1 \text{ kW} \quad \textbf{Ans.}$$

Example 11.14. *An open flat belt drive connects two parallel shafts 1.2 metres apart. The driving and the driven shafts rotate at 350 r.p.m. and 140 r.p.m. respectively and the driven pulley is 400 mm in diameter. The belt is 5 mm thick and 80 mm wide. The coefficient of friction between the belt and pulley is 0.3 and the maximum permissible tension in the belting is 1.4 MN/m². Determine:*

1. diameter of the driving pulley, 2. maximum power that may be transmitted by the belting, and 3. required initial belt tension.

Solution. Given : $x = 1.2$ m ; $N_1 = 350$ r.p.m. ; $N_2 = 140$ r.p.m. ; $d_2 = 400$ mm $= 0.4$ m ; $t = 5$ mm $= 0.005$ m ; $b = 80$ mm $= 0.08$ m ; $\mu = 0.3$; $\sigma = 1.4$ MN/m² $= 1.4 \times 10^6$ N/m²

1. ***Diameter of the driving pulley***

Let d_1 = Diameter of the driving pulley.

We know that $$\frac{N_2}{N_1} = \frac{d_1}{d_2} \quad \text{or} \quad d_1 = \frac{N_2 . d_2}{N_1} = \frac{140 \cdot 0.4}{350} = 0.16 \text{ m} \quad \textbf{Ans.}$$

2. ***Maximum power transmitted by the belting***

First of all, let us find the angle of contact of the belt on the smaller pulley (or driving pulley).

Let θ = Angle of contact of the belt on the driving pulley.

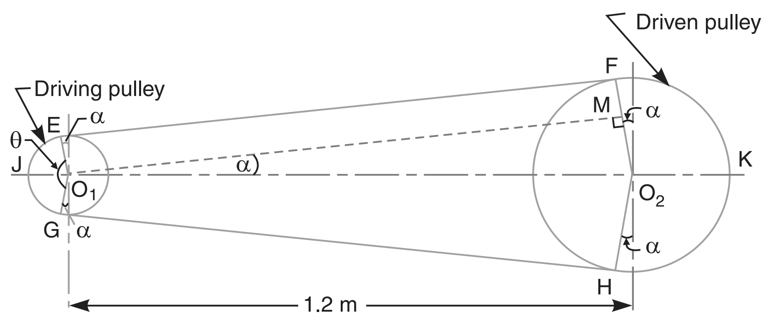


Fig. 11.18

From Fig. 11.18, we find that

$$\sin \alpha = \frac{O_2 M}{O_1 O_2} = \frac{r_2 - r_1}{x} = \frac{d_2 - d_1}{2x} = \frac{0.4 - 0.16}{2 \cdot 1.2} = 0.1$$

or $\alpha = 5.74°$

$\therefore$ $\theta = 180° - 2\,\alpha = 180° - 2 \times 5.74° = 168.52°$

$= 168.52 \times \pi / 180 = 2.94$ rad

Let T_1 = Tension in the tight side of the belt, and

T_2 = Tension in the slack side of the belt.

We know that

$$2.3 \log \left(\frac{T_1}{T_2} \right) = \propto . \theta = 0.3 \cdot 2.94 = 0.882$$

$$\log \left(\frac{T_1}{T_2} \right) = \frac{0.882}{2.3} = 0.3835 \text{ or } \frac{T_1}{T_2} = 2.42 \quad \text{...}(i)$$

...(Taking antilog of 0.3835)

We know that maximum tension to which the belt can be subjected,

$$T_1 = \sigma \times b \times t = 1.4 \times 10^6 \times 0.08 \times 0.005 = 560 \text{ N}$$

$\therefore$
$$T_2 = \frac{T_1}{2.42} = \frac{560}{2.42} = 231.4 \text{ N} \quad \text{...[From equation } (i)]$$

Velocity of the belt,
$$v = \frac{\pi d_1 . N_1}{60} = \frac{\pi \cdot 0.16 \cdot 350}{60} = 2.93 \text{ m/s}$$

$\therefore$ Power transmitted, $P = (T_1 - T_2)\, v = (560 - 231.4)\, 2.93 = 963$ W $= 0.963$ kW **Ans.**

3. *Required initial belt tension*

We know that the initial belt tension,

$$T_0 = \frac{T_1 + T_2}{2} = \frac{560 + 231.4}{2} = 395.7 \text{ N} \quad \textbf{Ans.}$$

Example 11.15. *An open belt running over two pulleys 240 mm and 600 mm diameter connects two parallel shafts 3 metres apart and transmits 4 kW from the smaller pulley that rotates at 300 r.p.m. Coefficient of friction between the belt and the pulley is 0.3 and the safe working tension is 10N per mm width. Determine :* **1.** *minimum width of the belt,* **2.** *initial belt tension, and* **3.** *length of the belt required.*

Solution. Given : $d_2 = 240$ mm $= 0.24$ m ; $d_1 = 600$ mm $= 0.6$ m ; $x = 3$ m ; $P = 4$ kW $= 4000$ W; $N_2 = 300$ r.p.m. ; $\mu = 0.3$; $T_1 = 10$ N/mm width

1. *Minimum width of belt*

We know that velocity of the belt,

$$v = \frac{\pi d_2 . N_2}{60} = \frac{\pi \cdot 0.24 \cdot 300}{60} = 3.77 \text{ m/s}$$

Let T_1 = Tension in the tight side of the belt, and

T_2 = Tension in the slack side of the belt.

$\therefore$ Power transmitted (P),

$$4000 = (T_1 - T_2)\, v = (T_1 - T_2)\, 3.77$$

or
$$T_1 - T_2 = 4000 / 3.77 = 1061 \text{ N} \quad \text{...}(i)$$

We know that for an open belt drive,

$$\sin\alpha = \frac{r_1 - r_2}{x} = \frac{d_1 - d_2}{2x} = \frac{0.6 - 0.24}{2 \cdot 3} = 0.06 \quad \text{or} \quad \alpha = 3.44°$$

and angle of lap on the smaller pulley,

$$\theta = 180° - 2\alpha = 180° - 2 \times 3.44° = 173.12°$$

$$= 173.12 \times \pi / 180 = 3.022 \text{ rad}$$

We know that

$$2.3 \log\left(\frac{T_1}{T_2}\right) = \propto . \theta = 0.3 \cdot 3.022 = 0.9066$$

$$\log\left(\frac{T_1}{T_2}\right) = \frac{0.9066}{2.3} = 0.3942 \quad \text{or} \quad \frac{T_1}{T_2} = 2.478 \qquad ...(ii)$$

...(Taking antilog of 0.3942)

From equations (*i*) and (*ii*),

$$T_1 = 1779 \text{ N}, \text{ and } T_2 = 718 \text{ N}$$

Since the safe working tension is 10 N per mm width, therefore minimum width of the belt,

$$b = \frac{T_1}{10} = \frac{1779}{10} = 177.9 \text{ mm} \quad \textbf{Ans.}$$

2. ***Initial belt tension***

We know that initial belt tension,

$$T_0 = \frac{T_1 + T_2}{2} = \frac{1779 + 718}{2} = 1248.5 \text{ N} \quad \textbf{Ans.}$$

3. ***Length of the belt required***

We know that length of the belt required,

$$L = \frac{\pi}{2}(d_1 - d_2) + 2x + \frac{(d_1 - d_2)^2}{4x}$$

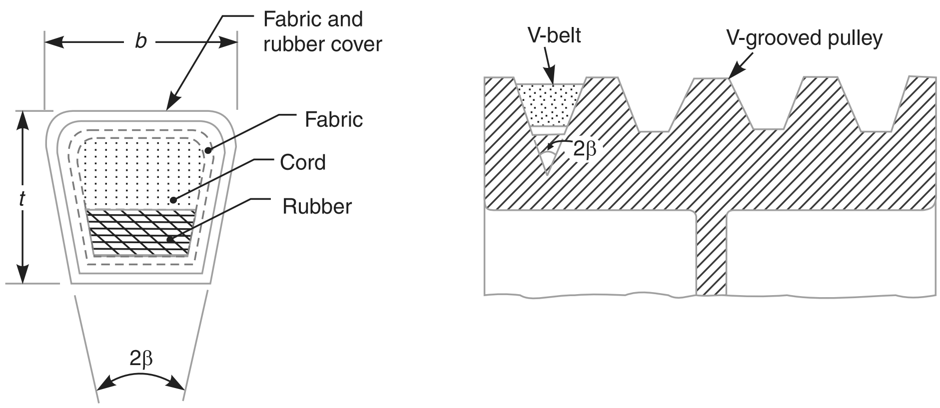
$$= \frac{\pi}{2}(0.6 + 0.24) + 2 \cdot 3 + \frac{(0.6 - 0.24)^2}{4 \cdot 3}$$

$$= 1.32 + 6 + 0.01 = 7.33 \text{ m} \quad \textbf{Ans.}$$

11.20. V-belt drive

We have already discussed that a V-belt is mostly used in factories and workshops where a great amount of power is to be transmitted from one pulley to another when the two pulleys are very near to each other.

The V-belts are made of fabric and cords moulded in rubber and covered with fabric and rubber, as shown in Fig. 11.19 (*a*). These belts are moulded to a trapezoidal shape and are made endless. These are particularly suitable for short drives *i.e.* when the shafts are at a short distance apart. The included angle for the V-belt is usually from 30° – 40°. In case of flat belt drive, the belt runs over the pulleys whereas in case of V-belt drive, the rim of the pulley is grooved in which the V-belt runs. The effect of the groove is to increase the frictional grip of the V-belt on the pulley and thus to reduce the tendency of slipping. In order to have a good grip on the pulley, the V-belt is in contact with the side faces of the groove and not at the bottom. The power is transmitted by the *wedging action between the belt and the V-groove in the pulley.



(*a*) Cross-section of a V-belt. (*b*) Cross-section of a V-grooved pulley.

Fig. 11.19. V-belt and V-grooved pulley.

A clearance must be provided at the bottom of the groove, as shown in Fig. 11.19 (*b*), in order to prevent touching to the bottom as it becomes narrower from wear. The V-belt drive, may be inclined at any angle with tight side either at top or bottom. In order to increase the power output, several V- belts may be operated side by side. It may be noted that in multiple V-belt drive, all the belts should stretch at the same rate so that the load is equally divided between them. When one of the set of belts break, the entire set should be replaced at the same time. If only one belt is replaced, the new unworn and unstressed belt will be more tightly stretched and will move with different velocity.

11.21. Advantages and Disadvantages of V-belt Drive Over Flat Belt Drive

Following are the advantages and disadvantages of the V-belt drive over flat belt drive.

Advantages

1. The V-belt drive gives compactness due to the small distance between the centres of pulleys.
2. The drive is positive, because the slip between the belt and the pulley groove is negligible.
3. Since the V-belts are made endless and there is no joint trouble, therefore the drive is smooth.
4. It provides longer life, 3 to 5 years.

* The wedging action of the V-belt in the groove of the pulley results in higher forces of friction. A little consideration will show that the wedging action and the transmitted torque will be more if the groove angle of the pulley is small. But a smaller groove angle will require more force to pull the belt out of the groove which will result in loss of power and excessive belt wear due to friction and heat. Hence a selective groove angle is a compromise between the two. Usually the groove angles of 32° to 38° are used.

5. It can be easily installed and removed.

6. The operation of the belt and pulley is quiet.

7. The belts have the ability to cushion the shock when machines are started.

8. The high velocity ratio (maximum 10) may be obtained.

9. The wedging action of the belt in the groove gives high value of limiting ratio of tensions. Therefore the power transmitted by V-belts is more than flat belts for the same coefficient of friction, arc of contact and allowable tension in the belts.

10. The V-belt may be operated in either direction with tight side of the belt at the top or bottom. The centre line may be horizontal, vertical or inclined.

Disadvantages

1. The V-belt drive cannot be used with large centre distances.

2. The V-belts are not so durable as flat belts.

3. The construction of pulleys for V-belts is more complicated than pulleys for flat belts.

4. Since the V-belts are subjected to certain amount of creep, therefore these are not suitable for constant speed application such as synchronous machines, and timing devices.

5. The belt life is greatly influenced with temperature changes, improper belt tension and mismatching of belt lengths.

6. The centrifugal tension prevents the use of V-belts at speeds below 5 m/s and above 50m/s.

11.22. Ratio of Driving Tensions for V-belt

A V-belt with a grooved pulley is shown in Fig. 11.20.

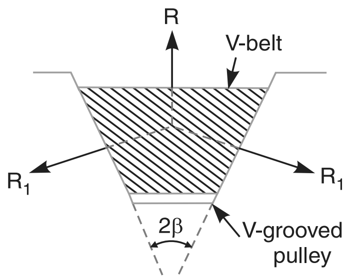


Fig. 11.20.

Let R_1 = Normal reaction between the belt and sides of the groove.

R = Total reaction in the plane of the groove.

2β = Angle of the groove.

μ = Coefficient of friction between the belt and sides of the groove.

Resolving the reactions vertically to the groove,

$$R = R_1 \sin\beta + R_1 \sin\beta = 2 R_1 \sin\beta$$

or
$$R_1 = \frac{R}{2\sin\beta}$$

We know that the frictional force

$$= 2\mu . R_1 = 2\mu \times \frac{R}{2\sin\beta} = \frac{\mu . R}{\sin\beta} = \mu . R \operatorname{cosec}\beta$$

Consider a small portion of the belt, as in Art. 11.14, subtending an angle $\delta\theta$ at the centre. The tension on one side will be T and on the other side $T + \delta T$. Now proceeding as in Art. 11.14, we get the frictional resistance equal to $\mu . R \operatorname{cosec}\beta$ instead of $\mu . R$. Thus the relation between T_1 and T_2 for the V-belt drive will be

$$2.3 \log\left(\frac{T_1}{T_2}\right) = \propto . \theta \operatorname{cosec}\beta$$

Example 11.17. *A belt drive consists of two V-belts in parallel, on grooved pulleys of the same size. The angle of the groove is 30°. The cross-sectional area of each belt is 750 mm² and μ. = 0.12. The density of the belt material is 1.2 Mg/m³ and the maximum safe stress in the material is 7 MPa. Calculate the power that can be transmitted between pulleys 300 mm diameter rotating at 1500 r.p.m. Find also the shaft speed in r.p.m. at which the power transmitted would be maximum.*

Solution. Given : $2\beta = 30°$ or $\beta = 15°$; $\alpha = 750$ mm² $= 750 \times 10^{-6}$ m² ; $\mu = 0.12$; $\rho = 1.2$ Mg/m³ $= 1200$ kg/m³ ; $\sigma = 7$ MPa $= 7 \times 10^6$ N/m² ; $d = 300$ mm $= 0.3$ m ; $N = 1500$ r.p.m.

Power transmitted

We know that velocity of the belt,

$$v = \frac{\pi d . N}{60} = \frac{\pi \cdot 0.3 \cdot 1500}{60} = 23.56 \text{ m/s}$$

and mass of the belt per metre length,

$$m = \text{Area} \times \text{length} \times \text{density} = 750 \times 10^{-6} \times 1 \times 1200 = 0.9 \text{ kg/m}$$

∴ Centrifugal tension,

$$T_C = m.v^2 = 0.9\,(23.56)^2 = 500 \text{ N}$$

We know that maximum tension in the belt,

$$T = \text{Maximum stress} \times \text{cross-sectional area of belt} = \sigma \times a$$
$$= 7 \times 10^6 \times 750 \times 10^{-6} = 5250 \text{ N}$$

∴ Tension in the tight side of the belt,

$$T_1 = T - T_C = 5250 - 500 = 4750 \text{ N}$$

Let T_2 = Tension in the slack side of the belt.

Since the pulleys are of the same size, therefore angle of contact, $\theta = 180° = \pi$ rad.

We know that

$$2.3 \log\left(\frac{T_1}{T_2}\right) = \propto . \theta \operatorname{cosec} \beta = 0.12 \cdot \pi \cdot \operatorname{cosec} 15 = 1.457$$

$$\log\left(\frac{T_1}{T_2}\right) = \frac{1.457}{2.3} = 0.6334 \quad \text{or} \quad \frac{T_1}{T_2} = 4.3$$

...(Taking antilog of 0.6334)

and $$T_2 = \frac{T_1}{4.3} = \frac{4750}{4.3} = 1105 \text{ N}$$

We know that power transmitted,

$$P = (T_1 - T_2)\, v \times 2 \qquad \text{...(∵ No. of belts = 2)}$$
$$= (4750 - 1105)\, 23.56 \times 2 = 171\ 752 \text{ W} = 171.752 \text{ kW} \quad \textbf{Ans.}$$

Shaft speed

Let N_1 = Shaft speed in r.p.m., and

v_1 = Belt speed in m/s.

We know that for maximum power, centrifugal tension,

$$T_C = T/3 \quad \text{or} \quad m\,(v_1)^2 = T/3 \quad \text{or} \quad 0.9\,(v_1)^2 = 5250/3 = 1750$$

∴ $$(v_1)^2 = 1750/0.9 = 1944.4 \quad \text{or} \quad v_1 = 44.1 \text{ m/s}$$

We know that belt speed (v_1),

$$44.1 = \frac{\pi d . N_1}{60} = \frac{\pi \cdot 0.3 \cdot N_1}{60} = 0.0157\, N_1$$

$\therefore$ $N_1 = 44.1 / 0.0157 = 2809$ r.p.m. **Ans.**

Example 11.18. *Power is transmitted using a V-belt drive. The included angle of V-groove is 30°. The belt is 20 mm deep and maximum width is 20 mm. If the mass of the belt is 0.35 kg per metre length and maximum allowable stress is 1.4 MPa, determine the maximum power transmitted when the angle of lap is 140°. μ = 0.15.*

Solution. Given : $2\beta = 30°$ or $\beta = 15°$; $t = 20$ mm = 0.02 m ; $b = 20$ mm = 0.02 m ; $m = 0.35$ kg/m ; $\sigma = 1.4$ MPa $= 1.4 \times 10^6$ N/m^2 ; $\theta = 140° = 140° \times \pi / 180 = 2.444$ rad ; $\mu = 0.15$

We know that maximum tension in the belt,

$$T = \sigma . b . t = 1.4 \times 10^6 \times 0.02 \times 0.02 = 560 \text{ N}$$

and for maximum power to be transmitted, velocity of the belt,

$$v = \sqrt{\frac{T}{3m}} = \sqrt{\frac{560}{3 \cdot 0.35}} = 23.1 \text{ m/s}$$

Let T_1 = Tension in the tight side of the belt, and

T_2 = Tension in the slack side of the belt.

We know that

$$2.3 \log\left(\frac{T_1}{T_2}\right) = \propto . \theta \operatorname{cosec} \beta = 0.15 \cdot 2.444 \cdot \operatorname{cosec} 15 = 1.416$$

$$\log\left(\frac{T_1}{T_2}\right) = \frac{1.416}{2.3} = 0.616 \text{ or } \frac{T_1}{T_2} = 4.13 \qquad ...(i)$$

...(Taking antilog of 0.616)

Centrifugal tension, $T_C = \frac{T}{3} = \frac{560}{3} = 187$ N

and $T_1 = T - T_C = 560 - 187 = 373$ N

$$T_2 = \frac{T_1}{4.13} = \frac{373}{4.13} = 90.3 \text{ N} \qquad \text{...[From equation } (i)]$$

We know that maximum power transmitted,

$$P = (T_1 - T_2)\, v = (373 - 90.3)\, 23.1 = 6530 \text{ W} = 6.53 \text{ kW}$$ **Ans.**

Example 11.19. *A compressor, requiring 90 kW is to run at about 250 r.p.m. The drive is by V-belts from an electric motor running at 750 r.p.m. The diameter of the pulley on the compressor shaft must not be greater than 1 metre while the centre distance between the pulleys is limited to 1.75 metre. The belt speed should not exceed 1600 m/min.*

Determine the number of V-belts required to transmit the power if each belt has a cross-sectional area of 375 mm², density 1000 kg/m³ and an allowable tensile stress of 2.5 MPa. The groove angle of the pulley is 35°. The coefficient of friction between the belt and the pulley is 0.25. Calculate also the length required of each belt.

Solution. Given : $P = 90$ kW ; $N_2 = 250$ r.p.m. ; $N_1 = 750$ r.p.m. ; $d_2 = 1$ m ; $x = 1.75$ m ; $v = 1600$ m/min = 26.67 m/s ; $a = 375$ mm^2 $= 375 \times 10^{-6}$ m^2 ; $\rho = 1000$ kg/m^3 ; $\sigma = 2.5$ MPa $= 2.5 \times 10^6$ N/m^2 ; $2\beta = 35°$ or $\beta = 17.5°$; $\mu = 0.25$

First of all, let us find the diameter of pulley on the motor shaft (d_1). We know that

$$\frac{N_2}{N_1} = \frac{d_1}{d_2} \quad \text{or} \quad d_1 = \frac{N_2.d_2}{N_1} = \frac{250 \times 1}{750} = 0.33\,\text{m}$$

We know that the mass of the belt per metre length,

$$m = \text{Area} \times \text{length} \times \text{density}$$
$$= 375 \times 10^{-6} \times 1 \times 1000 = 0.375 \text{ kg}$$

$\therefore$ Centrifugal tension, $T_C = m.v^2 = 0.375\,(26.67)^2 = 267$ N

and maximum tension in the belt,

$$T = \sigma.\,a = 2.5 \times 10^6 \times 375 \times 10^{-6} = 937.5 \text{ N}$$

$\therefore$ Tension in the tight side of the belt,

$$T_1 = T - T_C = 937.5 - 267 = 670.5 \text{ N}$$

Let T_2 = Tension in the slack side of the belt.

For an open belt drive, as shown in Fig. 11.21,

$$\sin\alpha = \frac{O_2M}{O_1\,O_2} = \frac{r_2 - r_1}{x} = \frac{d_2 - d_1}{2x} = \frac{1 - 0.33}{2 \cdot 1.75} = 0.1914$$

$\therefore$ $\alpha = 11°$

and angle of lap on smaller pulley (*i.e.* pulley on motor shaft),

$$\theta = 180° - 2\alpha = 180° - 2 \times 11° = 158°$$
$$= 158 \times \pi / 180 = 2.76 \text{ rad}$$

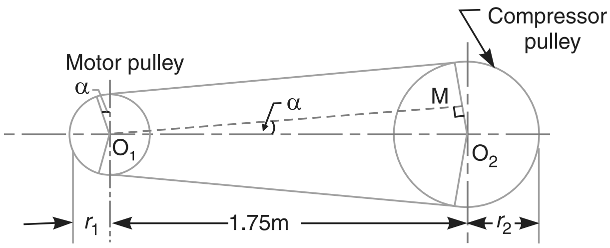


Fig. 11.21

We know that

$$2.3\log\left(\frac{T_1}{T_2}\right) = \propto.\,\theta\text{cosec}\,\beta = 0.25 \cdot 2.76 \cdot \text{cosec}17.5 = 2.295$$

$$\log\left(\frac{T_1}{T_2}\right) = \frac{2.295}{2.3} = 0.998 \quad \text{or} \quad \frac{T_1}{T_2} = 9.954 \qquad \text{...(Taking antilog of 0.998)}$$

and

$$T_2 = \frac{T_1}{9.954} = \frac{670.5}{9.954} = 67.36\,\text{N}$$

Number of V-belts

We know that power transmitted per belt

$$= (T_1 - T_2)\,v = (670.5 - 67.36)\,26.67 = 16\;086 \text{ W}$$
$$= 16.086 \text{ kW}$$

$$\therefore \quad \text{Number of V-belts} = \frac{\text{Total power transmitted}}{\text{Power transmitted per belt}} = \frac{90}{16.086} = 5.6 \text{ or } 6 \ \textbf{Ans.}$$

Length of each belt

We know that length of belt for an open belt drive,

$$L = \frac{\pi}{2}(d_2 + d_1) + 2x + \frac{(d_2 - d_1)^2}{4x}$$

$$= \frac{\pi}{2}(1 + 0.33) + 2 \cdot 1.75 + \frac{(1 - 0.33)^2}{4 \cdot 1.75}$$

$$= 2.1 + 3.5 + 0.064 = 5.664 \text{ m}$$ **Ans.**

11.23. Rope Drive

The rope drives are widely used where a large amount of power is to be transmitted, from one pulley to another, over a considerable distance. It may be noted that the use of flat belts is limited for the transmission of moderate power from one pulley to another when the two pulleys are not more than 8 metres apart. If large amounts of power are to be transmitted by the flat belt, then it would result in excessive belt cross-section. It may be noted that frictional grip in case of rope drives is more than that in V-drive. One of the main advantage of rope drives is that a number of separate drives may be taken from the one driving pulley. For example, in many spinning mills, the line shaft on each floor is driven by ropes passing directly from the main engine pulley on the ground floor.

The rope drives use the following two types of ropes :

1. Fibre ropes, and **2.** Wire ropes.

The fibre ropes operate successfully when the pulleys are about 60 metres apart, while the wire ropes are used when the pulleys are upto 150 metres apart.

11.24. Fibre Ropes

The ropes for transmitting power are usually made from fibrous materials such as hemp, manila and cotton. Since the hemp and manila fibres are rough, therefore the ropes made from these fibres are not very flexible and possesses poor mechanical properties. The hemp ropes have less strength as compared to manila ropes. When the hemp and manila ropes are bent over the sheave (or pulley), there is some sliding of fibres, causing the rope to wear and chafe internally. In order to minimise this defect, the rope fibres are lubricated with a tar, tallow or graphite. The lubrication also makes the rope moisture proof. The hemp ropes are suitable only for hand operated hoisting machinery and as tie ropes for lifting tackle, hooks etc.

The cotton ropes are very soft and smooth. The lubrication of cotton ropes is not necessary. But if it is done, it reduces the external wear between the rope and the grooves of its sheaves. It may be noted that manila ropes are more durable and stronger than cotton ropes. The cotton ropes are costlier than manila ropes.

Note : The diameter of manila and cotton ropes usually ranges from 38 mm to 50 mm. The size of the rope is usually designated by its circumference or **'girth'.**

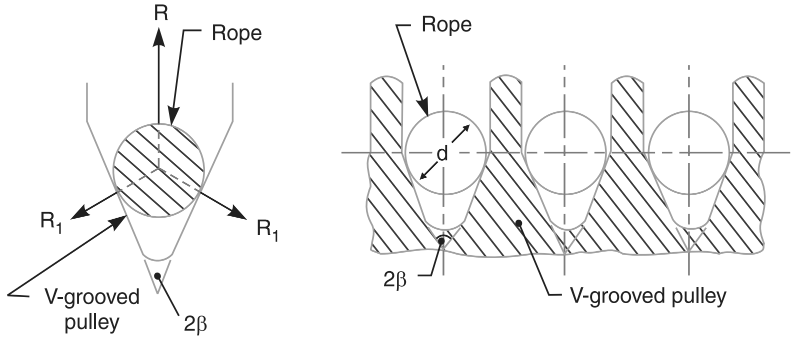
11.25. Advantages of Fibre Rope Drives

The fibre rope drives have the following advantages :

1. They give smooth, steady and quiet service.
2. They are little affected by out door conditions.
3. The shafts may be out of strict alignment.
4. The power may be taken off in any direction and in fractional parts of the whole amount.
5. They give high mechanical efficiency.

11.26. Sheave for Fibre Ropes

The fibre ropes are usually circular in cross-section as shown in Fig. 11.22 (*a*). The sheave for the fibre ropes is shown in Fig. 11.22 (*b*). The groove angle of the pulley for rope drives is usually 45°. The grooves in the pulleys are made narrow at the bottom and the rope is pinched between the edges of the V-groove to increase the holding power of the rope on the pulley.



(*a*) Cross-section of a rope. (*b*) Sheave (Grooved pulley) for ropes.

Fig. 11.22. Rope and sheave.

11.27. Wire Ropes

When a large amount of power is to be transmitted over long distances from one pulley to another (*i.e.* when the pulleys are upto 150 metres apart), then wire ropes are used. The wire ropes are

This electric hoist uses wire ropes.

widely used in elevators, mine hoists, cranes, conveyors, hauling devices and suspension bridges. The wire ropes run on grooved pulleys but they rest on the bottom of the *grooves and are not wedged between the sides of the grooves. The wire ropes have the following advantage over cotton ropes.

* The fibre ropes do not rest at the bottom of the groove.

1. These are lighter in weight, **2.** These offer silent operation, **3.** These can withstand shock loads, **4.** These are more reliable, **5.** They do not fail suddenly, **6.** These are more durable, **7.** The efficiency is high, and **8.** The cost is low.

11.28. Ratio of Driving Tensions for Rope Drive

The ratio of driving tensions for the rope drive may be obtained in the similar way as V-belts. We have discussed in Art. 11.22, that the ratio of driving tensions is

$$2.3\log\left(\frac{T_1}{T_2}\right) = \propto .\ \theta \operatorname{cosec} \beta$$

where, μ, θ and β have usual meanings.

Example 11.20. *A rope drive transmits 600 kW from a pulley of effective diameter 4 m, which runs at a speed of 90 r.p.m. The angle of lap is 160° ; the angle of groove 45° ; the coefficient of friction 0.28 ; the mass of rope 1.5 kg / m and the allowable tension in each rope 2400 N. Find the number of ropes required.*

Solution. Given : $P = 600$ kW ; $d = 4$ m ; $N = 90$ r.p.m. ; $\theta = 160° = 160 \times \pi / 180 = 2.8$ rad; $2\beta = 45°$ or $\beta = 22.5°$; $\mu = 0.28$; $m = 1.5$ kg / m ; $T = 2400$ N

We know that velocity of the rope,

$$v = \frac{\pi d . N}{60} = \frac{\pi \cdot\ 4 \cdot\ 90}{60} = 18.85 \text{ m/s}$$

∴ Centrifugal tension, $T_C = m.v^2 = 1.5\,(18.85)^2 = 533$ N

and tension in the tight side of the rope,

$$T_1 = T - T_C = 2400 - 533 = 1867 \text{ N}$$

Let T_2 = Tension in the slack side of the rope.

We know that

$$2.3\log\left(\frac{T_1}{T_2}\right) = \propto .\ \theta \operatorname{cosec}\beta = 0.28 \cdot 2.8 \cdot \operatorname{cosec} 22.5 = 2.05$$

$$\log\left(\frac{T_1}{T_2}\right) = \frac{2.05}{2.3} = 0.8913 \quad \text{or} \quad \frac{T_1}{T_2} = 7.786$$

...(Taking antilog of 0.8913)

and $$T_2 = \frac{T_1}{7.786} = \frac{1867}{7.786} = 240 \text{ N}$$

We know that power transmitted per rope

$$= (T_1 - T_2)\,v = (1867 - 240)\,18.85 = 30\ 670 \text{ W} = 30.67 \text{ kW}$$

∴ $$\text{Number of ropes} = \frac{\text{Total power transmitted}}{\text{Power transmitted per rope}} = \frac{600}{30.67} = 19.56 \text{ or } 20$$ **Ans.**

Example 11.21. *A pulley used to transmit power by means of ropes has a diameter of 3.6 metres and has 15 grooves of 45° angle. The angle of contact is 170° and the coefficient of friction between the ropes and the groove sides is 0.28. The maximum possible tension in the ropes is 960 N and the mass of the rope is 1.5 kg per metre length. What is the speed of pulley in r.p.m. and the power transmitted if the condition of maximum power prevail ?*

Solution. Given : $d = 3.6$ m ; No. of grooves = 15 ; $2\beta = 45°$ or $\beta = 22.5°$; $\theta = 170°$ $= 170\,\pi \times 180 = 2.967$ rad ; $\mu = 0.28$; $T = 960$ N ; $m = 1.5$ kg/m

Speed of the pulley

Let N = Speed of the pulley in r.p.m.

We know that for maximum power, velocity of the rope or pulley,

$$v = \sqrt{\frac{T}{3m}} = \sqrt{\frac{960}{3 \times 1.5}} = 14.6 \text{ m/s}$$

$$\therefore \quad N = \frac{v \cdot 60}{\pi d} = \frac{14.6 \cdot 60}{\pi \cdot 3.6} = 77.5 \text{ r.p.m.} \quad \textbf{Ans.} \qquad \ldots\left(\because v = \frac{\pi d N}{60}\right)$$

Power transmitted

We know that for maximum power, centrifugal tension,

$$T_C = T / 3 = 960 / 3 = 320 \text{ N}$$

∴ Tension in the tight side of the rope,

$$T_1 = T - T_C = 960 - 320 = 640 \text{ N}$$

Let T_2 = Tension in the slack side of the rope.

We know that $$2.3 \log\left(\frac{T_1}{T_2}\right) = \propto . \theta \operatorname{cosec} \beta = 0.28 \cdot 2.967 \cdot \operatorname{cosec} 22.5 = 2.17$$

$$\log\left(\frac{T_1}{T_2}\right) = \frac{2.17}{2.3} = 0.9438 \quad \text{or} \quad \frac{T_1}{T_2} = 8.78$$

...(Taking antilog of 0.9438)

and $$T_2 = \frac{T_1}{8.78} = \frac{640}{8.78} = 73 \text{ N}$$

∴ Power transmitted per rope = $(T_1 - T_2)\, v$ = (640 – 73) 14.6 = 8278 W = 8.278 kW

Since the number of grooves are 15, therefore total power transmitted

= 8.278 × 15 = 124.17 kW **Ans.**

Example 11.22. *Following data is given for a rope pulley transmitting 24 kW* :

Diameter of pulley = 400 mm ; Speed = 110 r.p.m.; angle of groove = 45° ; Angle of lap on smaller pulley = 160° ; Coefficient of friction = 0.28 ; Number of ropes = 10 ; Mass in kg/m length of ropes = 53 C^2 *; and working tension is limited to 122* C^2 *kN, where C is girth of rope in metres.*

Find initial tension and diameter of each rope.

Solution. Given : P_T = 24 kW ; d = 400 mm = 0.4 m ; N = 110 r.p.m. ; 2 β = 45° or β = 22.5°; θ = 160° = 160 × π / 180 = 2.8 rad ; n = 0.28 ; n = 10 ; m = 53 C^2 kg/m ; T = 122 C^2 kN = 122 × 10^3 C^2 N

Initial tension

We know that power transmitted per rope,

$$P = \frac{\text{Total power transmitted}}{\text{No. of ropes}} = \frac{P_T}{n} = \frac{24}{10} = 2.4 \text{ kW} = 2400 \text{ W}$$

and velocity of the rope, $$v = \frac{\pi d . N}{60} = \frac{\pi \cdot 0.4 \cdot 110}{60} = 2.3 \text{ m/s}$$

Let T = Tension in the tight side of the rope, and

T_2 = Tension in the slack side of the rope.

We know that power transmitted per rope (P)

$$2400 = (T_1 - T_2)\, v = (T_1 - T_2)\, 2.3$$

$$\therefore \quad T_1 - T_2 = 2400 / 2.3 = 1043.5 \text{ N} \quad \ldots(i)$$

We know that

$$2.3 \log\left(\frac{T_1}{T_2}\right) = \propto . \theta \operatorname{cosec} \beta = 0.28 \cdot 2.8 \cdot \operatorname{cosec} 22.5 = 2.05$$

$$\log\left(\frac{T_1}{T_2}\right) = \frac{2.05}{2.3} = 0.8913 \quad \text{or} \quad \frac{T_1}{T_2} = 7.786 \quad \ldots(ii)$$

...(Taking antilog of 0.8913)

From equations (*i*) and (*ii*),

$$T_1 = 1197.3 \text{ N, and } T_2 = 153.8 \text{ N}$$

We know that initial tension in each rope,

$$T_0 = \frac{T_1 + T_2}{2} = \frac{1197.3 + 153.8}{2} = 675.55 \text{ N} \quad \textbf{Ans.}$$

Diameter of each rope

Let d_1 = Diameter of each rope,

We know that centrifugal tension,

$$T_C = m.v^2 = 53\, C^2\, (2.3)^2 = 280.4\, C^2 \text{ N}$$

and working tension (T),

$$122 \times 10^3\, C^2 = T_1 + T_C = 1197.3 + 280.4\, C^2$$

$$122 \times 10^3\, C^2 - 280.4\, C^2 = 1197.3$$

$$\therefore \quad C^2 = 9.836 \times 10^{-3} \quad \text{or} \quad C = 0.0992 \text{ m} = 99.2 \text{ mm}$$

We know that girth (*i.e.* circumference) of rope (C),

$$99.2 = \pi\, d_1 \quad \text{or} \quad d_1 = 99.2 / \pi = 31.57 \text{ mm} \quad \textbf{Ans.}$$

11.29. Chain Drives

We have seen in belt and rope drives that slipping may occur. In order to avoid slipping, steel chains are used. The chains are made up of rigid links which are hinged together in order to provide the necessary flexibility for warping around the driving and driven wheels. The wheels have projecting teeth and fit into the corresponding recesses, in the links of the chain as shown in Fig. 11.23. The wheels and the chain are thus constrained to move together without slipping and ensures perfect velocity ratio. The toothed wheels are known as ***sprocket wheels*** or simply ***sprockets***. These wheels resemble to spur gears.

The chains are mostly used to transmit motion and power from one shaft to another, when the distance between the centres of the shafts is short such as in bicycles, motor cycles, agricultural machinery, road rollers, etc.

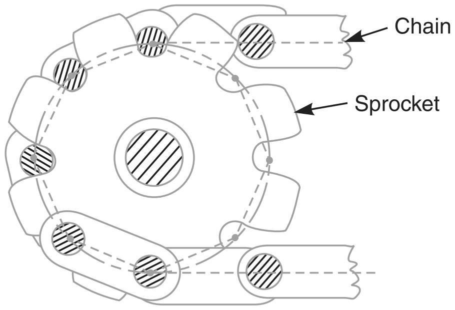


Fig. 11.23. Sprocket and chain.

11.30. Advantages and Disadvantages of Chain Drive Over Belt or Rope Drive

Following are the advantages and disadvantages of chain drive over belt or rope drive :

Advantages

1. As no slip takes place during chain drive, hence perfect velocity ratio is obtained.
2. Since the chains are made of metal, therefore they occupy less space in width than a belt or rope drive.
3. The chain drives may be used when the distance between the shafts is less.
4. The chain drive gives a high transmission efficiency (upto 98 per cent).
5. The chain drive gives less load on the shafts.
6. The chain drive has the ability of transmitting motion to several shafts by one chain only.

Disadvantages

1. The production cost of chains is relatively high.
2. The chain drive needs accurate mounting and careful maintenance.
3. The chain drive has velocity fluctuations especially when unduly stretched.

11.31. Terms Used in Chain Drive

The following terms are frequently used in chain drive.

1. ***Pitch of the chain*** **:** It is the distance between the hinge centre of a link and the corresponding hinge centre of the adjacent link as shown in Fig. 11.24. It is usually denoted by p.

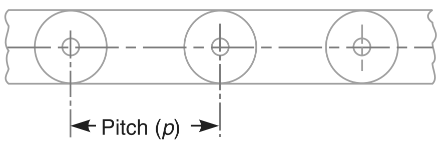


Fig. 11.24. Pitch of the chain.

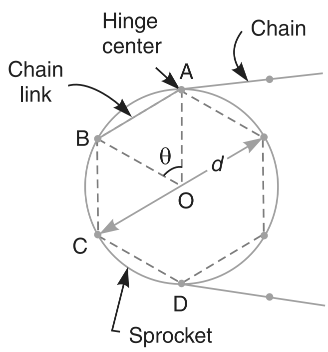


Fig. 11.25. Pitch circle diameter of the chain sprocket.

2. ***Pitch circle diameter of the chain sprocket*****.** It is the diameter of the circle on which the hinge centres of the chain lie, when the chain is wrapped round a sprocket as shown in Fig. 11.25. The points A, B, C, and D are the hinge centres of the chain and the circle drawn through these centres is called pitch circle and its diameter (d) is known as pitch circle diameter.

11.32. Relation Between Pitch and Pitch Circle Diameter

A chain wrapped round the sprocket is shown in Fig. 11.25. Since the links of the chain are rigid, therefore pitch of the chain does not lie on the arc of the pitch circle. The pitch length becomes a chord. Consider one pitch length *A B* of the chain subtending an angle θ at the centre of sprocket (or pitch circle).

Let d = Diameter of the pitch circle, and

T = Number of teeth on the sprocket.

From Fig. 11.25, we find that pitch of the chain,

$$p = AB = 2\,AO\sin\left(\frac{\theta}{2}\right) = 2 \cdot \frac{d}{2}\sin\left(\frac{\theta}{2}\right) = d\sin\left(\frac{\theta}{2}\right)$$

We know that
$$\theta = \frac{360°}{T}$$

$\therefore$
$$p = d\sin\left(\frac{360°}{2T}\right) = d\sin\left(\frac{180°}{T}\right)$$

or
$$d = p\,\text{cosec}\left(\frac{180°}{T}\right)$$

11.33. Relation Between Chain Speed and Angular Velocity of Sprocket

Since the links of the chain are rigid, therefore they will have different positions on the sprocket at different instants. The relation between the chain speed (v) and angular velocity of the sprocket (ω) also varies with the angular position of the sprocket. The extreme positions are shown in Fig. 11.26 (*a*) and (*b*).

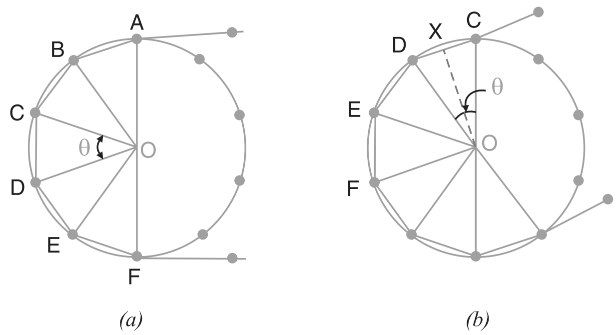


Fig. 11.26. Relation between chain speed and angular velocity of sprocket.

For the angular position of the sprocket as shown in Fig. 11.26 (*a*),

$$v = \omega \times OA$$

and for the angular position of the sprocket as shown in Fig. 11.26 (*b*),

$$v = \omega \cdot OX = \omega \cdot OC\cos\left(\frac{\theta}{2}\right) = \omega \cdot OA\cos\left(\frac{\theta}{2}\right) \quad \text{...}(\because OC = OA)$$

11.34. Kinematic of Chain Drive

Fig. 11.27 shows an arrangement of a chain drive in which the smaller or driving sprocket has 6 teeth and the larger or driven sprocket has 9 teeth. Though this is an impracticable case, but this is considered to bring out clearly the kinematic conditions of a chain drive. Let both the sprockets rotate anticlockwise and the angle subtended by the chain pitch at the centre of the driving and driven sprockets be α and ϕ respectively. The lines AB and A_1B_1 show the positions of chain having minimum and maximum inclination respectively with the line of centres O_1O_2 of the sprockets. The points A, B_2 and B are in one straight line and the points A_1, C and B_1 are in one straight line. It may be noted that the straight length of the chain between the two sprockets must be equal to exact number of pitches.

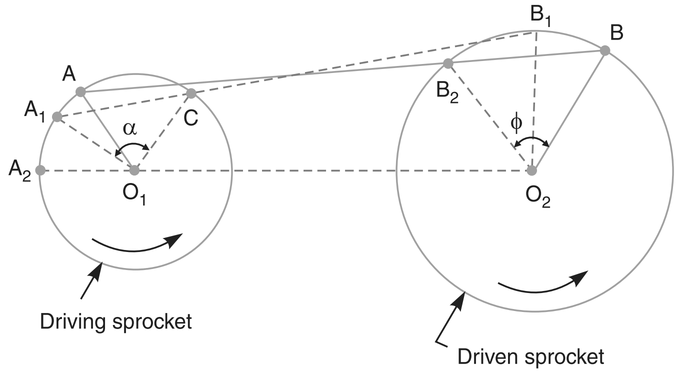


Fig. 11.27. Kinematic of chain drive.

Let us now consider the pin centre on the driving sprocket in position A. The length of the chain AB will remain straight as the sprockets rotate, until A reaches A_1 and B reaches B_1. As the driving sprocket continues to turn, the link A_1C of the chain turns about the pin centre C and the straight length of the chain between the two sprockets reduces to CB_1. When the pin centre C moves to the position A_1, the pin centre A_1 moves to the position A_2. During this time, each of the sprockets rotate from its original position by an angle corresponding to one chain pitch. During the first part of the angular displacement, the radius O_1A moves to O_1A_1 and the radius O_2B moves to O_2B_1. This arrangement is kinematically equivalent to the four bar chain O_1ABO_2.

During the second part of the angular displacement, the radius O_1A_1 moves to O_1A_2 and the radius O_2B_1 moves to O_2B_2. This arrangement is kinematically equivalent to the four bar chain $O_1CB_1O_2$. The ratio of the angular velocities, under these circumstances, cannot be constant. This may be easily shown as discussed below :

First of all, let us find the instantaneous centre for the two links O_1A and O_2B. This lies at point I which is the intersection of BA and O_2O_1 produced as shown in Fig. 11.28. If ω_1 is the angular velocity of the driving sprocket and ω_2 is the angular velocity of the driven sprocket, then

$$\omega_1 \times O_1I = \omega_2 \times O_2I$$

or
$$\frac{\omega_1}{\omega_2} = \frac{O_2I}{O_1I} = \frac{O_2O_1 + O_1I}{O_1I} = 1 + \frac{O_2O_1}{O_1I}.$$

The distance between the centres of two sprockets $O_1 O_2$ is constant for a given chain drive, but the distance $O_1 I$ varies periodically as the two sprockets rotate. This period corresponds to a rotation of the driving sprocket by an angle α. It is clear from the figure that the line AB has minimum inclination with line $O_1 O_2$. Therefore the distance $O_1 I$ is maximum and thus velocity ratio (ω_1 / ω_2) is minimum. When the chain occupies the position $A_1 B_1$, the inclination of line $A_1 B_1$ is maximum with the line $O_1 O_2$. Therefore the distance $O_1 I_1$ is minimum and thus the velocity ratio (ω_1 / ω_2) is maximum.

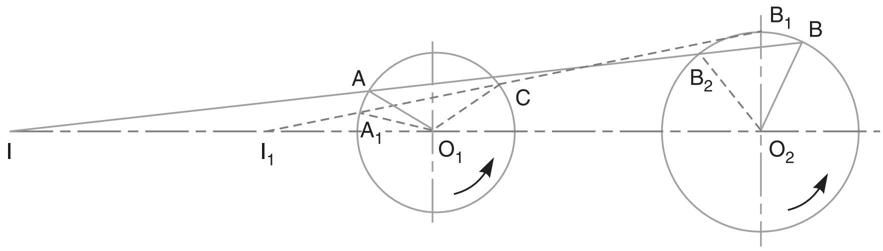


Fig. 11.28. Angular velocities of the two sprockets.

In actual practice, the smaller sprocket have a minimum of 18 teeth and hence the actual variation of velocity ratio (ω_1/ω_2) from the mean value is very small.

11.35. Classification of Chains

The chains, on the basis of their use, are classified into the following three groups :

1. Hoisting and hauling (or crane) chains,
2. Conveyor (or tractive) chains, and
3. Power transmitting (or driving) chains.

These chains are discussed, in detail, in the following pages.

11.36. Hoisting and Hauling Chains

These chains are used for hoisting and hauling purposes. The hoisting and hauling chains are of the following two types :

1. ***Chain with oval links*.** The links of this type of chain are of oval shape, as shown in Fig. 11.29 (*a*). The joint of each link is welded. The sprockets which are used for this type of chain have receptacles to receive the links. Such type of chains are used only at low speeds such as in chain hoists and in anchors for marine works.

(*a*) Chain with oval links. (*b*) Chain with square links.

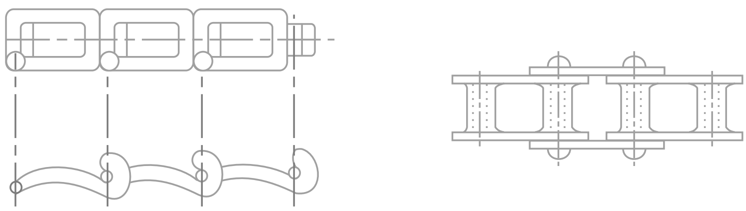
Fig. 11.29. Hoisting and hauling chains.

2. ***Chain with square links*.** The links of this type of chain are of square shape, as shown in Fig. 11.29 (*b*). Such type of chains are used in hoists, cranes, dredges. The manufacturing cost of this type of chain is less than that of chain with oval links, but in these chains, the kinking occurs easily on overloading.

11.37. Conveyor Chains

These chains are used for elevating and conveying the materials continuously. The conveyor chains are of the following two types :

1. Detachable or hook joint type chain, as shown in Fig. 11.30 (*a*), and
2. Closed joint type chain, as shown in Fig. 11.30 (*b*).



(*a*) Detachable or hook joint type chain. (*b*) Closed joint type chain.

Fig. 11.30. Conveyor chains.

The conveyor chains are usually made of malleable cast iron. These chains do not have smooth running qualities. The conveyor chains run at slow speeds of about 3 to 12 km.p.h.

11.38. Power Transmitting Chains

These chains are used for transmission of power, when the distance between the centres of shafts is short. These chains have provision for efficient lubrication. The power transmitting chains are of the following three types.

1. ***Block chain*.** A block chain, as shown in Fig. 11.31, is also known as **bush chain.** This type of chain was used in the early stages of development in the power transmission.

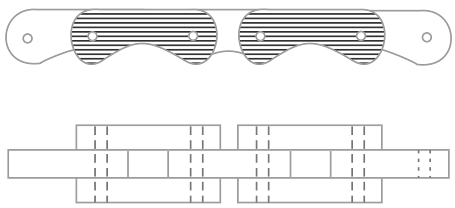


Fig. 11.31. Block chain.

It produces noise when approaching or leaving the teeth of the sprocket because of rubbing between the teeth and the links. Such type of chains are used to some extent as conveyor chain at small speed.

2. ***Bush roller chain*.** A bush roller chain, as shown in Fig. 11.32, consists of outer plates or pin link plates, inner plates or roller link plates, pins, bushes and rollers. A pin passes through the bush which is secured in the holes of the roller between the two sides of the chain. The rollers are free to rotate on the bush which protect the sprocket wheel teeth against wear.

A bush roller chain is extremely strong and simple in construction. It gives good service under severe conditions. There is a little noise with this chain which is due to impact of the rollers on the sprocket wheel teeth. This chain may be used where there is a little lubrication. When one of these chains elongates slightly due to wear and stretching of the parts, then the extended chain is of greater pitch than the pitch of the sprocket wheel teeth. The rollers then fit unequally into the cavities of the wheel. The result is that the total load falls on one teeth or on a few teeth. The stretching of the parts increase wear of the surfaces of the roller and of the sprocket wheel teeth.

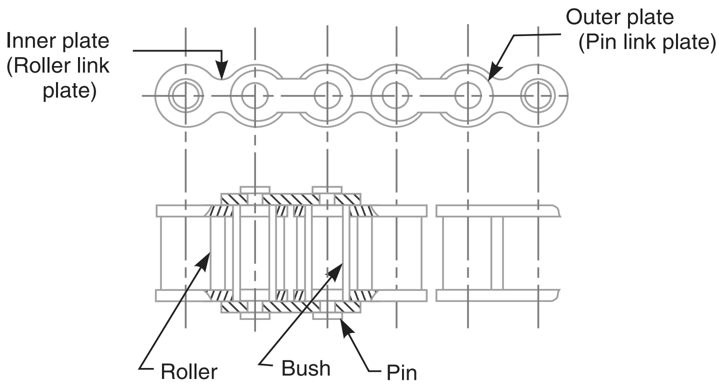


Fig. 11.32. Bush roller chain.

3. *Inverted tooth or silent chain*. An inverted tooth or silent chain is shown in Fig. 11.33. It is designed to eliminate the evil effects caused by stretching and to produce noiseless running. When the chain stretches and the pitch of the chain increases, the links ride on the teeth of the sprocket wheel at a slightly increased radius. This automatically corrects the small change in the pitch. There is no relative sliding between the teeth of the inverted tooth chain and the sprocket wheel teeth. When properly lubricated, this chain gives durable service and runs very smoothly and quietly.

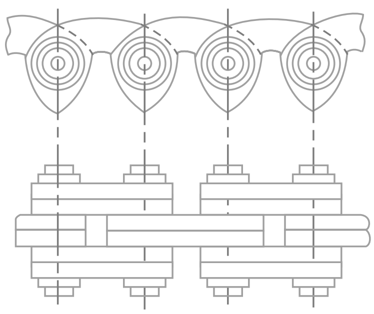


Fig. 11.33. Inverted tooth or silent chain.

11.39. Length of Chain

An open chain drive system connecting the two sprockets is shown in Fig. 11.34. We have already discussed in Art. 11.11 that the length of belt for an open belt drive connecting the two pulleys of radii r_1 and r_2 and a centre distance x, is

$$L = \pi(r_1 + r_2) + 2x + \frac{(r_1 - r_2)^2}{x} \qquad (i)$$

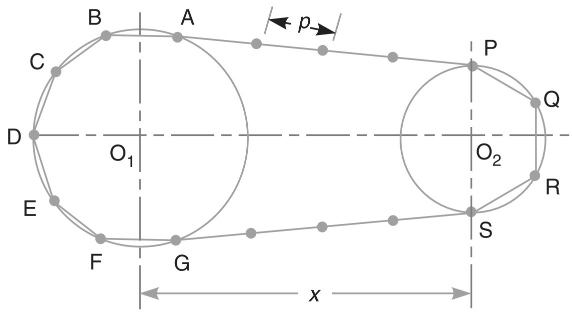


Fig. 11.34. Length of chain

If this expression is used for determining the length of chain, the result will be slightly greater than the required length. This is due to the fact that the pitch lines *A B C D E F G* and *P Q R S* of the sprockets are the parts of a polygon and not that of a circle. The exact length of the chain may be determined as discussed below :

Let T_1 = Number of teeth on the larger sprocket,

T_2 = Number of teeth on the smaller sprocket, and

p = Pitch of the chain.

We have discussed in Art. 11.32, that diameter of the pitch circle,

$$d = p \operatorname{cosec}\left(\frac{180°}{T}\right) \quad \text{or} \quad r = \frac{p}{2} \operatorname{cosec}\left(\frac{180°}{T}\right)$$

∴ For larger sprocket,

$$r_1 = \frac{p}{2} \operatorname{cosec}\left(\frac{180°}{T_1}\right)$$

and for smaller sprocket, $$r_2 = \frac{p}{2} \operatorname{cosec}\left(\frac{180°}{T_2}\right)$$

Since the term $\pi\,(r_1 + r_2)$ is equal to half the sum of the circumferences of the pitch circles, therefore the length of chain corresponding to

$$\pi(r_1 + r_2) = \frac{p}{2}(T_1 + T_2)$$

Substituting the values of r_1, r_2 and $\pi\,(r_1 + r_2)$ in equation (*i*), the length of chain is given by

$$L = \frac{p}{2}(T_1 + T_2) + 2x + \frac{\left[\frac{p}{2}\operatorname{cosec}\left(\frac{180°}{T_1}\right) - \frac{p}{2}\operatorname{cosec}\left(\frac{180°}{T_2}\right)\right]^2}{x}$$

If $x = m.p$, then

$$L = p\left[\frac{(T_1 + T_2)}{2} + 2m + \frac{\left[\operatorname{cosec}\left(\frac{180°}{T_1}\right) - \operatorname{cosec}\left(\frac{180°}{T_2}\right)\right]^2}{4\,m}\right] = p.K$$

where K = Multiplying factor

$$= \frac{(T_1 + T_2)}{2} + 2m + \frac{\left[\operatorname{cosec}\left(\frac{180°}{T_1}\right) - \operatorname{cosec}\left(\frac{180°}{T_2}\right)\right]^2}{4m}$$

The value of multiplying factor (K) may not be a complete integer. But the length of the chain must be equal to an integer number of times the pitch of the chain. Thus, the value of K should be rounded off to the next higher integral number.

OBJECTIVE TYPE QUESTIONS

1. The velocity ratio of two pulleys connected by an open belt or crossed belt is

   (*a*) directly proportional to their diameters

   (*b*) inversely proportional to their diameters

   (*c*) directly proportional to the square of their diameters

   (*d*) inversely proportional to the square of their diameters

2. Two pulleys of diameters d_1 and d_2 and at distance x apart are connected by means of an open belt drive. The length of the belt is

   (*a*) $\frac{\pi}{2}(d_1 + d_2) + 2x + \frac{(d_1 + d_2)^2}{4x}$ (*b*) $\frac{\pi}{2}(d_1 - d_2) + 2x + \frac{(d_1 - d_2)^2}{4x}$

   (*c*) $\frac{\pi}{2}(d_1 + d_2) + 2x + \frac{(d_1 - d_2)^2}{4x}$ (*d*) $\frac{\pi}{2}(d_1 - d_2) + 2x + \frac{(d_1 + d_2)^2}{4x}$

3. In a cone pulley, if the sum of radii of the pulleys on the driving and driven shafts is constant, then

   (*a*) open belt drive is recommended

   (*b*) cross belt drive is recommended

   (*c*) both open belt drive and cross belt drive are recommended

   (*d*) the drive is recommended depending upon the torque transmitted

4. Due to slip of the belt, the velocity ratio of the belt drive

   (*a*) decreases (*b*) increases (*c*) does not change

5. When two pulleys of different diameters are connected by means of an open belt drive, then the angle of contact taken into consideration should be of the

   (*a*) larger pulley (*b*) smaller pulley (*c*) average of two pulleys

6. The power transmitted by a belt is maximum when the maximum tension in the belt (T) is equal to

   (*a*) T_C (*b*) $2T_C$ (*c*) $3T_C$ (*d*) $4T_C$

   where T_C = Centrifugal tension.

7. The velocity of the belt for maximum power is

   (*a*) $\sqrt{\frac{T}{3m}}$ (*b*) $\sqrt{\frac{T}{4m}}$ (*c*) $\sqrt{\frac{T}{5m}}$ (*d*) $\sqrt{\frac{T}{6m}}$

   where m = Mass of the belt in kg per metre length.

8. The centrifugal tension in belts

(*a*) increases power transmitted

(*b*) decreases power transmitted

(*c*) have no effect on the power transmitted

(*d*) increases power transmitted upto a certain speed and then decreases

9. When the belt is stationary, it is subjected to some tension, known as initial tension. The value of this tension is equal to the

(*a*) tension in the tight side of the belt

(*b*) tension in the slack side of the belt

(*c*) sum of the tensions in the tight side and slack side of the belt

(*d*) average tension of the tight side and slack side of the belt

10. The relation between the pitch of the chain (p) and pitch circle diameter of the sprocket (d) is given by

(*a*) $p = d \sin\left(\dfrac{60°}{T}\right)$ (*b*) $p = d \sin\left(\dfrac{90°}{T}\right)$

(*c*) $p = d \sin\left(\dfrac{120°}{T}\right)$ (*d*) $p = d \sin\left(\dfrac{180°}{T}\right)$

where T = Number of teeth on the sprocket.

ANSWERS

1. (*b*)	**2.** (*c*)	**3.** (*b*)	**4.** (*a*)	**5.** (*b*)
6. (*c*)	**7.** (*a*)	**8.** (*c*)	**9.** (*d*)	**10.** (*d*)

MODULE-V

Brakes & Dynamometers : Classification of brakes, Analysis of simple block, Band and internal expanding shoe brake, Braking of a vehicle. Absorption and transmission dynamometers, Prony brake, Rope brake dynamometer, belt transmission, epicyclic train, torsion dynamometer.

Brakes and Dynamometers

Features

19.1. Introduction

A ***brake*** is a device by means of which artificial frictional resistance is applied to a moving machine member, in order to retard or stop the motion of a machine. In the process of performing this function, the brake absorbs either kinetic energy of the moving member or potential energy given up by objects being lowered by hoists, elevators etc. The energy absorbed by brakes is dissipated in the form of heat. This heat is dissipated in the surrounding air (or water which is circulated through the passages in the brake drum) so that excessive heating of the brake lining does not take place. The capacity of a brake depends upon the following factors :

1. The unit pressure between the braking surfaces,
2. The coefficient of friction between the braking surfaces,
3. The peripheral velocity of the brake drum,
4. The projected area of the friction surfaces, and
5. The ability of the brake to dissipate heat equivalent to the energy being absorbed.

The major functional difference between a clutch and a brake is that a clutch is used to keep the driving and driven member moving together, whereas brakes are used to stop a moving member or to control its speed.

19.2. Materials for Brake Lining

The material used for the brake lining should have the following characteristics :

1. It should have high coefficient of friction with minimum fading. In other words, the coefficient of friction should remain constant with change in temperature.
2. It should have low wear rate.
3. It should have high heat resistance.
4. It should have high heat dissipation capacity.
5. It should have adequate mechanical strength.
6. It should not be affected by moisture and oil.

The materials commonly used for facing or lining of brakes and their properties are shown in the following table.

Table 19.1. Properties of materials for brake lining.

Material for braking lining	*Coefficient of friction* (μ)			*Allowable pressure* (*p*)
	Dry	*Greasy*	*Lubricated*	*N/mm²*
Cast iron on cast iron	0.15 – 0.2	0.06 – 0.10	0.05 – 0.10	1.0 – 1.75
Bronze on cast iron	–	0.05 – 0.10	0.05 – 0.10	0.56 – 0.84
Steel on cast iron	0.20 – 0.30	0.07 – 0.12	0.06 – 0.10	0.84 – 1.40
Wood on cast iron	0.20 – 0.35	0.08 – 0.12	–	0.40 – 0.62
Fibre on metal	–	0.10 – 0.20	–	0.07 – 0.28
Cork on metal	0.35	0.25 – 0.30	0.22 – 0.25	0.05 – 0.10
Leather on metal	0.30 – 0.5	0.15 – 0.20	0.12 – 0.15	0.07 – 0.28
Wire asbestos on metal	0.35 – 0.5	0.25 – 0.30	0.20 – 0.25	0.20 – 0.55
Asbestos blocks on metal	0.40 – 0.48	0.25 – 0.30	–	0.28 – 1.1
Asbestos on metal (Short action)	–	–	0.20 – 0.25	1.4 – 2.1
Metal on cast iron (Short action)	–	–	0.05 – 0.10	1.4 – 2.1

19.3. Types of Brakes

The brakes, according to the means used for transforming the energy by the braking elements, are classified as :

1. Hydraulic brakes *e.g.* pumps or hydrodynamic brake and fluid agitator,
2. Electric brakes *e.g.* generators and eddy current brakes, and
3. Mechanical brakes.

Simple bicycle brakes.

The hydraulic and electric brakes cannot bring the member to rest and are mostly used where large amounts of energy are to be transformed while the brake is retarding the load such as in laboratory dynamometers, high way trucks and electric locomotives. These brakes are also used for retarding or controlling the speed of a vehicle for down-hill travel.

The mechanical brakes, according to the direction of acting force, may be divided into the following two groups :

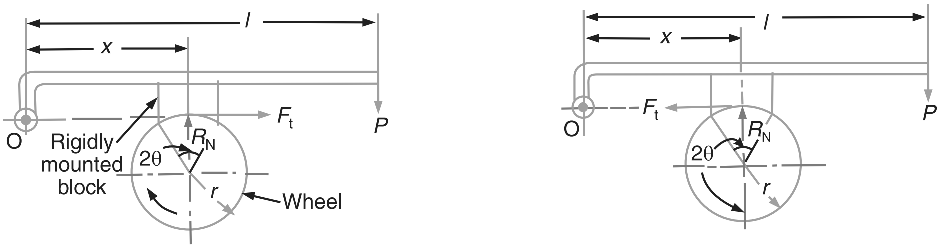
(*a*) ***Radial brakes.*** In these brakes, the force acting on the brake drum is in radial direction. The radial brakes may be sub-divided into ***external brakes*** and ***internal brakes***. According to the shape of the friction elements, these brakes may be ***block*** or ***shoe brakes*** and ***band brakes***.

(*b*) ***Axial brakes***. In these brakes, the force acting on the brake drum is in axial direction. The axial brakes may be disc brakes and cone brakes. The analysis of these brakes is similar to clutches.

Since we are concerned with only mechanical brakes, therefore, these are discussed, in detail, in the following pages.

19.4. Single Block or Shoe Brake

A single block or shoe brake is shown in Fig. 19.1. It consists of a block or shoe which is pressed against the rim of a revolving brake wheel drum. The block is made of a softer material than the rim of the wheel. This type of a brake is commonly used on railway trains and tram cars. The friction between the block and the wheel causes a tangential braking force to act on the wheel, which retard the rotation of the wheel. The block is pressed against the wheel by a force applied to one end of a lever to which the block is rigidly fixed as shown in Fig. 19.1. The other end of the lever is pivoted on a fixed fulcrum *O*.



(*a*) Clockwise rotation of brake wheel (*b*) Anticlockwise rotation of brake wheel.

Fig. 19.1. Single block brake. Line of action of tangential force passes through the fulcrum of the lever.

Let P = Force applied at the end of the lever,

R_N = Normal force pressing the brake block on the wheel,

r = Radius of the wheel,

2θ = Angle of contact surface of the block,

μ = Coefficient of friction, and

F_t = Tangential braking force or the frictional force acting at the contact surface of the block and the wheel.

If the angle of contact is less than 60°, then it may be assumed that the normal pressure between the block and the wheel is uniform. In such cases, tangential braking force on the wheel,

$$F_t = \mu . R_N \qquad ...(i)$$

and the braking torque, $T_B = F_t . r = \mu . R_N . r$ ... (*ii*)

Let us now consider the following three cases :

Case 1. When the line of action of tangential braking force (F_t) passes through the fulcrum *O* of the lever, and the brake wheel rotates clockwise as shown in Fig. 19.1 (*a*), then for equilibrium, taking moments about the fulcrum *O*, we have

$$R_N \times x = P \times l \quad \text{or} \quad R_N = \frac{P \times l}{x}$$

∴ Braking torque,

$$T_B = \mu . R_N . r = \mu \times \frac{P.l}{x} \times r = \frac{\mu . P . l . r}{x}$$

When brakes are on, the pads grip the wheel rim from either side, friction between the pads and the rim converts the cycle's kinetic energy into heat as they reduce its speed.

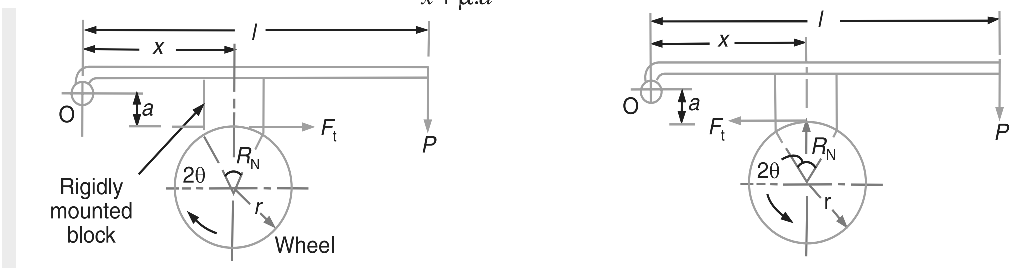
It may be noted that when the brake wheel rotates anticlockwise as shown in Fig. 19.1 (*b*), then the braking torque is same, *i.e.*

$$T_B = \mu.R_N.r = \frac{\mu.P.l.r}{x}$$

Case 2. When the line of action of the tangential braking force (F_t) passes through a distance '*a*' below the fulcrum *O*, and the brake wheel rotates clockwise as shown in Fig. 19.2 (*a*), then for equilibrium, taking moments about the fulcrum *O*,

$$R_N \times x + F_t \times a = P.l \quad \text{or} \quad R_N \times x + \mu\, R_N \times a = P.l \quad \text{or} \quad R_N = \frac{P.l}{x + \mu.a}$$

and braking torque, $\quad T_B = \mu R_N.r = \dfrac{\mu.p.l.r}{x + \mu.a}$



(*a*) Clockwise rotation of brake wheel. (*b*) Anticlockwise rotation of brake wheel.

Fig. 19.2. Single block brake. Line of action of F_t passes below the fulcrum.

When the brake wheel rotates anticlockwise, as shown in Fig. 19.2 (*b*), then for equilibrium,

$$R_N.x = P.l + F_t.a = P.l + \mu.R_N.a \qquad \text{...}(i)$$

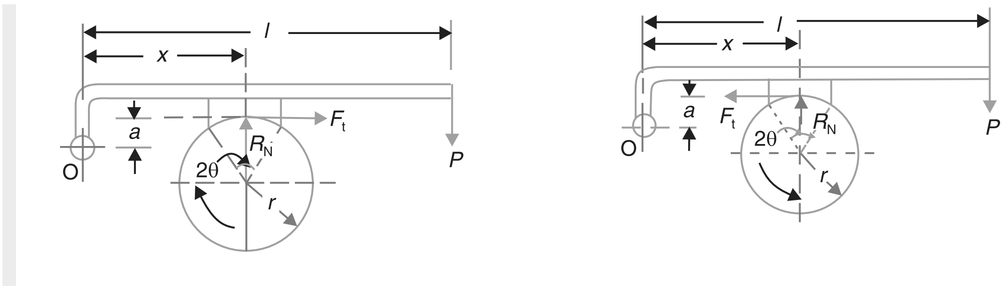
or $\quad R_N\,(x - \mu.a) = P.l \quad$ or $\quad R_N = \dfrac{P.l}{x - \mu.a}$

and braking torque, $\quad T_B = \mu.R_N.r = \dfrac{\mu.P.l.r}{x - \mu.a}$

Case 3. When the line of action of the tangential braking force (F_t) passes through a distance '*a*' above the fulcrum *O*, and the brake wheel rotates clockwise as shown in Fig. 19.3 (*a*), then for equilibrium, taking moments about the fulcrum *O*, we have

$$R_N.x = P.l + F_t\,.\,a = P.l + \mu.R_N.a \qquad \text{. . .}(ii)$$

or $\quad R_N\,(x - \mu.a) = P.l \quad$ or $\quad R_N = \dfrac{P.l}{x - \mu.a}$



(*a*) Clockwise rotation of brake wheel. (*b*) Anticlockwise rotation of brake wheel.

Fig. 19.3. Single block brake. Line of action of F_t passes above the fulcrum.

and braking torque, $\quad T_B = \mu.R_N.r = \dfrac{\mu.P.l.r}{x - \mu.a}$

When the brake wheel rotates anticlockwise as shown in Fig. 19.3 (*b*), then for equilibrium, taking moments about the fulcrum *O*, we have

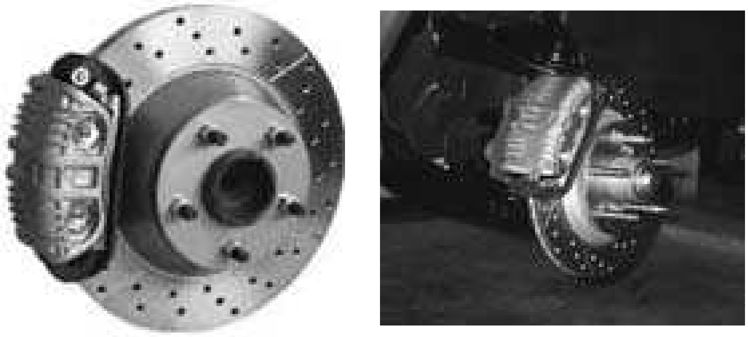
$$R_N \times x + F_t \times a = P.l \quad \text{or} \quad R_N \times x + \mu.R_N \times a = P.l \quad \text{or} \quad R_N = \frac{P.l}{x + \mu.a}$$

and braking torque, $T_B = \mu.R_N.r = \dfrac{\mu.P.l.r}{x+\mu.a}$

Notes : 1. From above we see that when the brake wheel rotates anticlockwise in case 2 [Fig. 19.2 (*b*)] and when it rotates clockwise in case 3 [Fig. 19.3 (*a*)], the equations (*i*) and (*ii*) are same, *i.e.*

$$R_N \times x = P.l + \mu.R_N.a$$

From this we see that the moment of frictional force ($\mu.R_N.a$) adds to the moment of force (*P.l*). In other words, the frictional force helps to apply the brake. Such type of brakes are said to be ***self energizing brakes***. When the frictional force is great enough to apply the brake with no external force, then the brake is said to be ***self-locking brake***.



Shoe brakes of a racing car

From the above expression, we see that if $x \le \mu.a$, then *P* will be negative or equal to zero. This means no external force is needed to apply the brake and hence the brake is self locking. Therefore the condition for the brake to be self locking is

$$x \le \mu.a$$

The self locking brake is used only in back-stop applications.

2. The brake should be self energizing and not the self locking.

3. In order to avoid self locking and to prevent the brake from grabbing, *x* is kept greater than $\mu . a$.

4. If A_b is the projected bearing area of the block or shoe, then the bearing pressure on the shoe,

$$p_b = R_N / A_b$$

We know that A_b = Width of shoe × Projected length of shoe = $w(2r\sin\theta)$

5. When a single block or shoe brake is applied to a rolling wheel, an additional load is thrown on the shaft bearings due to heavy normal force (R_N) and produces bending of the shaft.

In order to overcome this drawback, a double block or shoe brake is used, as discussed in Art. 19.6.

19.5. Pivoted Block or Shoe Brake

We have discussed in the previous article that when the angle of contact is less than 60°, then it may be assumed that the normal pressure between the block and the wheel is uniform. But when the angle of contact is greater than 60°, then the unit pressure normal to the surface of contact is less at the ends than at the centre. In such cases, the block or shoe is pivoted to the lever, as shown in Fig. 19.4, instead of being rigidly attached to the lever. This gives uniform wear of the brake lining in the direction of the applied force. The braking torque for a pivoted block or shoe brake (*i.e.* when $2\theta > 60°$) is given by

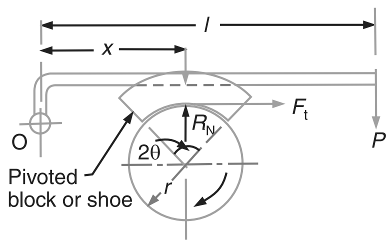


Fig. 19.4. Pivoted block or shoe brake.

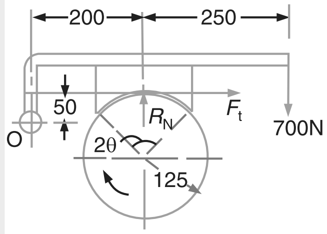
$$T_B = F_t \times r = \mu'.R_N.r$$

where μ' = Equivalent coefficient of friction = $\dfrac{4\mu\sin\theta}{2\theta + \sin 2\theta}$, and

μ = Actual coefficient of friction.

These brakes have more life and may provide a higher braking torque.

Example 19.1. *A single block brake is shown in Fig. 19.5. The diameter of the drum is 250 mm and the angle of contact is 90°. If the operating force of 700 N is applied at the end of a lever and the coefficient of friction between the drum and the lining is 0.35, determine the torque that may be transmitted by the block brake.*



All dimensions in mm.
Fig. 19.5

Solution. Given : $d = 250$ mm or $r = 125$ mm ; $2\theta = 90° = \pi/2$ rad ; $P = 700$ N ; $\mu = 0.35$

Since the angle of contact is greater than 60°, therefore equivalent coefficient of friction,

$$\mu' = \frac{4\mu \sin\theta}{2\theta + \sin 2\theta} = \frac{4 \times 0.35 \times \sin 45°}{\pi/2 + \sin 90°} = 0.385$$

Let R_N = Normal force pressing the block to the brake drum, and

F_t = Tangential braking force = $\mu'.R_N$

Taking moments about the fulcrum O, we have

$$700(250 + 200) + F_t \times 50 = R_N \times 200 = \frac{F_t}{\mu'} \times 200 = \frac{F_t}{0.385} \times 200 = 520\ F_t$$

or $520\ F_t - 50F_t = 700 \times 450$ or $F_t = 700 \times 450/470 = 670$ N

We know that torque transmitted by the block brake,

$T_B = F_t \times r = 670 \times 125 = 8\ 3750$ N-mm = 83.75N-m **Ans.**

Example 19.2. *Fig. 19.6 shows a brake shoe applied to a drum by a lever AB which is pivoted at a fixed point A and rigidly fixed to the shoe. The radius of the drum is 160 mm. The coefficient of friction at the brake lining is 0.3. If the drum rotates clockwise, find the braking torque due to the horizontal force of 600 N at B.*

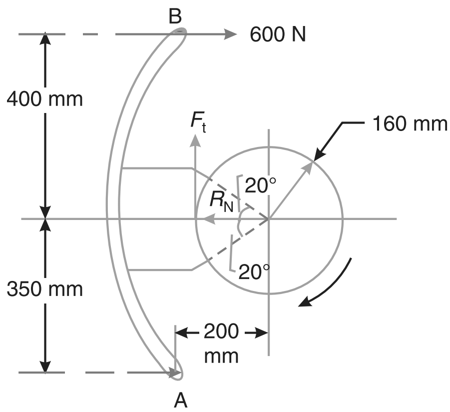


Fig. 19.6

Solution. Given : $r = 160$ mm = 0.16 m ; $\mu = 0.3$; $P = 600$ N

Since the angle subtended by the shoe at the centre of drum is 40°, therefore we need not to calculate the equivalent coefficient of friction μ'.

Let R_N = Normal force pressing the block to the brake drum, and

F_t = Tangential braking force = $\mu.R_N$

Taking moments about point A,

$R_N \times 350 + F_t\ (200 - 160) = 600\ (400 + 350)$

$$\frac{F_t}{0.3} \times 350 + 40F_t = 600 \times 750 \text{ or } 1207\ F_t = 450 \times 10^3$$

$\therefore$ $F_t = 450 \times 10^3/1207 = 372.8$ N

We know that braking torque,

$T_B = F_t \times r = 372.8 \times 0.16 = 59.6$ N-m **Ans.**

19.6. Double Block or Shoe Brake

When a single block brake is applied to a rolling wheel, an additional load is thrown on the shaft bearings due to the normal force (R_N). This produces bending of the shaft. In order to overcome this drawback, a double block or shoe brake, as shown in Fig. 19.9, is used. It consists of two brake blocks applied at the opposite ends of a diameter of the wheel which eliminate or reduces the unbalanced force on the shaft. The brake is set by a spring which pulls the upper ends of the brake arms together. When a force P is applied to the bell crank lever, the spring is compressed and the brake is released. This type of brake is often used on electric cranes and the force P is produced by an electromagnet or solenoid. When the current is switched off, there is no force on the bell crank lever and the brake is engaged automatically due to the spring force and thus there will be no downward movement of the load.

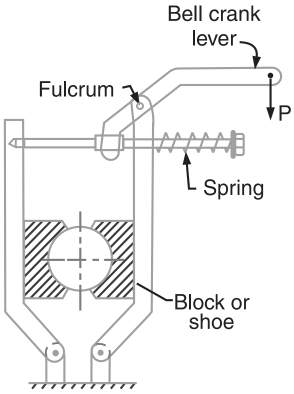


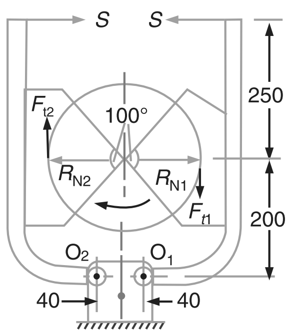
Fig. 19.9. Double block or shoe brake.

In a double block brake, the braking action is doubled by the use of two blocks and these blocks may be operated practically by the same force which will operate one. In case of double block or shoe brake, the braking torque is given by

$$T_B = (F_{t1} + F_{t2})\, r$$

where F_{t1} and F_{t2} are the braking forces on the two blocks.

Example 19.5. *A double shoe brake, as shown in Fig. 19.10, is capable of absorbing a torque of 1400 N-m. The diameter of the brake drum is 350 mm and the angle of contact for each shoe is 100°. If the coefficient of friction between the brake drum and lining is 0.4 ; find* **1.** *the spring force necessary to set the brake ; and* **2.** *the width of the brake shoes, if the bearing pressure on the lining material is not to exceed 0.3 N/mm².*



All dimensions in mm.
Fig. 19.10

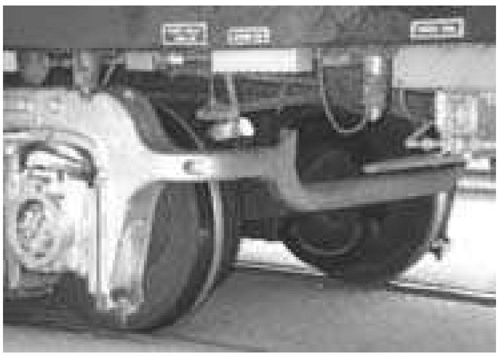
Solution. Given : T_B = 1400 N-m = 1400 × 10^3 N-mm ; d = 350 mm or r = 175 mm ; $2\theta = 100° = 100 \times \pi/180 = 1.75$ rad; $\mu = 0.4$; $p_b = 0.3$ N/mm²

1. ***Spring force necessary to set the brake***

Let S = Spring force necessary to set the brake.

R_{N1} and F_{t1} = Normal reaction and the braking force on the right hand side shoe, and

R_{N2} and F_{t2} = Corresponding values on the left hand side shoe.


Brakes on a railway coach.

Since the angle of contact is greater than 60°, therefore equivalent coefficient of friction,

$$\mu' = \frac{4\mu \sin\theta}{2\theta + \sin 2\theta} = \frac{4 \times 0.4 \times \sin 50°}{1.75 + \sin 100°} = 0.45$$

Taking moments about the fulcrum O_1, we have

$$S \times 450 = R_{N1} \times 200 + F_{t1}(175 - 40) = \frac{F_{t1}}{0.45} \times 200 + F_{t1} \times 135 = 579.4\, F_{t1}$$

$$\ldots \left(\text{Substituting } R_{N1} = \frac{F_{t1}}{\mu'}\right)$$

$\therefore$ $F_{t1} = S \times 450 / 579.4 = 0.776\, S$

Again taking moments about O_2, we have

$$S \times 450 + F_{t2}(175 - 40) = R_{N2} \times 200 = \frac{F_{t2}}{0.45} \times 200 = 444.4\, F_{t2}$$

$$\ldots \left(\text{Substituting } R_{N2} = \frac{F_{t2}}{\mu'}\right)$$

$444.4\, F_{t2} - 135 F_{t2} = S \times 450$ or $309.4\, F_{t2} = S \times 450$

$\therefore$ $F_{t2} = S \times 450 / 309.4 = 1.454\, S$

We know that torque capacity of the brake (T_B),

$$1400 \times 10^3 = (F_{t1} + F_{t2})\, r = (0.776\, S + 1.454\, S)\, 175 = 390.25\, S$$

$\therefore$ $S = 1400 \times 10^3 / 390.25 = 3587$ N **Ans.**

2. ***Width of the brake shoes***

Let b = Width of the brake shoes in mm.

We know that projected bearing area for one shoe,

$$A_b = b(2r \sin \theta) = b(2 \times 175 \sin 50°) = 268\, b \text{ mm}^2$$

Normal force on the right hand side of the shoe,

$$R_{N1} = \frac{F_{t1}}{\mu'} = \frac{0.776 \times S}{0.45} = \frac{0.776 \times 3587}{0.45} = 6186 \text{ N}$$

and normal force on the left hand side of the shoe,

$$R_{N2} = \frac{F_{t2}}{\mu'} = \frac{1.454 \times S}{0.45} = \frac{1.454 \times 3587}{0.45} = 11\,590 \text{ N}$$

We see that the maximum normal force is on the left hand side of the shoe. Therefore we shall find the width of the shoe for the maximum normal force *i.e.* R_{N2}.

We know that the bearing pressure on the lining material (p_b),

$$0.3 = \frac{R_{N2}}{A_b} = \frac{11\,590}{268\, b} = \frac{43.25}{b}$$

$\therefore$ $b = 43.25 / 0.3 = 144.2$ mm **Ans.**

19.7. Simple Band Brake

A band brake consists of a flexible band of leather, one or more ropes,or a steel lined with friction material, which embraces a part of the circumference of the drum. A band brake, as shown in Fig. 19.11, is called a ***simple band brake*** in which one end of the band is attached to a fixed pin or fulcrum of the lever while the other end is attached to the lever at a distance *b* from the fulcrum.

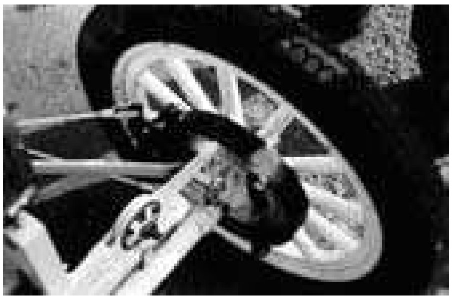
When a force *P* is applied to the lever at *C*, the lever turns about the fulcrum pin *O* and tightens the band on the drum and hence the brakes are applied. The friction between the band and the drum provides the braking force. The force *P* on the lever at *C* may be determined as discussed below :

Let T_1 = Tension in the tight side of the band,

T_2 = Tension in the slack side of the band,

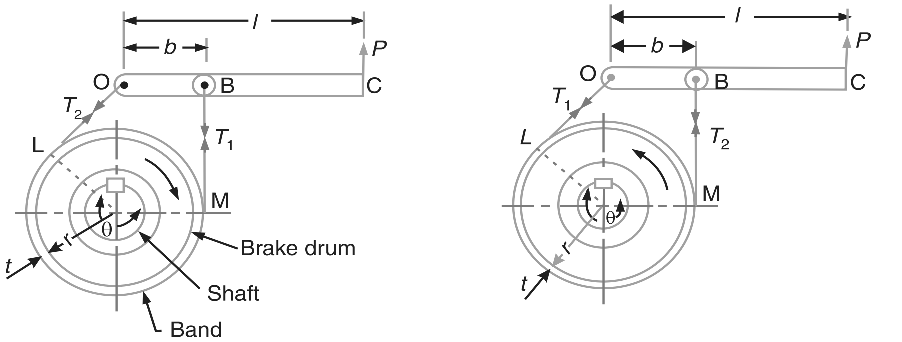
θ = Angle of lap (or embrace) of the band on the drum,
μ = Coefficient of friction between the band and the drum,
r = Radius of the drum,
t = Thickness of the band, and

r_e = Effective radius of the drum = $r + \dfrac{t}{2}$



Band brake

Bands of a brake shown separately



(*a*) Clockwise rotation of drum. (*b*) Anticlockwise rotation of drum.

Fig. 19.11. Simple band brake.

We know that limiting ratio of the tensions is given by the relation,

$$\frac{T_1}{T_2} = e^{\mu\theta} \quad \text{or} \quad 2.3 \log\left(\frac{T_1}{T_2}\right) = \mu.\theta$$

and braking force on the drum = $T_1 - T_2$

$\therefore$ Braking torque on the drum,

$$T_B = (T_1 - T_2)\, r \qquad \text{. . . (Neglecting thickness of band)}$$

$$= (T_1 - T_2)\, r_e \qquad \text{. . . (Considering thickness of band)}$$

Now considering the equilibrium of the lever OBC. It may be noted that when the drum rotates in the clockwise direction, as shown in Fig. 19.11 (*a*), the end of the band attached to the fulcrum O will be slack with tension T_2 and end of the band attached to B will be tight with tension T_1. On the other hand, when the drum rotates in the anticlockwise direction, as shown in Fig. 19.11 (*b*), the tensions in the band will reverse, *i.e.* the end of the band attached to the fulcrum O will be tight with tension T_1 and the end of the band attached to B will be slack with tension T_2. Now taking moments about the fulcrum O, we have

$$P.l = T_1.b \qquad \text{. . . (For clockwise rotation of the drum)}$$

and

$$P.l = T_2.b \qquad \text{. . . (For anticlockwise rotation of the drum)}$$

where l = Length of the lever from the fulcrum (OC), and

b = Perpendicular distance from O to the line of action of T_1 or T_2.

Notes : 1. When the brake band is attached to the lever, as shown in Fig. 19.11 (*a*) and (*b*), then the force (P) must act in the upward direction in order to tighten the band on the drum.

2. If the permissible tensile stress (σ) for the material of the band is known, then maximum tension in the band is given by

$$T_1 = \sigma.w.t$$

where w = Width of the band, and

t = thickness of the band.

Example 19.6. *A band brake acts on the 3/4th of circumference of a drum of 450 mm diameter which is keyed to the shaft. The band brake provides a braking torque of 225 N-m. One end of the band is attached to a fulcrum pin of the lever and the other end to a pin 100 mm from the fulcrum. If the operating force is applied at 500 mm from the fulcrum and the coefficient of friction is 0.25, find the operating force when the drum rotates in the* **(a)** *anticlockwise direction, and* **(b)** *clockwise direction.*

Solution. Given : d = 450 mm or r = 225 mm = 0.225 m ; T_B = 225 N-m ; $b = OB$ = 100 mm = 0.1 m ; l = 500 mm = 0.5 m ; μ = 0.25

Let P = Operating force.

Drums for band brakes.

(a) Operating force when drum rotates in anticlockwise direction

The band brake is shown in Fig. 19.11. Since one end of the band is attached to the fulcrum at O, therefore the operating force P will act upward and when the drum rotates anticlockwise, as shown in Fig. 19.11 (*b*), the end of the band attached to O will be tight with tension T_1 and the end of the band attached to B will be slack with tension T_2. First of all, let us find the tensions T_1 and T_2.

We know that angle of wrap,

$$\theta = \frac{3}{4}\text{th of circumference} = \frac{3}{4}\times 360° = 270°$$

$$= 270 \times \pi / 180 = 4.713 \text{ rad}$$

and
$$2.3 \log\left(\frac{T_1}{T_1}\right) = \mu.\theta = 0.25 \times 4.713 = 1.178$$

$\therefore$
$$\log\left(\frac{T_1}{T_2}\right) = \frac{1.178}{2.3} = 0.5123 \quad \text{or} \quad \frac{T_1}{T_2} = 3.253 \quad \ldots (i)$$

. . . (Taking antilog of 0.5123)

We know that braking torque (T_B),

$$225 = (T_1 - T_2)\, r = (T_1 - T_2)\, 0.225$$

$\therefore$ $T_1 - T_2 = 225 / 0.225 = 1000$ N . . . (*ii*)

From equations (*i*) and (*ii*), we have

$$T_1 = 1444 \text{ N}; \text{ and } \quad T_2 = 444 \text{ N}$$

Now taking moments about the fulcrum O, we have

$$P \times l = T_2.b \quad \text{or} \quad P \times 0.5 = 444 \times 0.1 = 44.4$$

$\therefore$ $P = 44.4 / 0.5 = 88.8$ N **Ans.**

(*b*) ***Operating force when drum rotates in clockwise direction***

When the drum rotates in clockwise direction, as shown in Fig.19.11 (*a*), then taking moments about the fulcrum *O*, we have

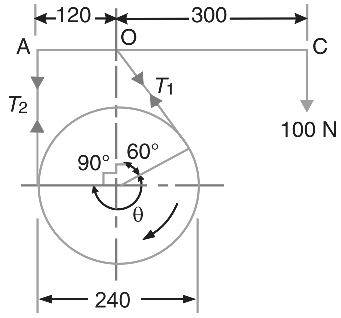
$$P \times l = T_1 . b \quad \text{or} \quad P \times 0.5 = 1444 \times 0.1 = 144.4$$

$$\therefore \quad P = 144.4 / 0.5 = 288.8 \text{ N } \textbf{Ans.}$$

Example 19.7. *The simple band brake, as shown in Fig. 19.12, is applied to a shaft carrying a flywheel of mass 400 kg. The radius of gyration of the flywheel is 450 mm and runs at 300 r.p.m.*

If the coefficient of friction is 0.2 and the brake drum diameter is 240 mm, find :

1. *the torque applied due to a hand load of 100 N,*
2. *the number of turns of the wheel before it is brought to rest, and*
3. *the time required to bring it to rest, from the moment of the application of the brake.*



All dimensions in mm.

Fig. 19.12

Solution. Given : $m = 400$ kg ; $k = 450$ mm $= 0.45$ m ; $N = 300$ r.p.m. or $\omega = 2\pi \times 300/60 = 31.42$ rad/s ; $\mu = 0.2$; $d = 240$ mm $= 0.24$ m or $r = 0.12$ m

1. ***Torque applied due to hand load***

First of all, let us find the tensions in the tight and slack sides of the band *i.e.* T_1 and T_2 respectively.

From the geometry of the Fig. 19.12, angle of lap of the band on the drum,

$$\theta = 360° - 150° = 210° = 210 \times \frac{\pi}{180} = 3.666 \text{ rad}$$

We know that

$$2.3 \log \left(\frac{T_1}{T_2} \right) = \mu.\theta = 0.2 \times 3.666 = 0.7332$$

$$\log \left(\frac{T_1}{T_2} \right) = \frac{0.7332}{2.3} = 0.3188 \quad \text{or} \quad \frac{T_1}{T_2} = 2.08 \qquad \text{. . . }(i)$$

... (Taking antilog of 0.3188)

Taking moments about the fulcrum *O*,

$$T_2 \times 120 = 100 \times 300 = 30\,000 \quad \text{or} \quad T_2 = 30\,000/120 = 250 \text{ N}$$

$$\therefore \quad T_1 = 2.08 T_2 = 2.08 \times 250 = 520 \text{ N} \qquad \text{. . . [From equation } (i)]$$

We know that torque applied,

$$T_B = (T_1 - T_2)\, r = (520 - 250)\, 0.12 = 32.4 \text{ N-m } \textbf{Ans.}$$

2. ***Number of turns of the wheel before it is brought to rest***

Let n = Number of turns of the wheel before it is brought to rest.

We know that kinetic energy of rotation of the drum

$$= \frac{1}{2} \times I.\omega^2 = \frac{1}{2} \times m.k^2.\omega^2 = \frac{1}{2} \times 400 (0.45)^2 (31.42)^2 = 40\,000 \text{ N-m}$$

This energy is used to overcome the work done due to the braking torque (T_B).

$$\therefore \quad 40\,000 = T_B \times 2\pi n = 32.4 \times 2\pi n = 203.6\, n$$

or

$$n = 40\,000 / 203.6 = 196.5 \textbf{ Ans.}$$

3. Time required to bring the wheel to rest

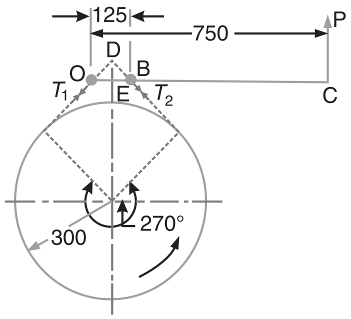
We know that the time required to bring the wheel to rest

$$= n / N = 196.5 / 300 = 0.655 \text{ min} = 39.3 \text{ s} \quad \textbf{Ans.}$$

Example 19.8. *A simple band brake operates on a drum of 600 mm in diameter that is running at 200 r.p.m. The coefficient of friction is 0.25. The brake band has a contact of 270°, one end is fastened to a fixed pin and the other end to the brake arm 125 mm from the fixed pin. The straight brake arm is 750 mm long and placed perpendicular to the diameter that bisects the angle of contact.*

1. *What is the pull necessary on the end of the brake arm to stop the wheel if 35 kW is being absorbed ? What is the direction for this minimum pull ?*

2. *What width of steel band of 2.5 mm thick is required for this brake if the maximum tensile stress is not to exceed 50 N/mm² ?*



All dimensions in mm

Fig. 19.13

Solution. Given : d = 600 mm or r = 300 mm ; N = 200 r.p.m. ; $\mu = 0.25$; $\theta = 270° = 270 \times \pi / 180 = 4.713$ rad ; Power = 35 kW = 35×10^3 W ; t = 2.5 mm ; σ = 50 N/mm²

1. Pull necessary on the end of the brake arm to stop the wheel

Let P = Pull necessary on the end of the brake arm to stop the wheel.

The simple band brake is shown in Fig. 19.13. Since one end of the band is attached to the fixed pin O, therefore the pull P on the end of the brake arm will act upward and when the wheel rotates anticlockwise, the end of the band attached to O will be tight with tension T_1 and the end of the band attached to B will be slack with tension T_2. First of all, let us find the tensions T_1 and T_2. We know that

$$2.3 \log\left(\frac{T_1}{T_2}\right) = \mu.\theta = 0.25 \times 4.713 = 1.178$$

$$\therefore \quad \log\left(\frac{T_1}{T_2}\right) = \frac{1.178}{2.3} = 0.5122 \quad \text{or} \quad \frac{T_1}{T_2} = 3.25 \qquad \text{... (Taking antilog of 0.5122) ... } (i)$$

Let $\quad T_B$ = Braking torque.

We know that power absorbed,

$$35 \times 10^3 = \frac{2\pi \times N.T_B}{60} = \frac{2\pi \times 200 \times T_B}{60} = 21\, T_B$$

$$\therefore \quad T_B = 35 \times 10^3 / 21 = 1667 \text{ N-m} = 1667 \times 10^3 \text{ N-mm}$$

We also know that braking torque (T_B),

$$1667 \times 10^3 = (T_1 - T_2)\ r = (T_1 - T_2)\ 300$$

$$\therefore \quad T_1 - T_2 = 1167 \times 10^3 / 300 = 5556 \text{ N} \qquad \text{...}(ii)$$

From equations (*i*) and (*ii*), we find that

$$T_1 = 8025 \text{ N}; \quad \text{and} \quad T_2 = 2469 \text{ N}$$

Now taking moments about O, we have

$$P \times 750 = T_2 \times {}^{*}OD = T_2 \times 62.5\sqrt{2} = 2469 \times 88.4 = 218\ 260$$

$\therefore \qquad P = 218260 / 750 = 291$ N **Ans.**

2. ***Width of steel band***

Let $\qquad w$ = Width of steel band in mm.

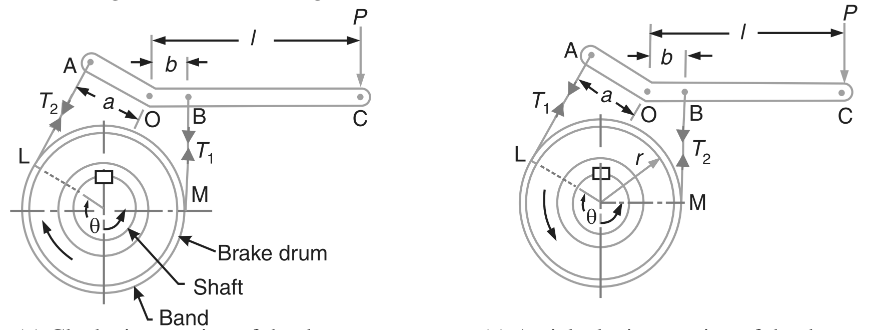
We know that maximum tension in the band (T_1),

$$8025 = \sigma.w.t = 50 \times w \times 2.5 = 125\, w$$

$\therefore \qquad w = 8025 / 125 = 64.2$ mm **Ans.**

19.8. Differential Band Brake

In a differential band brake, as shown in Fig. 19.14, the ends of the band are joined at A and B to a lever AOC pivoted on a fixed pin or fulcrum O. It may be noted that for the band to tighten, the length OA must be greater than the length OB.



(a) Clockwise rotation of the drum. $\qquad$ (a) Anticlockwise rotation of the drum.

Fig. 19.14. Differential band brake.

The braking torque on the drum may be obtained in the similar way as discussed in simple band brake. Now considering the equilibrium of the lever AOC. It may be noted that when the drum rotates in the clockwise direction, as shown in Fig. 19.14 (a), the end of the band attached to A will be slack with tension T_2 and end of the band attached to B will be tight with tension T_1. On the other hand, when the drum rotates in the anticlockwise direction, as shown in Fig. 19.14 (b), the end of the band attached to A will be tight with tension T_1 and end of the band attached to B will be slack with tension T_2. Now taking moments about the fulcrum O, we have

$$P.l + T_1.b = T_2.a$$

... (For clockwise rotation of the drum)

or $\qquad P.l = T_2.a - T_1.b \qquad$ ... (i)

and $\qquad P.l + T_2.b = T_1.a$

... (For anticlockwise rotation of the drum)

or $\qquad P.l = T_1.a - T_2.b \qquad$ ... (ii)

Tractors are specially made to move on rough terrain and exert high power at low speeds.

Note : This picture is given as additional information and is not a direct example of the current chapter.

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* OD = Perpendicular distance from O to the line of action of tension T_2.

$OE = EB = OB/2 = 125/2 = 62.5$ mm, and $\angle DOE = 45°$

$\therefore \quad OD = OE \sec 45° = 62.5\sqrt{2}$ mm

We have discussed in block brakes (Art. 19.4), that when the frictional force helps to apply the brake, it is said to be self energizing brake. In case of differential band brake, we see from equations (*i*) and (*ii*) that the moment $T_1.b$ and $T_2.b$ helps in applying the brake (because it adds to the moment $P.l$) for the clockwise and anticlockwise rotation of the drum respectively.

We have also discussed that when the force P is negative or zero, then brake is self locking. Thus for differential band brake and for clockwise rotation of the drum, the condition for self locking is

$$T_2.a \le T_1.b \quad \text{or} \quad T_2 / T_1 \le b / a$$

and for anticlockwise rotation of the drum, the condition for self locking is

$$T_1.a \le T_2.b \quad \text{or} \quad T_1 / T_2 \le b / a$$

Notes : 1. The condition for self locking may also be written as follows :
For clockwise rotation of the drum,

$$T_1.b \ge T_2.a \quad \text{or} \quad T_1 / T_2 \ge a / b$$

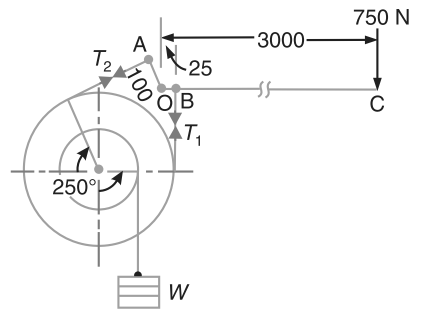
and for anticlockwise rotation of the drum,

$$T_2.b \ge T_1.a \quad \text{or} \quad T_1 / T_2 \ge a / b$$

2. When in Fig. 19.14 (*a*) and (*b*), the length OB is greater than OA, then the force P must act in the upward direction in order to apply the brake. The tensions in the band, *i.e.* T_1 and T_2 will remain unchanged.

Example 19.9. *In a winch, the rope supports a load W and is wound round a barrel 450 mm diameter. A differential band brake acts on a drum 800 mm diameter which is keyed to the same shaft as the barrel. The two ends of the bands are attached to pins on opposite sides of the fulcrum of the brake lever and at distances of 25 mm and 100 mm from the fulcrum. The angle of lap of the brake band is 250° and the coefficient of friction is 0.25. What is the maximum load W which can be supported by the brake when a force of 750 N is applied to the lever at a distance of 3000 mm from the fulcrum ?*

Solution. Given : $D = 450$ mm or $R = 225$ mm ; $d = 800$ mm or $r = 400$ mm ; $OB = 25$ mm ; $OA = 100$ mm ; $\theta = 250° = 250 \times \pi/180 = 4.364$ rad ; $\mu = 0.25$; $P = 750$ N ; $l = OC = 3000$ mm



All dimensions in mm.

Fig. 19.15

Since OA is greater than OB, therefore the operating force ($P = 750$ N) will act downwards.

First of all, let us consider that the drum rotates in clockwise direction.

We know that when the drum rotates in clockwise direction, the end of band attached to A will be slack with tension T_2 and the end of the band attached to B will be tight with tension T_1, as shown in Fig. 19.15. Now let us find out the values of tensions T_1 and T_2. We know that

$$2.3 \log \left(\frac{T_1}{T_2} \right) = \mu.\theta = 0.25 \times 4.364 = 1.091$$

$$\therefore \quad \log \left(\frac{T_1}{T_2} \right) = \frac{1.091}{2.3} = 0.4743 \quad \text{or} \quad \frac{T_1}{T_2} = 2.98 \qquad \text{... (Taking antilog of 0.4743)}$$

and $$T_1 = 2.98\, T_2 \qquad \text{... } (i)$$

Now taking moments about the fulcrum O,

$$750 \times 3000 + T_1 \times 25 = T_2 \times 100$$

or $T_2 \times 100 - 2.98\ T_2 \times 25 = 2250 \times 10^3$ ... ($\because\ T_1 = 2.98\ T_2$)

$$25.5\ T_2 = 2250 \times 10^3 \quad \text{or} \quad T_2 = 2250 \times 10^3/25.5 = 88 \times 10^3 \text{ N}$$

and $T_1 = 2.98 T_2 = 2.98 \times 88 \times 10^3 = 262 \times 10^3$ N

We know that braking torque,

$$T_B = (T_1 - T_2)\ r$$
$$= (262 \times 10^3 - 88 \times 10^3)\ 400 = 69.6 \times 10^6 \text{ N-mm} \quad ...(i)$$

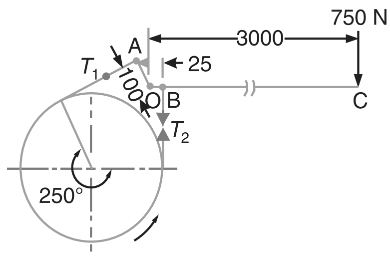
and the torque due to load W newtons,

$$T_W = W.R = W \times 225 = 225\ W \text{ N-mm} \quad ...(ii)$$

Since the braking torque must be equal to the torque due to load W newtons, therefore from equations (*i*) and (*ii*),

$$W = 69.6 \times 10^6/225 = 309 \times 10^3 \text{ N} = 309 \text{ kN}$$

Now let us consider that the drum rotates in anticlockwise direction. We know that when the drum rotates in anticlockwise direction, the end of the band attached to A will be tight with tension T_1 and end of the band attached to B will be slack with tension T_2, as shown in Fig. 19.16. The ratio of tensions T_1 and T_2 will be same as calculated above, *i.e.*



All dimensions in mm.
Fig. 19.16

$$\frac{T_1}{T_2} = 2.98 \quad \text{or } T_1 = 2.98\ T_2$$

Now taking moments about the fulcrum O,

$$750 \times 3000 + T_2 \times 25 = T_1 \times 100$$

or $2.98\ T_2 \times 100 - T_2 \times 25 = 2250 \times 10^3$ ... ($\because\ T_1 = 2.98\ T_2$)

$$273\ T_2 = 2250 \times 10^3 \quad \text{or} \quad T_2 = 2250 \times 10^3/273 = 8242 \text{ N}$$

and $T_1 = 2.98\ T_2 = 2.98 \times 8242 = 24\ 561$ N

$\therefore$ Braking torque, $T_B = (T_1 \times T_2)\ r$

$$= (24\ 561 - 8242)400 = 6.53 \times 10^6 \text{ N-mm} \quad ...(iii)$$

From equations (*ii*) and (*iii*),

$$W = 6.53 \times 10^6/225 = 29 \times 10^3 \text{ N} = 29 \text{ kN}$$

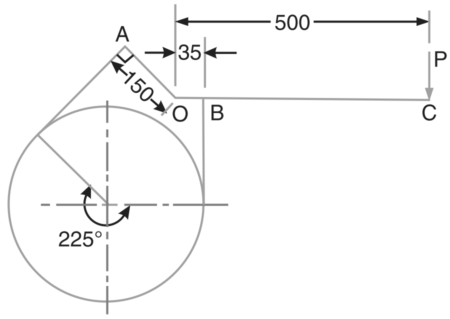
From above, we see that the maximum load (W) that can be supported by the brake is 309 kN, when the drum rotates in clockwise direction. **Ans.**

Example 19.10. *A differential band brake, as shown in Fig. 19.17, has an angle of contact of 225°. The band has a compressed woven lining and bears against a cast iron drum of 350 mm diameter. The brake is to sustain a torque of 350 N-m and the coefficient of friction between the band and the drum is 0.3. Find :* **1.** *The necessary force (P) for the clockwise and anticlockwise rotation of the drum; and* **2.** *The value of 'OA' for the brake to be self locking, when the drum rotates clockwise.*

Solution. Given: $\theta = 225° = 225 \times \pi/180 = 3.93$ rad ; $d = 350$ mm or $r = 175$ mm ; $T = 350$ N-m $= 350 \times 10^3$ N-mm

1. ***Necessary force (P) for the clockwise and anticlockwise rotation of the drum***

When the drum rotates in the clockwise direction, the end of the band attached to A will be slack with tension T_2 and the end of the band attached to B will be tight with tension T_1, as shown in Fig. 19.18. First of all, let us find the values of tensions T_1 and T_2.



All dimensions in mm.

Fig. 19.17

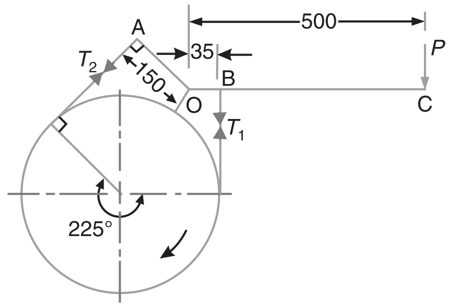


Fig. 19.18

We know that

$$2.3\log\left(\frac{T_1}{T_2}\right) = \mu.\theta = 0.3 \times 3.93 = 1.179$$

$$\therefore \quad \log\left(\frac{T_1}{T_2}\right) = \frac{1.179}{2.3} = 0.5126 \quad \text{or} \quad \frac{T_1}{T_2} = 3.255 \qquad \text{... (Taking antilog of 0.5126) ... } (i)$$

and braking torque (T_B),

$$350 \times 10^3 = (T_1 - T_2)r = (T_1 - T_2)\ 175$$

$$\therefore \quad T_1 - T_2 = 350 \times 10^3/175 = 2000 \text{ N} \qquad \text{... } (ii)$$

From equations (*i*) and (*ii*), we find that

$$T_1 = 2887 \text{ N ; and } T_2 = 887 \text{ N}$$

Now taking moments about the fulcrum *O*, we have

$$P \times 500 = T_2 \times 150 - T_1 \times 35 = 887 \times 150 - 2887 \times 35 = 32 \times 10^3$$

$$\therefore \quad P = 32 \times 10^3/500 = 64 \text{ N } \textbf{Ans.}$$

When the drum rotates in the anticlockwise direction, the end of the band attached to A will be tight with tension T_1 and end of the band attached to *B* will be slack with tension T_2, as shown in Fig. 19.19. Taking moments about the fulcrum *O*, we have

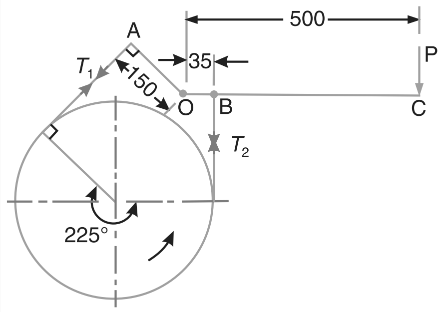


Fig. 19.19

$$P \times 500 = T_1 \times 150 - T_2 \times 35$$
$$= 2887 \times 150 - 887 \times 35$$
$$= 402 \times 10^3$$
$$P = 402 \times 10^3/500 = 804 \text{ N } \textbf{Ans.}$$

2. ***Value of 'OA' for the brake to be self locking, when the drum rotates clockwise***

The clockwise rotation of the drum is shown in Fig 19.18.

For clockwise rotation of the drum, we know that

$$P \times 500 = T_2 \times OA - T_1 \times OB$$

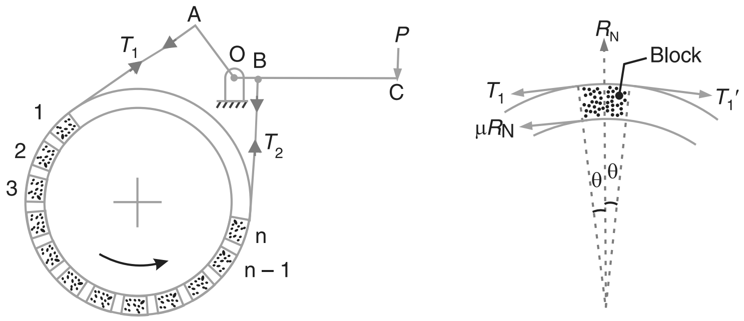
For the brake to be self locking, *P* must be equal to zero. Therefore

$$T_2 \times OA = T_1 \times OB$$

and
$$OA = \frac{T_1 \times OB}{T_2} = \frac{2887 \times 35}{887} = 114 \text{ mm } \textbf{Ans.}$$

19.9. Band and Block Brake

The band brake may be lined with blocks of wood or other material, as shown in Fig. 19.20 (*a*). The friction between the blocks and the drum provides braking action. Let there are '*n*' number of blocks, each subtending an angle 2θ at the centre and the drum rotates in anticlockwise direction.



(*a*) (*b*)

Fig. 19.20. Band and block brake.

Let T_1 = Tension in the tight side,

T_2 = Tension in the slack side,

μ = Coefficient of friction between the blocks and drum,

T_1' = Tension in the band between the first and second block,

T_2', T_3' etc.= Tensions in the band between the second and third block, between the third and fourth block etc.

Consider one of the blocks (say first block) as shown in Fig. 19.20 (*b*). This is in equilibrium under the action of the following forces :

1. Tension in the tight side (T_1),

2. Tension in the slack side (T_1') or tension in the band between the first and second block,

3. Normal reaction of the drum on the block (R_N), and

4. The force of friction ($\mu.R_N$).

Resolving the forces radially, we have

$$(T_1 + T_1')\sin\theta = R_N \qquad \text{... } (i)$$

Resolving the forces tangentially, we have

$$(T_1 + T_1')\cos\theta = \mu.R_N \qquad \text{... } (ii)$$

Dividing equation (*ii*) by (*i*), we have

$$\frac{(T_1 - T_1')\cos\theta}{(T_1 + T_1')\sin\theta} = \frac{\mu.R_N}{R_N}$$

or

$$(T_1 - T_1') = \mu\tan\theta\,(T_1 + T_1')$$

$\therefore$

$$\frac{T_1}{T_1'} = \frac{1 + \mu\tan\theta}{1 - \mu\tan\theta}$$

Similarly, it can be proved for each of the blocks that

$$\frac{T_1'}{T_2'} = \frac{T_2'}{T_3'} = \frac{T_3'}{T_4'} = \text{........}\frac{T_{n-1}}{T_2} = \frac{1 + \mu\tan\theta}{1 - \mu\tan\theta}$$

$$\therefore \qquad \frac{T_1}{T_2} = \frac{T_1}{T_1'} \times \frac{T_1'}{T_2'} \times \frac{T_2'}{T_3'} \times \text{.........} \times \frac{T_{n-1}}{T_2} = \left(\frac{1+\mu\tan\theta}{1-\mu\tan\theta}\right)^n \qquad \text{... (iii)}$$

Braking torque on the drum of effective radius r_e,

$$T_B = (T_1 - T_2)\, r_e$$
$$= (T_1 - T_2)\, r \qquad \text{... [Neglecting thickness of band]}$$

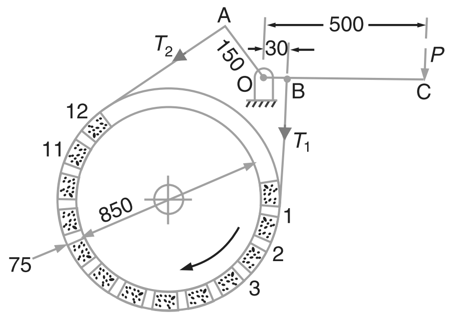
Note : For the first block, the tension in the tight side is T_1 and in the slack side is T_1' and for the second block, the tension in the tight side is T_1' and in the slack side is T_2'. Similarly for the third block, the tension in the tight side is T_2' and in the slack side is T_3' and so on. For the last block, the tension in the tight side is T_{n-1} and in the slack side is T_2.

Example 19.11. *In the band and block brake shown in Fig. 19.21, the band is lined with 12 blocks each of which subtends an angle of 15° at the centre of the rotating drum. The thickness of the blocks is 75 mm and the diameter of the drum is 850 mm. If, when the brake is in action, the greatest and least tensions in the brake strap are T_1 and T_2, show that*

$$\frac{T_1}{T_2} = \left(\frac{1+\mu\tan 7.5°}{1-\mu\tan 7.5°}\right)^{12}, \text{ where } \mu \text{ is the}$$

coefficient of friction for the blocks.

With the lever arrangement as shown in Fig.19.21, find the least force required at C for the blocks to absorb 225 kW at 240 r.p.m. The coefficient of friction between the band and blocks is 0.4.



All dimensions in mm.

Fig. 19.21

Solution. Given : $n = 12$; $2\theta = 15°$ or $\theta = 7.5°$; $t = 75$ mm $= 0.075$ m ; $d = 850$ mm $= 0.85$ m ; Power $= 225$ kW $= 225 \times 10^3$ W ; $N = 240$ r.p.m.; $\mu = 0.4$

Since $OA > OB$, therefore the force at C must act downward. Also, the drum rotates clockwise, therefore the end of the band attached to A will be slack with tension T_2 (least tension) and the end of the band attached to B will be tight with tension T_1 (greatest tension).

Consider one of the blocks (say first block) as shown in Fig. 19.22. This is in equilibrium under the action of the following four forces :

1. Tension in the tight side (T_1),

2. Tension in the slack side (T_1') or the tension in the band between the first and second block,

3. Normal reaction of the drum on the block (R_N), and

4. The force of friction ($\mu.R_N$).

Resolving the forces radially, we have

$$(T_1 + T_1')\sin 7.5° = R_N \qquad \text{... (i)}$$

Resolving the forces tangentially, we have

$$(T_1 - T_1')\cos 7.5° = \mu.R_N \qquad \text{... (ii)}$$

Dividing equation (*ii*) by (*i*), we have

$$\frac{(T_1 - T_1')\cos 7.5°}{(T_1 + T_1')\sin 7.5°} = \mu \quad \text{or} \quad \frac{T_1 - T_1'}{T_1 + T_1'} = \mu\tan 7.5°$$

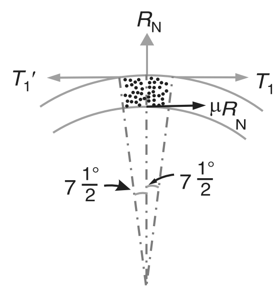


Fig. 19.22

$\therefore \quad T_1 - T_1' = T_1.\mu \tan 7.5° + T_1'.\mu \tan 7.5°$

or $\quad T_1(1 - \mu \tan 7.5°) = T_1'(1 + \mu \tan 7.5°)$

$$\therefore \quad \frac{T_1}{T_1'} = \left(\frac{1 + \mu \tan 7.5°}{1 - \mu \tan 7.5°}\right)$$

Similarly, for the other blocks, the ratio of tensions $\frac{T_1'}{T_2'} = \frac{T_2'}{T_3'}$ etc. remains constant.

Therefore for 12 blocks having greatest tension T_1 and least tension T_2 is

$$\frac{T_1}{T_2} = \left(\frac{1 + \mu \tan 7.5°}{1 - \mu \tan 7.5°}\right)^{12}$$

Least force required at C

Let $\quad P$ = Least force required at C.

We know that diameter of band,

$$D = d + 2t = 0.85 + 2 \times 0.075 = 1 \text{ m}$$

$$\therefore \quad \text{Power absorbed} = \frac{(T_1 - T_2)\pi D.N}{60}$$

or $$T_1 - T_2 = \frac{\text{Power} \times 60}{\pi\, DN} = \frac{225 \times 10^3 \times 60}{\pi \times 1 \times 240} = 17\ 900 \text{ N} \quad \text{... } (iii)$$

We have proved that

$$\frac{T_1}{T_2} = \left(\frac{1 + \mu \tan 7.5°}{1 - \mu \tan 7.5°}\right)^{12} = \left(\frac{1 + 0.4 \times 0.1317}{1 - 0.4 \times 0.1317}\right)^{12} = \left(\frac{1.0527}{0.9473}\right)^{12} = 3.55 \quad \text{... } (iv)$$

From equations (*iii*) and (*iv*), we find that

$$T_1 = 24\ 920 \text{ N, and } T_2 = 7020 \text{ N}$$

Now taking moments about O, we have

$$P \times 500 = T_2 \times 150 - T_1 \times 30 = 7020 \times 150 - 24\ 920 \times 30 = 305\ 400$$

$$\therefore \quad P = 305\ 400 / 500 = 610.8 \text{ N } \textbf{Ans.}$$

Example 19.12. *A band and block brake, having 14 blocks each of which subtends an angle of 15° at the centre, is applied to a drum of 1 m effective diameter. The drum and flywheel mounted on the same shaft has a mass of 2000 kg and a combined radius of gyration of 500 mm. The two ends of the band are attached to pins on opposite sides of the brake lever at distances of 30 mm and 120 mm from the fulcrum. If a force of 200 N is applied at a distance of 750 mm from the fulcrum, find:*

1. maximum braking torque, 2. angular retardation of the drum, and 3. time taken by the system to come to rest from the rated speed of 360 r.p.m.

The coefficient of friction between blocks and drum may be taken as 0.25.

Solution. Given : $n = 14$; $2\theta = 15°$ or $\theta = 7.5°$; $d = 1$ m or $r = 0.5$ m ; $m = 2000$ kg ; $k = 500$ mm = 0.5 m ; $P = 200$ N ; $N = 360$ r.p.m. ; $l = 750$ mm ; $\mu = 0.25$

1. ***Maximum braking torque***

The braking torque will be maximum when $OB > OA$ and the drum rotates anticlockwise as shown in Fig. 19.23. The force P must act upwards and the end of the band attached to A is tight under tension T_1 and the end of the band attached to B is slack under tension T_2.

Taking moments about O,

$$200 \times 750 + T_1 \times 30 = T_2 \times 120$$

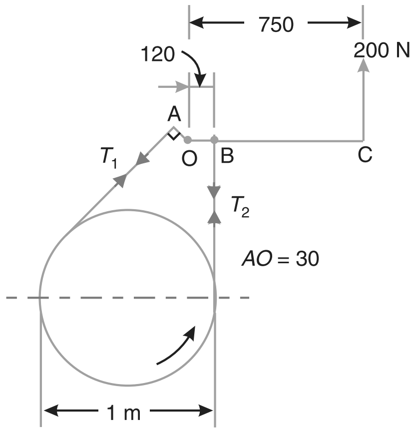
$$12\, T_2 - 3T_1 = 15\,000 \quad \ldots (i)$$

We know that $\dfrac{T_1}{T_2} = \left(\dfrac{1+\mu \tan\theta}{1-\mu\tan\theta}\right)^n$

$$= \left(\frac{1+0.25\tan 7.5°}{1-0.25\tan 7.5°}\right)^{14}$$

$$= \left(\frac{1+0.25\times 0.1317}{1-.025\times 0.1317}\right)^{14}$$

$$= (1.068)^{14} = 2.512 \ldots (ii)$$



All dimensions in mm
Fig. 19.23

From equations (*i*) and (*ii*),

$$T_1 = 8440 \text{ N, and } T_2 = 3360 \text{ N}$$

We know that maximum braking torque,

$$T_B = (T_1 - T_2)\, r = (8440 - 3360)0.5 = 2540 \text{ N-m} \quad \textbf{Ans.}$$

2. ***Angular retardation of the drum***

Let α = Angular retardation of the drum.

We know that braking torque (T_B),

$$2540 = I.\alpha = m.k^2.\alpha = 2000(0.5)^2\alpha = 500\alpha$$

$\therefore$ $\alpha = 2540/500 = 5.08$ rad/s^2 **Ans.**

3. ***Time taken by the system to come to rest***

Let t = Required time.

Since the system is to come to rest from the rated speed of 360 r.p.m., therefore

Initial angular speed, $\omega_1 = 2\pi\times 360/60 = 37.7$ rad/s

and final angular speed, $\omega_2 = 0$

We know that $\omega_2 = \omega_1 - \alpha.t$ ... (– ve sign due to retardation)

$\therefore$ $t = \omega_1/\alpha = 37.7/5.08 = 7.42$ s **Ans.**

19.10. Internal Expanding Brake

An internal expanding brake consists of two shoes S_1 and S_2 as shown in Fig. 19.24. The outer surface of the shoes are lined with some friction material (usually with Ferodo) to increase the coefficient of friction and to prevent wearing away of the metal. Each shoe is pivoted at one end about a fixed fulcrum O_1 and O_2 and made to contact a cam at the other end. When the cam rotates, the shoes are pushed outwards against the rim of the drum. The friction between the shoes and the drum produces the braking torque and hence reduces the speed of the drum. The shoes are normally held in off position by a spring as shown in Fig. 19.24. The drum encloses the entire mechanism to keep out dust and moisture. This type of brake is commonly used in motor cars and light trucks.

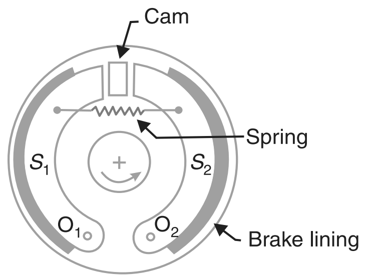


Fig. 19.24. Internal expanding brake.

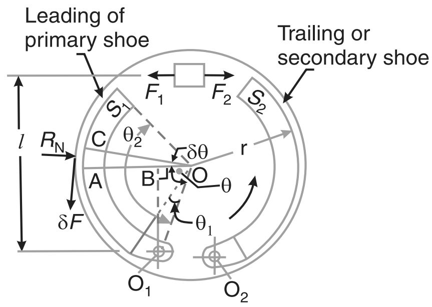


Fig. 19.25. Forces on an internal expanding brake.

We shall now consider the forces acting on such a brake, when the drum rotates in the anticlockwise direction as shown in Fig. 19.25. It may be noted that for the anticlockwise direction, the left hand shoe is known as ***leading*** or ***primary shoe*** while the right hand shoe is known as ***trailing*** or ***secondary shoe***.

Let r = Internal radius of the wheel rim,

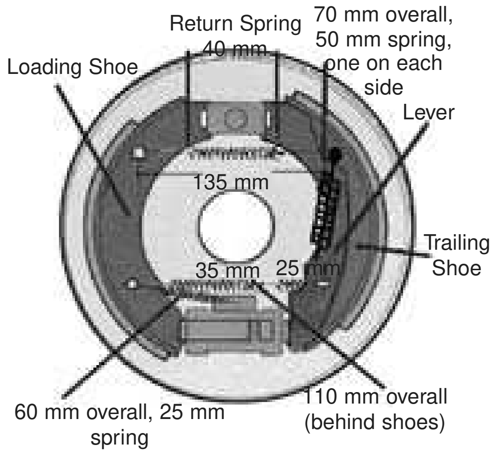
b = Width of the brake lining,

p_1 = Maximum intensity of normal pressure,

p_N = Normal pressure,

F_1 = Force exerted by the cam on the leading shoe, and

F_2 = Force exerted by the cam on the trailing shoe.



Internal expanding brake.

Consider a small element of the brake lining AC subtending an angle $\delta\theta$ at the centre. Let OA makes an angle θ with OO_1 as shown in Fig. 19.25. It is assumed that the pressure distribution on the shoe is nearly uniform, however the friction lining wears out more at the free end. Since the shoe turns about O_1, therefore the rate of wear of the shoe lining at A will be proportional to the radial displacement of that point. The rate of wear of the shoe lining varies directly as the perpendicular distance from O_1 to OA, *i.e.* O_1B. From the geometry of the figure,

$$O_1B = OO_1 \sin\theta$$

and normal pressure at A,

$$p_N \propto \sin\theta \quad \text{or} \quad p_N = p_1 \sin\theta$$

∴ Normal force acting on the element,

$$\delta R_N = \text{Normal pressure} \times \text{Area of the element}$$

$$= p_N (b.r.\delta\theta) = p_1 \sin\theta (b.r.\delta\theta)$$

and braking or friction force on the element,

$$\delta F = \mu \times \delta R_N = \mu.p_1 \sin\theta (b.r.\delta\theta)$$

∴ Braking torque due to the element about O,

$$\delta T_B = \delta F \times r = \mu.p_1 \sin\theta (b.r.\delta\theta) r = \mu.\, p_1 b\, r^2 (\sin\theta.\delta\theta)$$

and total braking torque about O for whole of one shoe,

$$T_B = \mu\, p_1 b\, r^2 \int_{\theta_1}^{\theta_2} \sin\theta\, d\theta = \mu\, p_1 b\, r^2 \left[-\cos\theta\right]_{\theta_1}^{\theta_2}$$

$$= \mu\, p_1 b r^2 (\cos\theta_1 - \cos\theta_2)$$

Moment of normal force δR_N of the element about the fulcrum O_1,

$$\delta M_N = \delta R_N \times O_1 B = \delta R_N (OO_1 \sin\theta)$$

$$= p_1 \sin\theta (b.r.\delta\theta)(OO_1 \sin\theta) = p_1 \sin^2\theta (b.r.\delta\theta) OO_1$$

$\therefore$ Total moment of normal forces about the fulcrum O_1,

$$M_N = \int_{\theta_1}^{\theta_2} p_1 \sin^2\theta (b.r.\delta\theta) OO_1 = p_1.b.r.OO_1 \int_{\theta_1}^{\theta_2} \sin^2\theta\, d\theta$$

Internal exparding brake.

$$= p_1.b.r.OO_1 \int_{\theta_1}^{\theta_2} \frac{1}{2}(1-\cos 2\theta) d\theta \qquad \ldots \left[\because \sin^2\theta = \frac{1}{2}(1-\cos 2\theta)\right]$$

$$= \frac{1}{2} p_1.b.r.OO_1 \left[\theta - \frac{\sin 2\theta}{2}\right]_{\theta_1}^{\theta_2}$$

$$= \frac{1}{2} p_1.b.r.OO_1 \left[\theta_2 - \frac{\sin 2\theta_2}{2} - \theta_1 + \frac{\sin 2\theta_1}{2}\right]$$

$$= \frac{1}{2} p_1.b.r.OO_1 \left[(\theta_2 - \theta_1) + \frac{1}{2}(\sin 2\theta_1 - \sin 2\theta_2)\right]$$

Moment of frictional force δF about the fulcrum O_1,

$$\delta M_F = \delta F \times AB = \delta F (r - OO_1 \cos\theta) \qquad \ldots (\because AB = r - OO_1 \cos\theta)$$

$$= \mu\, p_1 \sin\theta (b.r.\delta\theta)(r - OO_1 \cos\theta)$$

$$= \mu.p_1.b.r(r \sin\theta - OO_1 \sin\theta\cos\theta)\delta\theta$$

$$= \mu.p_1.b.r\left(r\sin\theta - \frac{OO_1}{2}\sin 2\theta\right)\delta\theta \qquad \ldots (\because 2\sin\theta\cos\theta = \sin 2\theta)$$

$\therefore$ Total moment of frictional force about the fulcrum O_1,

$$M_F = \mu\, p_1 b\, r \int_{\theta_1}^{\theta_2} \left(r\sin\theta - \frac{OO_1}{2}\sin 2\theta\right) d\theta$$

$$= \mu\, p_1 b\, r \left[-r\cos\theta + \frac{OO_1}{4}\cos 2\theta\right]_{\theta_1}^{\theta_2}$$

$$= \mu\, p_1 b\, r \left[-r\cos\theta_2 + \frac{OO_1}{4}\cos 2\theta_2 + r\cos\theta_1 - \frac{OO_1}{4}\cos 2\theta_1\right]$$

$$= \mu\, p_1 b\, r \left[r(\cos\theta_1 - \cos\theta_2) + \frac{OO_1}{4}(\cos 2\theta_2 - \cos 2\theta_1)\right]$$

Now for leading shoe, taking moments about the fulcrum O_1,

$$F_1 \times l = M_N - M_F$$

and for trailing shoe, taking moments about the fulcrum O_2,

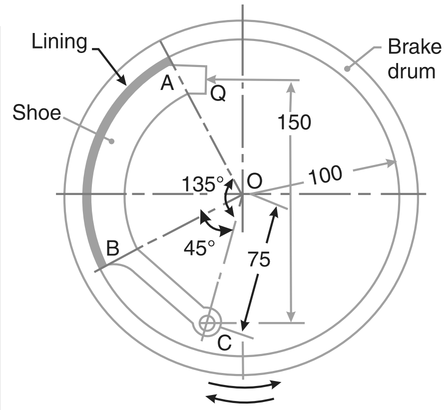
$$F_2 \times l = M_N + M_F$$

Note : If $M_F > M_N$, then the brake becomes self locking.

Example 19.13. *The arrangement of an internal expanding friction brake, in which the brake shoe is pivoted at 'C' is shown in Fig. 19.26. The distance 'CO' is 75 mm, O being the centre of the drum. The internal radius of the brake drum is 100 mm. The friction lining extends over an arc AB, such that the angle AOC is 135° and angle BOC is 45°. The brake is applied by means of a force at Q, perpendicular to the line CQ, the distance CQ being 150 mm.*

The local rate of wear on the lining may be taken as proportional to the normal pressure on an element at an angle of 'θ' with OC and may be taken as equal to $p_1 \sin\theta$, where p_1 is the maximum intensity of normal pressure.

The coefficient of friction may be taken as 0.4 and the braking torque required is 21 N-m. Calculate the force Q required to operate the brake when **1.** *The drum rotates clockwise, and* **2.** *The drum rotates anticlockwise.*



All dimensions in mm

Fig. 19.26

Solution. Given : $OC = 75$ mm ; $r = 100$ mm ; $\theta_2 = 135° = 135 \times \pi/180 = 2.356$ rad ; $\theta_1 = 45° = 45 \times \pi/180 = 0.786$ rad ; $l = 150$ mm ; $\mu = 0.4$; $T_B = 21$ N-m $= 21 \times 10^3$ N-mm

1. ***Force 'Q' required to operate the brake when drum rotates clockwise***

We know that total braking torque due to shoe (T_B),

$$21 \times 10^3 = \mu . p_1 . b . r^2 (\cos\theta_1 - \cos\theta_2)$$

$$= 0.4 \times p_1 \times b (100)^2 (\cos 45° - \cos 135°) = 5656\ p_1 . b$$

$$\therefore \quad p_1 . b = 21 \times 10^3 / 5656 = 3.7$$

Total moment of normal forces about the fulcrum C,

$$M_N = \frac{1}{2} p_1 . b . r . OC \left[(\theta_2 - \theta_1) + \frac{1}{2} (\sin 2\theta_1 - \sin 2\theta_2) \right]$$

$$= \frac{1}{2} \times 3.7 \times 100 \times 75 \left[(2.356 - 0.786) + \frac{1}{2} (\sin 90° - \sin 270°) \right]$$

$$= 13\ 875\ (1.57 + 1) = 35\ 660 \text{ N-mm}$$

and total moment of friction force about the fulcrum C,

$$M_F = \mu . p_1 . b . r \left[r(\cos\theta_1 - \cos\theta_2) + \frac{OC}{4} (\cos 2\theta_2 - \cos 2\theta_1) \right]$$

$$= 0.4 \times 3.7 \times 100 \left[100 (\cos 45° - \cos 135°) + \frac{75}{4} (\cos 270° - \cos 90°) \right]$$

$$= 148 \times 141.4 = 20\ 930 \text{ N-mm}$$

Taking moments about the fulcrum C, we have

$$Q \times 150 = M_N + M_F = 35\,660 + 20\,930 = 56\,590$$

$$\therefore \quad Q = 56\,590 / 150 = 377 \text{ N} \quad \textbf{Ans.}$$

2. ***Force 'Q' required to operate the brake when drum rotates anticlockwise***

Taking moments about the fulcrum C, we have

$$Q \times 150 = M_N - M_F = 35\,660 - 20\,930 = 14\,730$$

$$\therefore \quad Q = 14\,730/150 = 98.2 \text{ N} \quad \textbf{Ans.}$$

19.11. Braking of a Vehicle

In a four wheeled moving vehicle, the brakes may be applied to

1. the rear wheels only,
2. the front wheels only, and
3. all the four wheels.

In all the above mentioned three types of braking, it is required to determine the retardation of the vehicle when brakes are applied. Since the vehicle retards, therefore it is a problem of dynamics. But it may be reduced to an equivalent problem of statics by including the inertia force in the system of forces actually applied to the vehicle. The inertia force is equal and opposite to the braking force causing retardation.

Now, consider a vehicle moving up an inclined plane, as shown in Fig. 19.27.

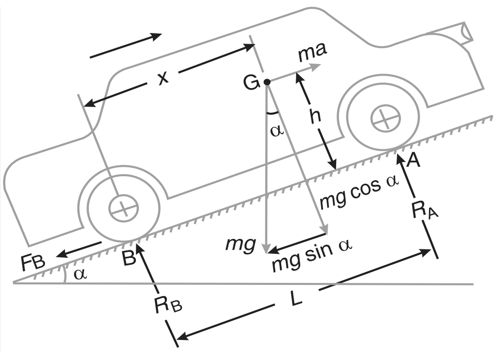


Fig. 19.27. Motion of vehicle up the inclined plane and brakes are applied to rear wheels only.

Let α = Angle of inclination of the plane to the horizontal,

m = Mass of the vehicle in kg (such that its weight is $m.g$ newtons),

h = Height of the C.G. of the vehicle above the road surface in metres,

x = Perpendicular distance of C.G. from the rear axle in metres,

L = Distance between the centres of the rear and front wheels (also called wheel base) of the vehicle in metres,

R_A = Total normal reaction between the ground and the front wheels in newtons,

R_B = Total normal reaction between the ground and the rear wheels in newtons,

μ = Coefficient of friction between the tyres and road surface, and

a = Retardation of the vehicle in m/s^2.

We shall now consider the above mentioned three cases of braking, one by one. In all these cases, the braking force acts in the opposite direction to the direction of motion of the vehicle.

1. ***When the brakes are applied to the rear wheels only***

It is a common way of braking the vehicle in which the braking force acts at the rear wheels only.

Let F_B = Total braking force (in newtons) acting at the rear wheels due to the application of the brakes. Its maximum value is $\mu.R_B$.

The various forces acting on the vehicle are shown in Fig. 19.27. For the equilibrium of the vehicle, the forces acting on the vehicle must be in equilibrium.

Resolving the forces parallel to the plane,

$$F_B + m.g.\sin\alpha = m.a \qquad \ldots (i)$$

Resolving the forces perpendicular to the plane,

$$R_A + R_B = m.g \cos\alpha \qquad \ldots (ii)$$

Taking moments about G, the centre of gravity of the vehicle,

$$F_B \times h + R_B \times x = R_A (L - x) \qquad \ldots (iii)$$

Substituting the value of $F_B = \mu.R_B$, and $R_A = m.g \cos\alpha - R_B$ [from equation (*ii*)] in the above expression, we have

$$\mu.R_B \times h + R_B \times x = (m.g \cos\alpha - R_B)(L - x)$$

$$R_B (L + \mu.h) = m.g \cos\alpha (L - x)$$

$$\therefore \qquad R_B = \frac{m.g \cos\alpha (L - x)}{L + \mu.h}$$

and

$$R_A = m.g \cos\alpha - R_B = m.g \cos\alpha - \frac{m.g \cos\alpha (L - x)}{L + \mu.h}$$

$$= \frac{m.g \cos\alpha (x + \mu.h)}{L + \mu.h}$$

We know from equation (*i*),

$$a = \frac{F_B + m.g \sin\alpha}{m} = \frac{F_B}{m} + g \sin\alpha = \frac{\mu.R_B}{\text{m}} + g \sin\alpha$$

$$= \frac{\mu.g \cos\alpha (L - x)}{L + \mu.h} + g \sin\alpha \qquad \ldots \text{(Substituting the value of } R_B\text{)}$$

Notes : 1. When the vehicle moves on a level track, then $\alpha = 0$.

$$\therefore \qquad R_B = \frac{m.g(L - x)}{L + \mu.h}\,;\; R_A = \frac{m.g(x + \mu.h)}{L + \mu.h} \text{ and } a = \frac{\mu.g(L - x)}{L + \mu.h}$$

2. If the vehicle moves down the plane, then equation (*i*) becomes

$$F_B - m.g \sin\alpha = m.a$$

$$\therefore \qquad a = \frac{F_B}{m} - g.\sin\alpha = \frac{\mu.R_B}{m} - g.\sin\alpha = \frac{\mu.g \cos\alpha (L - x)}{L + \mu.h} - g \sin\alpha$$

2. *When the brakes are applied to front wheels only*

It is a very rare way of braking the vehicle, in which the braking force acts at the front wheels only.

Let F_A = Total braking force (in newtons) acting at the front wheels due to the application of brakes. Its maximum value is $\mu.R_A$.

The various forces acting on the vehicle are shown in Fig. 19.28.

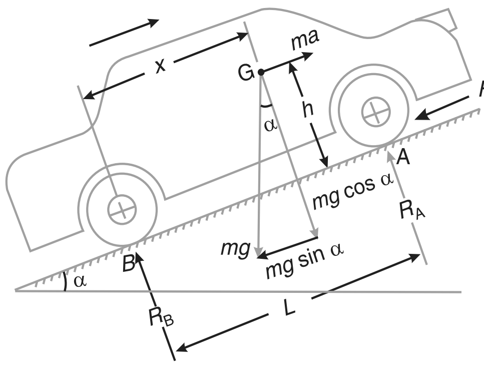


Fig. 19.28. Motion of the vehicle up the inclined plane and brakes are applied to front wheels only.

Resolving the forces parallel to the plane,

$$F_A + m.g \sin\alpha = m.a \qquad \ldots (i)$$

Resolving the forces perpendicular to the plane,

$$R_A + R_B = m.g \cos\alpha \qquad \ldots (ii)$$

Taking moments about G, the centre of gravity of the vehicle,

$$F_A \times h + R_B \times x = R_A (L - x)$$

Substituting the value of $F_A = \mu.R_A$ and $R_B = m.g\cos\alpha - R_A$ [from equation (*ii*)] in the above expression, we have

$$\mu.R_A \times h + (m.g\cos\alpha - R_A)\,x = R_A (L - x)$$

$$\mu.R_A \times h + m.g\cos\alpha \times x = R_A \times L$$

$$\therefore \quad R_A = \frac{m.g\cos\alpha \times x}{L - \mu.h}$$

and

$$R_B = m.g\cos\alpha - R_A = m.g\cos\alpha - \frac{m.g\cos\alpha \times x}{L - \mu.h}$$

$$= m.g\cos\alpha\left(1 - \frac{x}{L - \mu.h}\right) = m.g\cos\alpha\left(\frac{L - \mu.h - x}{L - \mu.h}\right)$$

We know from equation (*i*),

$$a = \frac{F_A + m.g\sin\alpha}{m} = \frac{\mu.R_A + m.g\sin\alpha}{m}$$

$$= \frac{\mu.m.g\cos\alpha \times x}{(L - \mu.h)m} + \frac{m.g\sin\alpha}{m} \quad \text{. . . (Substituting the value of } R_A)$$

$$= \frac{\mu.g\cos\alpha \times x}{L - \mu.h} + g\sin\alpha$$

Notes : 1. When the vehicle moves on a level track, then $\alpha = 0$.

$$\therefore \quad R_A = \frac{m.g \times x}{L - \mu.h}; \quad R_B = \frac{m.g(L - \mu.h - x)}{L - \mu.h}; \quad \text{and} \quad a = \frac{\mu.g \cdot x}{L - \mu.h}$$

2. When the vehicle moves down the plane, then equation (*i*) becomes

$$F_A - m.g\sin\alpha = m.a$$

$$\therefore \quad a = \frac{F_A}{\text{m}} - g.\sin\alpha = \frac{\mu.R_A}{m} - g.\sin\alpha = \frac{\mu.g\cos\alpha \times x}{L - \mu.h} - g\sin\alpha$$

3. *When the brakes are applied to all the four wheels*

This is the most common way of braking the vehicle, in which the braking force acts on both the rear and front wheels.

Let F_A = Braking force provided by the front wheels = $\mu.R_A$, and

F_B = Braking force provided by the rear wheels = $\mu.R_B$.

A little consideration will show that when the brakes are applied to all the four wheels, the braking distance (*i.e.* the distance in which the vehicle is brought to rest after applying the brakes) will be the least. It is due to this reason that the brakes are applied to all the four wheels.

The various forces acting on the vehicle are shown in Fig. 19.29.

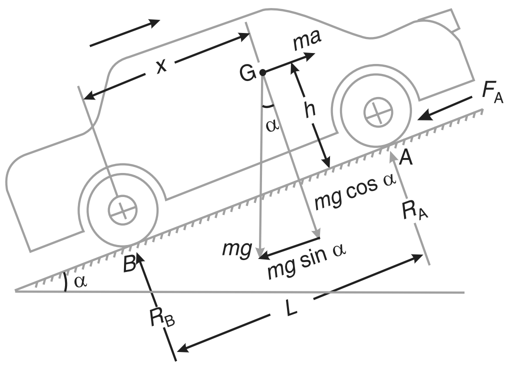


Fig. 19.29. Motion of the vehicle up the inclined plane and the brakes are applied to all the four wheels.

Resolving the forces parallel to the plane,

$$F_A + F_B + m.g \sin\alpha = m.a \qquad \ldots (i)$$

Resolving the forces perpendicular to the plane,

$$R_A + R_B = m.g \cos\alpha \qquad \ldots (ii)$$

Taking moments about G, the centre of gravity of the vehicle,

$$(F_A + F_B)h + R_B \times x = R_A (L - x) \qquad \ldots (iii)$$

Substituting the value of $F_A = \mu.R_A$, $F_B = \mu.R_B$ and $R_B = m.g \cos\alpha - R_A$ [From equation (*ii*)] in the above expression,

$$\mu(R_A + R_B)h + (m.g \cos\alpha - R_A)x = R_A(L - x)$$

$$\mu(R_A + m.g \cos\alpha - R_A)h + (m.g \cos\alpha - R_A)x = R_A(L - x)$$

$$\mu.m.g \cos\alpha \times h + m.g \cos\alpha \times x = R_A \times L$$

$$\therefore \qquad R_A = \frac{m.g \cos\alpha(\mu.h + x)}{L}$$

and

$$R_B = m.g \cos\alpha - R_A = m.g \cos\alpha - \frac{mg \cos\alpha(\mu.h + x)}{L}$$

$$= m.g \cos\alpha\left[1 - \frac{\mu.h + x}{L}\right] = m.g \cos\alpha\left(\frac{L - \mu.h - x}{L}\right)$$

Now from equation (*i*),

$$\mu.R_A + \mu\, R_B + m.g \sin\alpha = m.a$$

$$\mu\,(R_A + R_B) + m.g \sin\alpha = m.a$$

$$\mu.m.g.\cos\alpha + m.g \sin\alpha = m.a \qquad \ldots \text{[From equation } (ii)]$$

$$\therefore \qquad a = g(\mu.\cos\alpha + \sin\alpha)$$

Notes : 1. When the vehicle moves on a level track, then $\alpha = 0$.

$$\therefore \qquad R_A = \frac{m.g(\mu.h + x)}{L}\,;\; R_B = m.g\left(\frac{L - \mu.h - x}{L}\right); \text{ and } a = g.\mu$$

2. If the vehicle moves down the plane, then equation (*i*) may be written as

$$F_A + F_B - m.g \sin\alpha = m.a$$

or

$$\mu(R_A + R_B) - m.g \sin\alpha = m.a$$

$$\mu.m.g \cos\alpha - m.g \sin\alpha = m.a$$

and

$$a = g(\mu.\cos\alpha - \sin\alpha)$$

Example 19.14. *A car moving on a level road at a speed 50 km/h has a wheel base 2.8 metres, distance of C.G. from ground level 600 mm, and the distance of C.G. from rear wheels 1.2 metres. Find the distance travelled by the car before coming to rest when brakes are applied,*

1. *to the rear wheels,* ***2.*** *to the front wheels, and* ***3.*** *to all the four wheels.*

The coefficient of friction between the tyres and the road may be taken as 0.6.

Solution. Given : $u = 50$ km/h $= 13.89$ m/s ; $L = 2.8$ m ; $h = 600$ mm $= 0.6$ m ; $x = 1.2$ m ; $\mu = 0.6$

Let $\qquad s$ = Distance travelled by the car before coming to rest.

1. When brakes are applied to the rear wheels

Since the vehicle moves on a level road, therefore retardation of the car,

$$a = \frac{\mu.g(L - x)}{L + \mu.h} = \frac{0.6 \times 9.81(2.8 - 1.2)}{2.8 + 0.6 \times 0.6} = 2.98 \text{ m/s}^2$$

We know that for uniform retardation,

$$s = \frac{u^2}{2a} = \frac{(13.89)^2}{2 \times 2.98} = 32.4 \text{ m} \textbf{ Ans.}$$

2. ***When brakes are applied to the front wheels***

Since the vehicle moves on a level road, therefore retardation of the car,

$$a = \frac{\mu . g . x}{L - \mu . h} = \frac{0.6 \times 9.18 \times 1.2}{2.8 - 0.6 \times 0.6} = 2.9 \text{ m/s}^2$$

We know that for uniform retardation,

$$s = \frac{u^2}{2a} = \frac{(13.89)^2}{2 \times 2.9} = 33.26 \text{ m} \textbf{ Ans.}$$

3. ***When the brakes are applied to all the four wheels***

Since the vehicle moves on a level road, therefore retardation of the car,

$$a = g . \mu = 9.81 \times 0.6 = 5.886 \text{ m/s}^2$$

We know that for uniform retardation,

$$s = \frac{u^2}{2a} = \frac{(13.89)^2}{2 \times 5.886} = 16.4 \text{ m} \textbf{ Ans.}$$

Example 19.15. *A vehicle moving on a rough plane inclined at 10° with the horizontal at a speed of 36 km/h has a wheel base 1.8 metres. The centre of gravity of the vehicle is 0.8 metre from the rear wheels and 0.9 metre above the inclined plane. Find the distance travelled by the vehicle before coming to rest and the time taken to do so when* **1.** *The vehicle moves up the plane, and* **2.** *The vehicle moves down the plane.*

The brakes are applied to all the four wheels and the coefficient of friction is 0.5.

Solution. Given : $\alpha = 10°$; $u = 36$ km / h $= 10$ m / s ; $L = 1.8$ m ; $x = 0.8$ m ; $h = 0.9$ m ; $\mu = 0.5$

Let s = Distance travelled by the vehicle before coming to rest, and

t = Time taken by the vehicle in coming to rest.

1. ***When the vehicle moves up the plane and brakes are applied to all the four wheels***

Since the vehicle moves up the inclined plane, therefore retardation of the vehicle,

$$a = g\,(\mu \cos \alpha + \sin \alpha)$$

$$= 9.81\,(0.5 \cos 10° + \sin 10°) = 9.81(0.5 \times 0.9848 + 0.1736) = 6.53 \text{ m/s}^2$$

We know that for uniform retardation,

$$s = \frac{u^2}{2a} = \frac{(10)^2}{2 \times 6.53} = 7.657 \text{ m} \textbf{ Ans.}$$

and final velocity of the vehicle (v),

$$0 = u + a.t = 10 - 6.53\,t \qquad \text{...(Minus sign due to retardation)}$$

$$\therefore \quad t = 10 / 6.53 = 1.53 \text{ s} \textbf{ Ans.}$$

2. ***When the vehicle moves down the plane and brakes are applied to all the four wheels***

Since the vehicle moves down the inclined plane, therefore retardation of the vehicle,

$$a = g\,(\mu \cos \alpha - \sin \alpha)$$

$$= 9.81(0.5 \cos 10° - \sin 10°) = 9.81(0.5 \times 0.9848 - 0.1736) = 3.13 \text{ m/s}^2$$

We know that for uniform retardation,

$$s = \frac{u^2}{2a} = \frac{(10)^2}{2 \times 3.13} = 16 \text{ m } \textbf{Ans.}$$

and final velocity of the vehicle (v),

$$0 = u + a.t = 10 - 3.13\, t \qquad \text{. . . (Minus sign due to retardation)}$$

$$\therefore \quad t = 10/3.13 = 3.2 \text{ s } \textbf{Ans.}$$

Example 19.16. *The wheel base of a car is 3 metres and its centre of gravity is 1.2 metres ahead the rear axle and 0.75 m above the ground level. The coefficient of friction between the wheels and the road is 0.5. Determine the maximum deceleration of the car when it moves on a level road, if the braking force on all the wheels is the same and no wheel slip occurs.*

Solution. Given : $L = 3$ m ; $x = 1.2$ m ; $h = 0.75$ m ; $\mu = 0.5$

Let a = Maximum deceleration of the car,

m = Mass of the car,

F_A and F_B = Braking forces at the front and rear wheels respectively, and

R_A and R_B = Normal reactions at the front and rear wheels respectively.

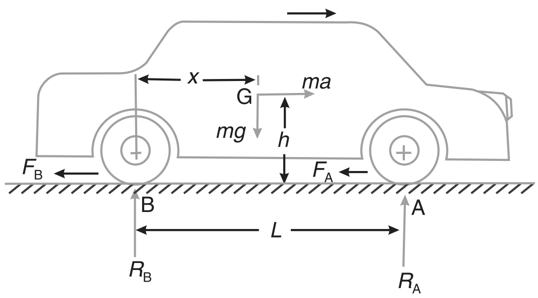


Fig. 19.30

The various forces acting on the car are shown in Fig. 19.30.

We shall consider the following two cases:

(*a*) *When the slipping is imminent at the rear wheels*

We know that when the brakes are applied to all the four wheels and the vehicle moves on a level road, then

$$R_B = m.g\left(\frac{L - \mu.h - x}{L}\right) = m \times 9.81\left(\frac{3 - 0.5 \times 0.75 - 1.2}{3}\right) = 4.66\, m \text{ N}$$

and $\quad F_A + F_B = m.a \quad$ or $\quad 2\mu.\ R_B = m.a \qquad$ . . . ($\because$ $F_B = F_A$ and $F_B = \mu.R_B$)

$\therefore \quad 2 \times 0.5 \times 4.66\, m = m.a \quad$ or $\quad a = 4.66$ m/s^2

(*b*) *When the slipping is imminent at the front wheels*

We know that when the brakes are applied to all the four wheels and the vehicle moves on a level road, then

$$R_A = \frac{m.g(\mu.h + x)}{L} = \frac{m \times 9.81(0.5 \times 0.75 + 1.2)}{3} = 5.15\, m \text{ N}$$

and $\quad F_A + F_B = m.a \quad$ or $\quad 2\mu\ .\ R_A = m.a \qquad$ . . . ($\because$ $F_A = F_B$ and $F_A = \mu\ .\ R_A$)

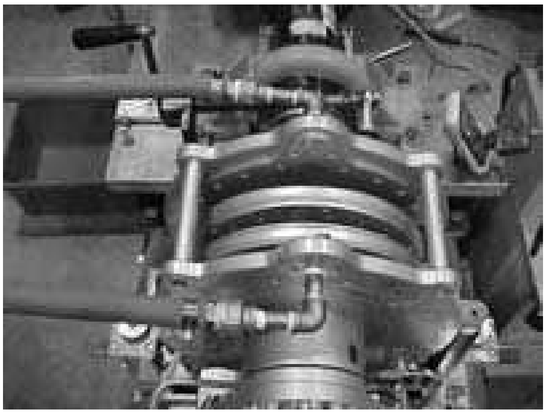
$\therefore \quad 2 \times 0.5 \times 5.15\, m = m.a \quad$ or $\quad a = 5.15$ m/s^2

Hence the maximum possible deceleration is 4.66 m/s^2 and slipping would occur first at the rear wheels. **Ans.**

19.12. Dynamometer

A dynamometer is a brake but in addition it has a device to measure the frictional resistance. Knowing the frictional resistance, we may obtain the torque transmitted and hence the power of the engine.

19.13. Types of Dynamometers



Dynamometers measure the power of the engines.

Following are the two types of dynamometers, used for measuring the brake power of an engine.

1. Absorption dynamometers, and **2.** Transmission dynamometers.

In the ***absorption dynamometers***, the entire energy or power produced by the engine is absorbed by the friction resistances of the brake and is transformed into heat, during the process of measurement. But in the ***transmission dynamometers***, the energy is not wasted in friction but is used for doing work. The energy or power produced by the engine is transmitted through the dynamometer to some other machines where the power developed is suitably measured.

19.14. Classification of Absorption Dynamometers

The following two types of absorption dynamometers are important from the subject point of view :

1. Prony brake dynamometer, and **2.** Rope brake dynamometer.

These dynamometers are discussed, in detail, in the following pages.

19.15. Prony Brake Dynamometer

A simplest form of an absorption type dynamometer is a prony brake dynamometer, as shown in Fig. 19.31. It consists of two wooden blocks placed around a pulley fixed to the shaft of an engine whose power is required to be measured. The blocks are clamped by means of two bolts and nuts, as shown in Fig. 19.31. A helical spring is provided between the nut and the upper block to adjust the pressure on the pulley to control its speed. The upper block has a long lever attached to it and carries a weight W at its outer end. A counter weight is placed at the other end of the lever which balances the brake when unloaded. Two stops S, S are provided to limit the motion of the lever.

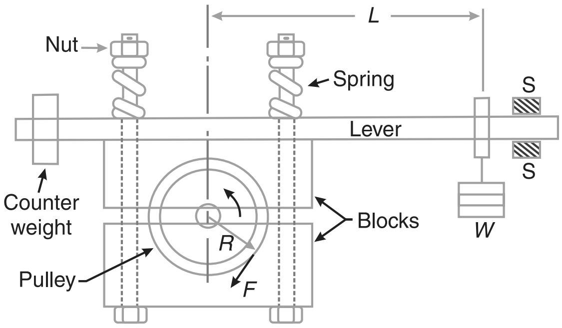


Fig. 19.31. Prony brake dynamometer.

When the brake is to be put in operation, the long end of the lever is loaded with suitable weights W and the nuts are tightened until the engine shaft runs at a constant speed and the lever is in horizontal position. Under these conditions, the moment due to the weight W must balance the moment of the frictional resistance between the blocks and the pulley.

Let W = Weight at the outer end of the lever in newtons,

L = Horizontal distance of the weight W from the centre of the pulley in metres,

F = Frictional resistance between the blocks and the pulley in newtons,

R = Radius of the pulley in metres, and

N = Speed of the shaft in r.p.m.

Another dynamo

We know that the moment of the frictional resistance or torque on the shaft,

$T = W.L = F.R$ N-m

Work done in one revolution

= Torque × Angle turned in radians

$= T \times 2\pi$ N-m

∴ Work done per minute

$= T \times 2\pi N$ N-m

We know that brake power of the engine,

$$B.P. = \frac{\text{Work done per min.}}{60} = \frac{T \times 2\pi N}{60} = \frac{W.L \times 2\pi N}{60} \text{ watts}$$

Notes : 1. From the above expression, we see that while determining the brake power of engine with the help of a prony brake dynamometer, it is not necessary to know the radius of the pulley, the coefficient of friction between the wooden blocks and the pulley and the pressure exerted by tightening of the nuts.

2. When the driving torque on the shaft is not uniform, this dynamometer is subjected to severe oscillations.

19.16. Rope Brake Dynamometer

It is another form of absorption type dynamometer which is most commonly used for measuring the brake power of the engine. It consists of one, two or more ropes wound around the flywheel or rim of a pulley fixed rigidly to the shaft of an engine. The upper end of the ropes is attached to a spring balance while the lower end of the ropes is kept in position by applying a dead weight as shown in Fig. 19.32. In order to prevent the slipping of the rope over the flywheel, wooden blocks are placed at intervals around the circumference of the flywheel.

In the operation of the brake, the engine is made to run at a constant speed. The frictional torque, due to the rope, must be equal to the torque being transmitted by the engine.

Let W = Dead load in newtons,

S = Spring balance reading in newtons,

D = Diameter of the wheel in metres,

d = diameter of rope in metres, and

N = Speed of the engine shaft in r.p.m.

∴ Net load on the brake

$= (W - S)$ N

We know that distance moved in one revolution

$= \pi(D + d)$ m

∴ Work done per revolution

$$= (W - S)\,\pi\,(D + d) \text{ N-m}$$

and work done per minute

$$= (W - S)\,\pi\,(D + d)\,N \text{ N-m}$$

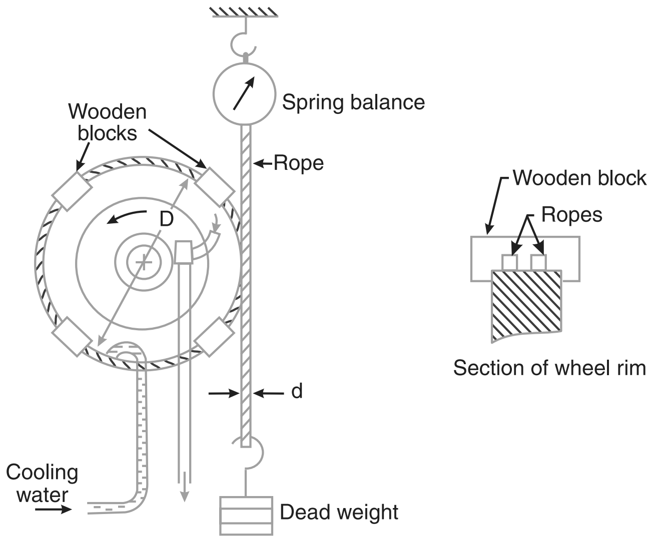


Fig. 19.32. Rope brake dynamometer.

∴ Brake power of the engine,

$$\text{B.P} = \frac{\text{Work done per min}}{60} = \frac{(W - S)\,\pi\,(D + d)N}{60} \text{ watts}$$

If the diameter of the rope (d) is neglected, then brake power of the engine,

$$\text{B.P.} = \frac{(W - S)\,\pi\,D\,N}{60} \text{ watts}$$

An engine is being readied for testing on a dynamometer

Note: Since the energy produced by the engine is absorbed by the frictional resistances of the brake and is transformed into heat, therefore it is necessary to keep the flywheel of the engine cool with soapy water. The flywheels have their rims made of a channel section so as to receive a stream of water which is being whirled round by the wheel. The water is kept continually flowing into the rim and is drained away by a sharp edged scoop on the other side, as shown in Fig. 19.32.

Example 19.17. *In a laboratory experiment, the following data were recorded with rope brake:*

Diameter of the flywheel 1.2 m; diameter of the rope 12.5 mm; speed of the engine 200 r.p.m.; dead load on the brake 600 N; spring balance reading 150 N. Calculate the brake power of the engine.

Solution. Given : D = 1.2 m ; d = 12.5 mm = 0.0125 m ; N = 200 r.p.m ; W = 600 N ; S = 150 N

We know that brake power of the engine,

$$\text{B.P.} = \frac{(W - S)\,\pi\,(D + d)N}{60} = \frac{(600 - 150)\,\pi\,(1.2 + 0.0125)200}{60} = 5715 \text{ W}$$

$= 5.715$ kW **Ans.**

19.17. Classification of Transmission Dynamometers

The following types of transmission dynamometers are important from the subject point of view :

1. Epicyclic-train dynamometer, **2.** Belt transmission dynamometer, and **3.** Torsion dynamometer.

We shall now discuss these dynamometers, in detail, in the following pages.

19.18. Epicyclic-train Dynamometer

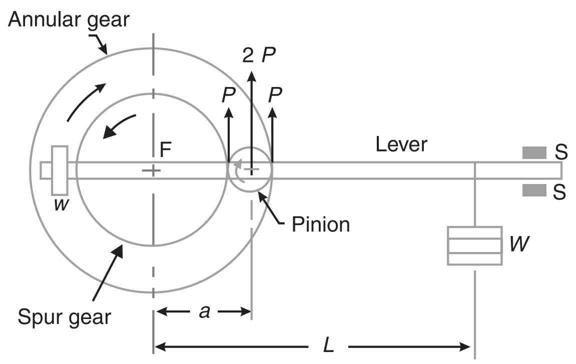


Fig. 19.33. Epicyclic train dynamometer.

An epicyclic-train dynamometer, as shown in Fig. 19.33, consists of a simple epicyclic train of gears, *i.e.* a spur gear, an annular gear (a gear having internal teeth) and a pinion. The spur gear is keyed to the engine shaft (*i.e.* driving shaft) and rotates in anticlockwise direction. The annular gear is also keyed to the driving shaft and rotates in clockwise direction. The pinion or the intermediate gear meshes with both the spur and annular gears. The pinion revolves freely on a lever which is pivoted to the common axis of the driving and driven shafts. A weight w is placed at the smaller end of the lever in order to keep it in position. A little consideration will show that if the friction of the pin on which the pinion rotates is neglected, then the tangential effort P exerted by the spur gear on the pinion and the tangential reaction of the annular gear on the pinion are equal.

Since these efforts act in the upward direction as shown, therefore total upward force on the lever acting through the axis of the pinion is $2P$. This force tends to rotate the lever about its fulcrum and it is balanced by a dead weight W at the end of the lever. The stops S, S are provided to control the movement of the lever.

For equilibrium of the lever, taking moments about the fulcrum F,

$$2P \times a = W.L \quad \text{or} \quad P = W.L/2a$$

Let R = Pitch circle radius of the spur gear in metres, and

N = Speed of the engine shaft in r.p.m.

$\therefore$ Torque transmitted, $T = P.R$

and power transmitted $= \dfrac{T \times 2\pi N}{60} = \dfrac{P.R \times 2\pi N}{60}$ watts

19.19. Belt Transmission Dynamometer-Froude or Throneycroft Transmission Dynamometer

When the belt is transmitting power from one pulley to another, the tangential effort on the driven pulley is equal to the difference between the tensions in the tight and slack sides of the belt. A belt dynamometer is introduced to measure directly the difference between the tensions of the belt, while it is running.

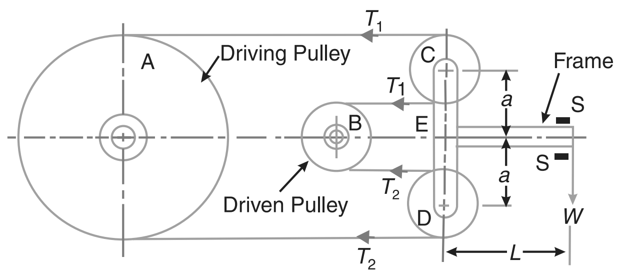


Fig. 19.34. Froude or Throneycroft transmission dynamometer.

A belt transmission dynamometer, as shown in Fig. 19.34, is called a Froude or Throneycroft transmission dynamometer. It consists of a pulley *A* (called driving pulley) which is rigidly fixed to the shaft of an engine whose power is required to be measured. There is another pulley *B* (called driven pulley) mounted on another shaft to which the power from pulley *A* is transmitted. The pulleys *A* and *B* are connected by means of a continuous belt passing round the two loose pulleys *C* and *D* which are mounted on a *T*-shaped frame. The frame is pivoted at *E* and its movement is controlled by two stops *S*,*S*. Since the tension in the tight side of the belt (T_1) is greater than the tension in the slack side of the belt (T_2), therefore the total force acting on the pulley *C* (*i.e.* $2T_1$) is greater than the total force acting on the pulley *D* (*i.e.* $2T_2$). It is thus obvious that the frame causes movement about *E* in the anticlockwise direction. In order to balance it, a weight *W* is applied at a distance *L* from *E* on the frame as shown in Fig. 19.34.

Now taking moments about the pivot *E*, neglecting friction,

$$2T_1 \times a = 2T_2 \times a + W.L \qquad \text{or} \qquad T_1 - T_2 = \frac{W.L}{2a}$$

Let D = diameter of the pulley *A* in metres, and
N = Speed of the engine shaft in r.p.m.

$\therefore$ Work done in one revolution $= (T_1 - T_2)\,\pi\, D$ N-m

and workdone per minute $= (T_1 - T_2)\,\pi\, DN$ N-m

$\therefore$ Brake power of the engine, $\text{B.P.} = \dfrac{(T_1 - T_2)\,\pi\, DN}{60}$ watts

Example 19.18. *The essential features of a transmission dynamometer are shown in Fig. 19.35. A is the driving pulley which runs at 600 r.p.m. B and C are jockey pulleys mounted on a horizontal beam pivoted at D, about which point the complete beam is balanced when at rest. E is the driven pulley and all portions of the belt between the pulleys are vertical. A, B and C are each 300 mm diameter and the thickness and weight of the belt are neglected. The length DF is 750 mm.*

Find : **1.** *the value of the weight W to maintain the beam in a horizontal position when 4.5 kW is being transmitted, and* **2.** *the value of W, when the belt just begins to slip on pulley A. The coefficient of friction being 0.2 and maximum tension in the belt 1.5 kN.*

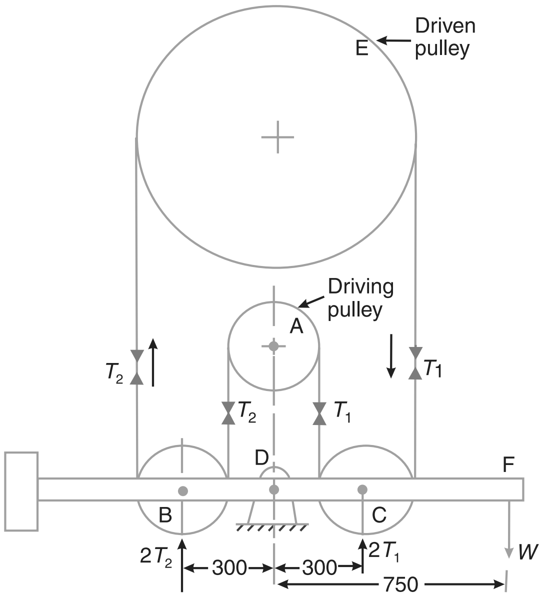


Fig. 19.35. All dimensions in mm.

Solution. Given : $N_A = 600$ r.p.m. : $D_A = D_B = D_C = 300$ mm = 0.3 m

1. ***Value of the weight W to maintain the beam in a horizontal position***

Given : Power transmitted (P) = 4.5 kW = 4500 W

Let T_1 = Tension in the tight side of the belt on pulley A, and

T_2 = Tension in the slack side of the belt on pulley A.

$\therefore$ Force acting upwards on the pulley $C = 2T_1$

and force acting upwards on the pulley $B = 2T_2$

Now taking moments about the pivot D,

$$W \times 750 = 2T_1 \times 300 - 2T_2 \times 300 = 600\ (T_1 - T_2)$$

$$\therefore \quad T_1 - T_2 = W \times 750 / 600 = 1.25\ W \text{ N}$$

We know that the power transmitted (P),

$$4500 = \frac{(T_1 - T_2)\,\pi\, D_A N_A}{60} = \frac{1.25 W \times \pi \times 0.3 \times 600}{60} = 11.78\, W$$

$$\therefore \quad W = 4500 / 11.78 = 382 \text{ N} \quad \textbf{Ans.}$$

2. ***Value of W, when the belt just begins to slip on A***

Given : $\mu = 0.2$; $T_1 = 1.5$ kN = 1500 N

We know that

$$2.3 \log\left(\frac{T_1}{T_2}\right) = \mu.\theta = 0.2 \times \pi = 0.6284 \quad \ldots (\because \theta = 180° = \pi \text{ rad})$$

$$\log\left(\frac{T_1}{T_2}\right) = \frac{0.6284}{2.3} = 0.2732 \quad \text{or} \quad \frac{T_1}{T_2} = 1.876 \quad \ldots \text{(Taking antilog of 0.2732)}$$

$$\therefore \quad T_2 = T_1 / 1.876 = 1500 / 1.876 = 800 \text{ N}$$

Now taking moments about the pivot D,

$$W \times 750 = 2T_1 \times 300 - 2T_2 \times 300 = 2 \times 1500 \times 300 - 2 \times 800 \times 300$$

$$= 420 \times 10^3$$

$$\therefore \quad W = 420 \times 10^3 / 750 = 560 \text{ N} \quad \textbf{Ans.}$$

19.20. Torsion Dynamometer

A torsion dynamometer is used for measuring large powers particularly the power transmitted along the propeller shaft of a turbine or motor vessel. A little consideration will show that when the power is being transmitted, then the driving end of the shaft twists through a small angle relative to the driven end of the shaft. The amount of twist depends upon many factors such as torque acting on the shaft (T), length of the shaft (l), diameter of the shaft (D) and modulus of rigidity (C) of the material of the shaft. We know that the torsion equation is

$$\frac{T}{J} = \frac{C.\theta}{l}$$

where θ = Angle of twist in radians, and

J = Polar moment of inertia of the shaft.

For a solid shaft of diameter D, the polar moment of inertia

$$J = \frac{\pi}{32} \times D^4$$

and for a hollow shaft of external diameter D and internal diameter d, the polar moment of inertia,

$$J = \frac{\pi}{32}(D^4 - d^4)$$

From the above torsion equation,

$$T = \frac{C.J}{l} \times \theta = k.\theta$$

where $k = C.J/l$ is a constant for a particular shaft. Thus, the torque acting on the shaft is proportional to the angle of twist. This means that if the angle of twist is measured by some means, then the torque and hence the power transmitted may be determined.

We know that the power transmitted

$$P = \frac{T \times 2\pi N}{60} \text{ watts, where } N \text{ is the speed in r.p.m.}$$

A number of dynamometers are used to measure the angle of twist, one of which is discussed in Art. 19.21. Since the angle of twist is measured for a small length of the shaft, therefore some magnifying device must be introduced in the dynamometer for accurate measurement.

Example 19.19. *A torsion dynamometer is fitted to a propeller shaft of a marine engine. It is found that the shaft twists 2° in a length of 20 metres at 120 r.p.m. If the shaft is hollow with 400 mm external diameter and 300 mm internal diameter, find the power of the engine. Take modulus of rigidity for the shaft material as 80 GPa.*

Solution. Given : θ = 2° = 2 × π /180 = 0.035 rad ; l = 20 m ; N = 120 r.p.m. ; D = 400 mm = 0.4 m ; d = 300 mm = 0.3 m ; C = 80 GPa = 80×10^9 N/m^2

We know that polar moment of inertia of the shaft,

$$J = \frac{\pi}{32}(D^4 - d^4) = \frac{\pi}{32}\left[(0.4)^4 - (0.3)^4\right] = 0.0017 \text{m}^4$$

and torque applied to the shaft,

$$T = \frac{C.J}{l} \times \theta = \frac{80 \times 10^9 \times 0.0017}{20} \times 0.035 = 238 \times 10^3 \text{ N-m}$$

We know that power of the engine,

$$P = \frac{T \times 2\pi N}{60} = \frac{238 \times 10^3 \times 2\pi \times 120}{60} = 2990 \times 10^3 \text{ W} = 2990 \text{ kW}$$ **Ans.**

DO YOU KNOW ?

1. Distinguish between brakes and dynamometers.
2. Discuss the various types of the brakes.
3. Show that, in a band and block brake, the ratio of the maximum and minimum tensions in the brake straps is

$$\frac{T_0}{T_n} = \left(\frac{1+\mu\tan\theta}{1-\mu\tan\theta}\right)^n$$

where T_0 = Maximum tension,
T_n = Minimum tension
μ = Coefficient of friction between the blocks and drum, and
2θ = Angle subtended by each block at the centre of the drum.

4. Describe with the help of a neat sketch the principles of operation of an internal expanding shoe. Derive the expression for the braking torque.
5. What are the leading and trailing shoes of an internal expanding shoe brake ?
6. What is the difference between absorption and transmission dynamometers ? What are torsion dynamometers ?
7. Describe the construction and operation of a prony brake or rope brake absorption dynamometer.
8. Describe with sketches one form of torsion dynamometer and explain with detail the calculations involved in finding the power transmitted.
9. Explain with neat sketches the Bevis-Gibson flash light dynamometer.

OBJECTIVE TYPE QUESTIONS

1. The brakes commonly used in railway trains is
   (*a*) shoe brake (*b*) band brake
   (*c*) band and block brake (*d*) internal expanding brake
2. The brake commonly used in motor cars is
   (*a*) shoe brake (*b*) band brake
   (*c*) band and block brake (*d*) internal expanding brake
3. In a differential band brake, as shown in Fig. 19.45, the length OA is greater than OB. In order to apply the brake, the force P at C should
   (*a*) be zero (*b*) act in upward direction
   (*c*) act in downward direction
4. For the brake to be self locking, the force P at C as shown in Fig. 19.45, should
   (*a*) be zero
   (*b*) act in upward direction
   (*c*) act in downward direction

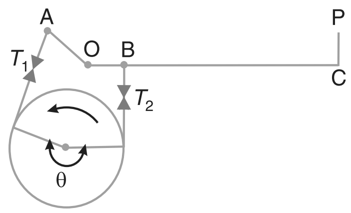


Fig. 19.45

5. When brakes are applied to all the four wheels of a moving car, the distance travelled by the car before it is brought to rest, will be
   (*a*) maximum (*b*) minimum
6. Which of the following is an absorption type dynamometer ?
   (*a*) prony brake dynamometer (*b*) rope brake dynamometer
   (*c*) epicyclic-train dynamometer (*d*) torsion dynamometer

ANSWERS

1. (*a*) **2.** (*d*) **3.** (*c*) **4.** (*a*) **5.** (*b*) **6.** (*a*), (*b*)